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Conformally Transformed Einstein Generalized m-th root with Curvature Properties

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Abstract

The purpose of the present paper is to study the confromal transformation of generalized $m-$th root Finsler metric. The spray co-effecients, Riemannian curvature and Ricci curvature of conformally transformed generalized m-th root metric are shown to be rational function of direction. Further, under certain conditions it is shown that a conformally transformed generalized $m-$th root metric is locally dually flat if and only if the conformal transformation is homothetic. Moreover, the condition for the conformally transformed metrics to be Einstein then, it is Ricci flat and Isotropic mean Berwald curvature are also found.

Key Words : Finsler space, Generalized $m-$th metric, conformal transformation, locally dually flat, Einstein metric, Ricci curvature and Isotropic mean Berwald curvature.
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1. Introduction

The theory of $m-$th root metric has been first developed by the author H. Shimada [20], applied to biology as an ecological metric by P. L. Antonelli [2] and it has been studied by several authors ([3], [21], [22], [23] and [18]). It is regarded as a generalization of Riemannian metric in the sense that the second root metric is a Riemannian metric. The third and fourth metrics are called the cubic and quadratic metric respectively [13]. In four dimension, the special fourth root metric in the form $F = \sqrt[4]{y^1y^2y^3y^4}$ is called the Berwald-Moor metric([9] and [10]), which is considered by physicists as an important subject for a possible model of space time.

Recent studies shows that m-th root Finsler metric play a very important role in physics, space time and general relativity ([6] and [17]). The authors Z. Shen and B. Li [13] were studied the geometric properties of locally projectively flat fourth root metric in the form $F = \sqrt[4]{a_{ijkl}(x)y^iy^jy^ky^l}$ and generalized fourth root metric in the form $\sqrt{\sqrt{a_{ijkl}(x)y^iy^jy^ky^l} + b_{ij}y^iy^j}$.

In [23], the authors Yaoyong Yu and Ying You were proved the m-th root Einstein Finsler metric is Ricci flat. In 2011, A. Tayebi and B. Najafi have been characterized locally dually flat. In [21], A. Tayebi, Peyghan and Shahbazi have found a condition, which is a generalized m-th root metric is projectively related to an m-th root metric. In 2013, the authors Abolfazl Taleshian and Dordi Mohamad Saghali were got the result every generalized m-th root metric with almost vanishing H-curvature has vanishing H-curvature and also expressed a necessary and sufficient condition for the metric $F = \sqrt[m]{A}$ is locally projectively flat and locally dually flat.

This paper is organized as following: we find the spray coefficients of conformally transformed generalized m-th root Finsler metric and it shows that, it is rational function of y. Then, we prove that conformally transformed generalized m-th root metric is locally dually flat if and only if the conformal transformation is homothetic. Further, according to [23], we will study the Einstein metric is Ricci flat. Finally, we show that conformally transformed given metric is weakly Berwald metric.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM be the tangent space at $x \in M$ and $TM = \bigcup_{x \in M} T_xM$ be the tangent bundle of M. Each element of TM has the form (x, y) where $x \in M$ and $y \in T_xM$ called the supporting element and let $TM_0 = TM \backslash \{0\}$.

A Finsler metric on a manifold M is a function ([6]) $F : TM \longrightarrow [0, \infty)$ which has the following properties: (i) F is C^∞ on TM_0, (ii) F is positively 1-homogeneous on the fiber of tangent bundle TM, (iii) for any tangent vector $y \in T_xM$, the following quadratic form g_y on T_xM is positive definite given by

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t}[F^2(y + su + tv)]|_{s,t=0}, \ \ u, v \in T_xM.$$

Let $x \in M$ and $F_x = F/_{T_xM}$. To measure the non-Euclidian feature of F_x, define $C_y : T_x \otimes T_x \otimes T_x \longrightarrow \mathbb{R}$ by

$$C_y(u,v,w) = \frac{1}{2}\frac{d}{dt}[g_y + tw(u,v)]|_{t=0}, \ \ u,v,w \in T_xM.$$

The family $C = \{C_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if F is Riemannian.

Given Finsler manifold (M,F), then global vector field G is induced by F on TM_0, which is a standard coordinate (x^i, y^i) for TM_0 is given by

$$G = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where, $G^i(y)$ are local functions on TM given by

$$G^i = \frac{1}{4}g^{il}\left\{\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial[F^2]}{\partial x^l}\right\}, \ \ y \in T_xM.$$

Here, G^i is called the associated spray to (M,F). The projection of on integral curve of G is called a geodesics in M. In local co-ordinates a curve $c(t)$ is a geodesics if and only if its co-ordinates $c^i(t)$ satisfy, $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_xM$, define $B_y : T_xM \otimes T_xM \otimes T_xM \longrightarrow T_xM$ and $E_y : T_xM \otimes T_xM \longrightarrow \mathbb{R}$ by $B_y(u,v,w) = B^i_{jkl}(y)u^jv^kw^l$ and $E_y(u,v) = E_{jk}(y)u^jv^k$ where,

$$B^i_{jkl}(y) = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y), \ \ E_{jk}(y) = \frac{1}{2}B^m_{jkm}(y),$$

$$u = u^i\frac{\partial}{\partial x^i}|_x, \ \ v = v^i\frac{\partial}{\partial x^i}|_x, \ \text{and} \ w = w^i\frac{\partial}{\partial x^i}|_x.$$

Here, B and E are called the Berwald curvature and mean Berwald curvature if $B = 0$ and $E = 0$ respectively.

Let F be a Finsler metric defined by $F = \sqrt{A^{\frac{2}{m}} + B}$, where A and B are given by,

$$A = a_{i_1 i_2,\ldots,i_m}(x)y^{i_1}y^{i_2},\ldots,y^{i_m} \ \text{ and } B = b_{ij}(x)y^iy^j, \tag{2.1}$$

with $a_{i_1 i_2,\ldots,i_m}$ and b_{ij} are symmetric in all its indices [21]. Then, F is called generalized m-th root metric. Clearly, A is homogeneous of degree m in y. Put

$$A_i = \frac{\partial A}{\partial y^i}, \ A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \ A_{x^i} = \frac{\partial A}{\partial x^i}, \ A_0 = A_{x^i}y^i, \ A_{0l} = A_{x^i y^l}y^i,$$

$$B_i = \frac{\partial B}{\partial y^i}, \ B_{ij} = \frac{\partial^2 B}{\partial y^i \partial y^j}, \ B_{x^i} = \frac{\partial B}{\partial x^i}, \ B_0 = B_{x^i}y^i, \ B_{0l} = B_{x^i y^l}y^i.$$

The normalized supporting element l_i of F is given by

$$l_i = F_{y^i} = \frac{\partial F}{\partial y^i} = \frac{1}{2F}\left(\frac{2}{m}A^{\frac{2-m}{m}}A_i + B_i\right). \tag{2.2}$$

Consider the conformal transformation $\bar{F}(x,y) = e^{2\sigma(x)}F(x,y)$ of generalized m-th root metric $F = \sqrt{A^{\frac{2}{m}} + B}$. Clearly, $\bar{F}$ is also an generalized m-th root Finsler metric on M^n. Throughout the paper we call the Finsler metric $\bar{F}$ as to conformally transformed generalized m-th root metric and $(M^n, \bar{F}) = \bar{F}^n$ as conformally transformed Finsler space. The quantities corresponding to the transformed Finsler space $\bar{F}^n$ will be denoted as, by putting bar on the top of that quantity, for instance,

$$\begin{aligned}
&\bar{A} = e^{m\sigma}A,\ \bar{A_i} = e^{m\sigma}A_i,\ \text{and},\ \bar{A_{ij}} = e^{m\sigma}A_{ij},\\
&\bar{B} = e^{m\sigma}B,\ \bar{B_i} = e^{m\sigma}B_i,\ \text{and},\ \bar{B_{ij}} = e^{m\sigma}B_{ij}.
\end{aligned}$$

3. Fundamental tensor and Spray coefficients of Conformally transformed Generalized m-th root metric

The fundamental metric tensor g_{ij} of Finsler space F^n is given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} = FF_{y^i y^j} + F_{y^i}F_{y^j}. \tag{3.1}$$

In view of equation (2.1) and (2.2) in (3.1) we have

$$g_{ij} = \frac{A^{\frac{2}{m}-2}}{m^2}[mAA_{ij} + (2-m)A_iA_j] + b_{ij}. \tag{3.2}$$

The contravariant metric tensor g^{ij} of Finsler space F^n is given by

$$g^{ij} = A^{-\frac{2}{m}}\left(mAA^{ij} + \frac{(m-2)}{(m-1)}y^iy^j\right), \tag{3.3}$$

where matrix (A^{ij}) denotes inverse of (A_{ij}) [23]. Here, we have used $A^{ij}A_j = A^i = y^i$.

The covariant and contravariant metric tensor of conformally transformed Finsler space $\bar{F}^n$ are given as: $\bar{g}_{ij} = e^{2\sigma}g_{ij}$ and $\bar{g}^{ij} = e^{2\sigma}g^{ij}$.

Therefore, the covariant metric tensor $\bar{g}_{ij}$ and contravariant metric tensor $\bar{g}^{ij}$ of conformally transformed generalized m-th root Finsler space $\bar{F}^n$ are given as;

$$\bar{g}_{ij} = e^{2\sigma}\left(\frac{A^{\frac{2}{m}-2}}{m^2}[mAA_{ij} + (2-m)A_iA_j] + b_{ij}\right), \tag{3.4}$$

and

$$\bar{g}^{ij} = e^{-2\sigma}\left(A^{-\frac{2}{m}}[mAA^{ij} + \frac{(m-2)}{(m-1)}y^iy^j]\right). \tag{3.5}$$

The geodesic curves of $\bar{F}^n$ are characterized by a system of equations:

$$\frac{d^2x^i}{dt^2} + \bar{G}^i(x, \frac{dx}{dt}) = 0,$$

where,

$$\bar{G}^i = \frac{1}{4}\bar{g}^{il}\left\{[\bar{F}^2]_{x^ky^l}y^k - [\bar{F}^2]_{x^l}\right\}. \tag{3.6}$$

are called the spray coefficients of $\bar{F}$.
The spray coefficients $\bar{G}^i$ of $\bar{F}^n$ can be written as;

$$\begin{aligned}\bar{G}^i &= \frac{1}{4}e^{-2\sigma}g^{il}\left\{\frac{\partial^2(e^{2\sigma}F^2)}{\partial x^k\partial y^l}y^k - \frac{\partial(e^{2\sigma}F^2)}{\partial x^l}\right\},\\ &= \frac{1}{4}e^{-2\sigma}g^{il}\{e^{2\sigma}(F^2_{x^ky^l}y^k - F^2_{x^l}) + 2FF_{y^l}e^{2\sigma}2\sigma_{x^k}y^k - F^2e^{2\sigma}2\sigma_{x^l}\}.\end{aligned}$$

which implies that,

$$\bar{G}^i = G^i + \frac{1}{2}g^{il}\{2FF_{y^l}\sigma_{x^k}y^k - F^2\sigma_{x^l}\}. \tag{3.7}$$

By direct computation, G^i is given by

$$G^i = \xi^{il}[(\frac{2-m}{m})A_lA_0A^{-1} + A_{0l} + A_{x^l} + \frac{m}{2}A^{\frac{m-2}{m}}(B_{0l} - B_{x^l})], \tag{3.8}$$

where,

$$\xi^{il} = \frac{A^{il}}{2} + (\frac{m-2}{2m(m-1)})A^{-1}y^iy^l.$$

Further, the equation (3.5) in (3.7) we have

$$\begin{aligned}\bar{G}^i &= G^i + \frac{1}{2}A^{-\frac{2}{m}}\left(mAA^{il} + \frac{(m-2)}{(m-1)}y^iy^l\right)\{2Fl_j\sigma_{x^k}y^k - F^2\sigma_{x^l}\},\\ &= G^i + \frac{1}{2}A^{-\frac{2}{m}}\left(mAA^{il} + \frac{(m-2)}{(m-1)}y^iy^l\right)F^2(2\sigma_{x^k} - \sigma_{x^l}),\\ &= G^i + \frac{1}{2}A^{-\frac{2}{m}}(A^{\frac{2}{m}} + B)\left(mAA^{il} + \frac{(m-2)}{(m-1)}y^iy^l\right)(2\sigma_{x^k} - \sigma_{x^l}).\end{aligned}$$

Put, $\zeta = (2\sigma_{x^k} - \sigma_{x^l})$.
Thus,

$$\bar{G}^i = G^i + \frac{1}{2}(1 + A^{-\frac{2}{m}}B)\left(mAA^{il}\zeta + \frac{m-2}{m-1}\zeta y^iy^l\right). \tag{3.9}$$

Hence, we state the following:

Proposition 3.1. The spray coefficients $\bar{G}^i$ of the conformally transformed Finsler space $\bar{F}^n$ are given by (3.9) , where G^i are spray coefficients of Finsler space F^n.

In view of equation (3.8) and in [23] we observed that, G^i are rational functions of y . Hence from equation (3.9), we have;

Corollary 3.1. The spray coefficients $\bar{G}^i$ of the conformally transformed Finsler space $\bar{F}^n$ are rational functions of y.

4. Locally Dually flat Conformally transformed generalized m-th root metric

In [4], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they studied the information geometry on Riemannian manifolds. In Finsler geometry,Z. Shen extended the notion of locally dually flatness for Finsler metrics [19]. Dually at Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which play a very important role in studying at Finsler information structure. In [21] the authors, A. Tayebi, E. Peyghan and M. Shahbazi were characterize locally dually flat generalized m-th root Finsler metric.

A transformed Finsler metric $\bar{F} = \bar{F}(x, y)$ on a manifold M^n is said to be locally dually flat, if at any point there is a standard coordinate system (x^i, y^i) in TM such that $[\bar{F}^2]_{x^k y^l} y^k = 2[\bar{F}^2]_{x^l}$. In this case, the coordinate (x^i) is called an adapted local coordinate system [19]. And simple example is: Every locally Minkowskian metric is locally dually flat.

Consider the conformal transformation $\bar{F} = e^{2\sigma} F$, where F is an generalized m-th root metric.

Since, $[\bar{F}^2]_{x^k} = 2e^{2\sigma}\sigma_k + e^{2\sigma} F^2_{x^k} = e^{2\sigma}[F^2_{x^k} + 2F^2\sigma_k]$, where, $\sigma_k = \dfrac{\partial\sigma}{\partial x^k}$.

$$[\bar{F}^2]_{x^k y^l} = e^{2\sigma}[F^2_{x^k y^l} + 2F^2_{y^l}\sigma_k],$$

$$[\bar{F}^2]_{x^k y^l} y^k = e^{2\sigma}[F^2_{x^k y^l} y^k + 2F l_l \sigma_k y^k].$$

Therefore,

$$2\bar{F}^2_{x^k} - \bar{F}^2_{x^k y^l} y^k = e^{2\sigma}[2F^2_{x^l} + 4F^2\sigma_l - F^2_{x^k y^l} y^k - 2y_l\sigma_0],$$

where, $\sigma_0 = \sigma_k y^k$.

Thus, $\bar{F}$ is locally dually flat metric if and only if

$$\begin{aligned} & e^{2\sigma}\left\{2\left(\frac{2}{m}A^{\frac{2}{m}-1}A_{x^l}+B_{x^l}\right)+4(A^{\frac{2}{m}}+B)\sigma_l\right\} \\ & -e^{2\sigma}\left\{\frac{2}{m}A^{\frac{2}{m}-1}\left((\frac{2-m}{m})A_lA_0A^{-1}+A_{0l}+\frac{m}{2}A^{\frac{m-2}{m}}B_{0l}\right)\right. \\ & \left.-2y_l\sigma_0\right\}=0, \end{aligned} \tag{4.1}$$

where, $A_0 = A_{x^k}y^k$ and $A_{0l} = A_{x^k y^l}y^k$.

The equation (4.1) can be written as;

$$\begin{aligned} A_{x^l} = & \frac{m}{4}A^{\frac{m-2}{m}}\left\{\frac{2}{m}A^{\frac{2}{m}-1}\left(\frac{2-m}{m}A_0A_lA^{-1}+A_{0l}\right)\right. \\ & \left.-4A^{\frac{2}{m}}+(B_{ol}-4B\sigma_l)+2y_l\sigma_0\right\}. \end{aligned} \tag{4.2}$$

Hence, we get the following

Theorem 4.1. Let $\bar{F}$ be a conformally transformed generalized m-th root Finsler metric on a manifold M^n. Then, $\bar{F}$ is locally dually flat metric if and only if the equation (4.2) holds.

Corollary 4.2. Let $\bar{F}$ be a conformally transformed generalized m-th root metric. Then, $\bar{F}$ is locally dually flat if and only if conformal transformation is homothetic.

Proof. From theorem 4.1, since $\bar{F}$ is locally dually flat if and only if

$$2y_l\sigma_0 - 4B\sigma_l = 0. \tag{4.3}$$

Contracting equation (4.3) by y^l yields;

$$\sigma_0 F^2 - 4B\sigma_0 = 0.$$

which implies that, $\sigma_0 = 0$.

Hence from equation (4.3), we have $\sigma_l = 0$. i.e., $\frac{\partial\sigma}{\partial x^l} = 0$. So, σ is constant. Therefore, the transformation is homothetic. The converse is trivial.

5. Conformally transformed Einstein Generalized m-th root metric

In Finsler geometry, the flag curvature is an analogue of sectional curvature in Riemannian geometry. A natural problem is to study and characterize

Finsler metrics of constant flag curvature. There are only three local Riemannian metrics of constant sectional curvature, up to a scaling. However there are lots of non-Riemannian Finsler metrics of constant flag curvature. For example, the Funk metric is positively complete and non-reversible with $K = -\frac{1}{4}$ and the Hilbert-Klein metric is complete and reversible with $K = -1$ [11] .

In this section, we characterize the conformally transformed Einstein generalized m-th root Finsler metric is Ricci flat.

For a Finsler metric $\bar{F}$, the Riemannian curvature $\bar{R}_y : T_xM \longrightarrow T_xM$ is defined by

$$\bar{R}_y(u) = \bar{R}^i_k(x,y)u^k \frac{\partial}{\partial x^i}, \ \ u = u^k \frac{\partial}{\partial x^i},$$

where,

$$\bar{R}^i_k = 2\frac{\partial \bar{G}^i}{\partial x^k} - y^j \frac{\partial^2 \bar{G}^i}{\partial x^j \partial y^k} + 2\bar{G}^j \frac{\partial^2 \bar{G}^i}{\partial y^j \partial y^k} - \frac{\partial \bar{G}^i}{\partial y^j}\frac{\partial \bar{G}^i}{\partial y^k}. \tag{5.1}$$

The Finsler metric $\bar{F}$ is said to be of scalar flag curvature if there is a scalar function $\bar{K} = \bar{K}(x,y)$ such that

$$\bar{R}^i_k = \bar{K}(x,y)\bar{F}^2 \left\{ \delta^i_k - \frac{\bar{F}_{y^k} y^i}{\bar{F}} \right\}. \tag{5.2}$$

Moreover, $\bar{F}$ is said to be of constant flag curvature if $\bar{K}$ in equation (5.2) is constant.

The Ricci curvature of a transformed Finsler metric $\bar{F}$ on a manifold is a scalar function $\overline{Ric} : TM \longrightarrow \mathbb{R}$ defines to be the trace of $\bar{R}_y$. i.e.,

$$\overline{Ric}(y) = \bar{R}^k_k.$$

satisfying the homogeneity $\overline{Ric}(\lambda y) = \lambda \overline{Ric}(y)$ for $\lambda > 0$. A Finsler metric $\bar{F}$ is an n-dimensional manifold M^n is said to be Einstein metric if there is a scalar function $\bar{K} = \bar{K}(x)$ on M^n such that $\overline{Ric} = \bar{K}(n-1)\bar{F}^2$. A Finsler metric is said to be Ricci flat if $\overline{Ric} = 0$.

By formula (5.2) and corollary we get the following:

Lemma 5.1. $\bar{R}^i_k$ and $\overline{Ric} = \bar{R}^k_k$ are rational functions in y.

Proposition 5.2. Let $\bar{F}$ be a non-Riemannian conformally transformed generalized m-th root Finsler metric with $m > 2$ on a manifold M^n of dimensions $n > 1$. If $\bar{F}$ is Einstein metric then it is Ricci flat.

Proof. In view of equation (5.2) and Ricci curvature obviously it is Einstein.

If $\bar{F}$ is an Einstein metric, i.e., $\overline{Ric} = \bar{K}(n-1)\bar{F}^2$ and $\bar{F}^2$ is an irrational function, as $m > 2$ and $\overline{Ric}$ are rational function of y. Therefore, $\bar{K} = 0$ and so $\overline{Ric} = 0$. Hence, $\bar{F}$ is Ricci flat.

Corollary 5.3. Let $\bar{F} = e^{2\sigma}F$ be a non-Riemannian conformally transformed generalized m-th root Finsler metric with $m > 2$ on manifold M^n. If $\bar{F}$ is of constant flag curvature $\bar{K}$ then $\bar{K}$=0

Proof. By Schur lemma and corollary 3.6 of [7], immediately, we obtain the proof of the corollary.

6. Conformally transformed Generalized m-th root metric with Isotropic E-curvature

Let $\bar{G}^i$ be spray coefficients of a Finsler space $\bar{F}^n$ then the Berwald curvature of $\bar{F}^n$ is defined as

$$\bar{B}^i_{jkl} = \frac{\partial^3 \bar{G}^i}{\partial y^j \partial y^k \partial y^l}.$$

A transformed Finsler metric $\bar{F}$ is called a Berwald metric if spray coefficients $\bar{G}^i$ are quadratic in $y \in T_xM$, for any $x \in M^n$ or equivalently the Berwald curvature vanishes. The E-curvature is defined by the trace of the Berwald curvature. i.e., $\bar{E}_{ij} = \frac{1}{2}\bar{B}^m_{ijm}$. A Finsler metric $\bar{F}$ on an n-dimensional manifold M^n is said to be isotropic mean Berwald curvature or of isotropic E-curvature if

$$\bar{E}_{ij} = \frac{c(n+1)}{2\bar{F}}\bar{h}_{ij}, \tag{6.1}$$

where, $\bar{h}_{ij} = \bar{g}_{ij} - \bar{g}_{ip}y^p\bar{g}_{jq}y^q$ is the angular metric and $c = c(x)$ is a scalar function on M^n. If $c = 0$ then $\bar{F}$ is called weakly Berwald metric.

From equation (3.2), we have

$$\bar{g}_{ij} = e^{2\sigma}g_{ij} = e^{2\sigma}\left\{\frac{A^{\frac{2}{m}} - 2}{m^2}[mAA_{ij} + (2-m)A_iA_j] + b_{ij}\right\}. \tag{6.2}$$

The angular metric is given by

$$\bar{h}_{ij} = \bar{g}_{ij} - \bar{l}_i\bar{l}_j.$$

Since $\bar{h}_{ij} = e^{2\sigma}h_{ij}$. Therefore,

$$\bar{h}_{ij} = e^{2\sigma}\left\{\frac{A^{\frac{2}{m}-2}}{m^2}(mAA_{ij} + (2-m)A_iA_j) + b_{ij} - (A^{\frac{2}{m}} + b_{ij}y^iy^j)y_iy_j\right\} \tag{6.3}$$

From equation (6.3) in (6.1), we have

$$\bar{E}_{ij} = \frac{c(n+1)}{2\bar{F}} e^{2\sigma} \left\{ \frac{A^{\frac{2}{m}-2}}{m^2} (mAA_{ij} + (2-m)A_iA_j) + b_{ij} - (A^{\frac{2}{m}} \right.$$

$$\left. + b_{ij}y^iy^j)y_iy_j \right\},$$

$$= \frac{c(n+1)}{2F} \left\{ \frac{A^{\frac{2}{m}-2}}{m^2} (mAA_{ij} + (2-m)A_iA_j) + b_{ij} - (A^{\frac{2}{m}} \right.$$

$$\left. + b_{ij}y^iy^j)y_iy_j \right\},$$

Since, $\bar{F} = e^{2\sigma}F$ the above equation yields;

$$\bar{E}_{ij} = \frac{(n+1)}{2F} \left(\frac{A^{\frac{2}{m}-2}}{m^2} \right) c\{mAA_{ij} + (2-m)A_iA_j\}. \tag{6.4}$$

In view of equation (3.9), we clear that $\bar{E}_{ij}$ are rational functions of y. Thus, from equation (6.4), we have either $c = 0$ or

$$mAA_{ij} + (2-m)A_iA_j = 0. \tag{6.5}$$

Suppose $c \neq 0$. Contracting the equation (6.5) with A^{ik} yields

$$mA\delta_i^k + (2-m)A_iy^k = 0,$$

which implies that $A(m(n-1)+2) = 0$. Which is impossible, since $h_{ij} = 0$.

Therefore, $c = 0$ and consequently $\bar{E}_{ij} = 0$. Thus, we state the following

Proposition 6.3. Let $\bar{F}$ be a conformally transformed generalized m-th root Finsler metric is of isotropic mean Berwald curvature. Then, $\bar{F}$ is weakly Berwald metric.

7. Conclusion

An m-th root metric $F = \sqrt[m]{A}$, where $A = a_{i_1,i_2\ldots i_m}(x)y^{i_1}\ldots.y^{i_m}$, is regarded as a direct generalization of Riemannian metric in a sense, i.e., the second root metric is a Riemannian metric. The theory of m-th root metric has been developed by M. Matsumoto- H. Shimada. The m-th root Finsler metric play a very important role in physiscs, space time and general relativity.

Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas, including statistical inference, control system theory and multiterminal information theory. The notion of dually flat Riemannian metric, which comes under the information geometry on Riemannian manifolds. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which play a very important role in studying flat Finsler information structure. Then, work extended in Finsler geometry, the notion of locally dual flatness to Finsler metrics.

As we know, in general Einstein metrics is said to Ricci tensor is proportionality of metric tensor, i.e., $Ric \propto g_{ij}$, which are a natural extension of those in Riemannian geometry and they have good properties in Riemann geometry for some class of Finsler metrics. Some research have been progressed to generalized m-th root finsler metric is conformal to m-th root metric and also the curvature properties of an mention metrics.

Especially in this study, we consider the generalized m-th root Finsler metric $F = \sqrt{A^{\frac{2}{m}} + B}$, first we find the spray coefficients of conformally transformed generalized m-th root Finsler metric and it shows that, it is rational function of y. Then, we proved conformally transformed generalized m-th root metric is locally dually flat if and only if the conformal transformation is homothetic. Further, to refer [23], we characterized the Einstein metric is Ricci flat. Finally, we get conformally transformed metric is weakly Berwald metric.

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