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On Radon Measure Manifold

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Abstract

In this paper the concept of Radon measure manifold is studied in terms of Radon measure charts and Radon measure atlases with some measurable/extended topological properties that remain invariant under measurable homeomorphism and Radon measure structure-invariant map.

Keywords : Radon measure chart, Radon measure atlas, Radon measure manifold, measurable/extended regular property, measurable/extended normal property.
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1. Introduction

In this paper Radon measure is assigned on measurable chart and measurable atlas [7][8] to introduce the concepts of Radon measure chart and Radon measure atlas and also we show that some measurable topological properties on Radon measure manifold are invariant with respect to measurable homeomorphism [8] and Radon measure structure-invariant transformation. For this we use Inverse Function Theorem on measure manifold [8] and the concepts of measurable regular and measurable normal properties.

2. Preliminaries

We use the following concepts to develop the study on Radon measure manifold.

Definition 2.1. Measurable/extended Hausdorff property on a Measure Space [8]
A measure space $(\mathbb{R}^n, \tau, \Sigma, \mu)$ is said to be measurable Hausdorff if for every points $p, q \in (\mathbb{R}^n, \tau, \Sigma, \mu)$ with $p \neq q$ $\exists$ Borel open sets $A, B \in \Sigma$ such that $p \in A$ and $p \notin B$, $q \in B$ and $q \notin A$ such that $A \cap B = \emptyset \Rightarrow \mu(A \cap B) = \mu(\emptyset) = 0$.

Definition 2.2. Second Countable Measure Space [8]
A measure space $(\mathbb{R}^n, \tau, \Sigma, \mu)$ is second countable provided there is a countable base for all $q \in (\mathbb{R}^n, \tau, \Sigma, \mu)$ satisfying the following conditions:
(i) For each $q_i \in (\mathbb{R}^n, \tau, \Sigma, \mu)$ there exists $B_i \in \mathfrak{B}_{\mathfrak{q}}$ and $A_i \in \Sigma$ such that $q_i \in B_i \subseteq A_i$ for $i = 1, 2, 3, ...$
(ii) for $B_i \subseteq A_i$, $\mu(B_i) \leq \mu(A_i)$. (Monotonicity Property)

Definition 2.3. Measurable Regular space (e-Regular space)
A measurable space $(\mathbb{R}^n, \tau, \Sigma)$ is said to be e-regular if given any point $p \in (\mathbb{R}^n, \tau)$ and F_σ-set $F \in \Sigma$ in $(\mathbb{R}^n, \tau, \Sigma)$ such that $p \notin F$, $\exists$ disjoint G_δ-sets A and $B \in \Sigma$ such that for all $p \in A, F \subset B$ and $A \cap B = \emptyset$.

In the main results we show that some measurable topological properties on Radon measure manifold are invariant with respect to measurable homeomorphism and Radon measure structure-invariant transformation. For this we use Inverse Function Theorem on measure manifold [8] and the concepts of Radon e-regular and e-normal spaces [10].

Definition 2.4. Inverse Function Theorem on Measure Manifold [8]
Let $F : (M, \tau, \Sigma, \mu) \longrightarrow (M_1, \tau_1, \Sigma_1, \mu_1)$ be a C^∞ measurable homeomorphism and measure - invariant map of measure manifolds and suppose that $F_{*p} : T_p(M) \longrightarrow T_{f(p)}(M_1)$ is a linear isomorphism at some point p of M. Then there exists a measure chart (U, ϕ) of p in M such that the restriction of F to (U, ϕ) is a measurable diffeomorphism onto a measure chart (V, ψ) of $F(p)$ in M_1. This implies for every C^∞ function F which is measurable homeomorphism and measure-invariant has a C^∞ F^{-1} which is also measurable homeomorphism and measure-invariant.

Definition 2.5. Measurable Homeomorphism [8]
Let (M, τ) be a Hausdorff second-countable topological space and $(\mathbb{R}^n, \tau, \Sigma, \mu)$ be a measure space. Then the function $\phi : U \subset M \longrightarrow (\mathbb{R}^n, \tau, \Sigma, \mu)$ is called measurable Homeomorphism if
(i) ϕ is bijective and bi continuous,
(ii) ϕ and ϕ^{-1} are measurable.

Definition 2.6. Radon measure conditions on Borel subset of $(\mathbb{R}^n, \tau, \Sigma, \mu_R)$
(**i**) A Radon measure μ_R for a measurable subset $A \in \mathcal{B} \in (\mathbb{R}^n, \tau, \Sigma, \mu_R)$ (where $\mathcal{B}$ is a collection of Borel subsets of $(\mathbb{R}^n, \tau, \Sigma, \mu_R)$ is a positive Borel measure $\mu : \mathcal{B} \longrightarrow [0, \infty]$ which is finite on Borel compact subsets $K_i \in \mathcal{B} \in (\mathbb{R}^n, \tau, \Sigma, \mu_R)$ and is inner regular in the sense that for every measurable compact subset K_i

$\in A$, we have,

$\mu_R(A) = \sup\{\mu_R(K_i)\colon \forall\, i \in I; K_i \subseteq A; \forall\, K_i \in \mathcal{K}\}$, (1)

where $\mathcal{K} \subset A$ is the collection of Borel compact subsets $K_i \in (\mathbb{R}^n, \tau, \Sigma, \mu_R)$.

(**ii**) Also μ_R is outer regular on a family $\mathcal{O}$ of measurable/Borel open subsets if, for every measurable subset A we have,

$\mu_R(A) = inf\{\mu_R(O_i)\colon \forall\, i \in I; O_i \supseteq A; \forall\, O_i \in \mathcal{O}\}$. (2)

Now the measurable subset A is Radon measurable in $(\mathbb{R}^n, \tau, \Sigma, \mu_R)$.

Therefore, A is a **Radon measure subset** of $(\mathbb{R}^n, \tau, \Sigma, \mu_R)$.

Then $(\mathbb{R}^n, \tau, \Sigma, \mu_R)$ is called a **Radon measure space**.

If any Hausdorff second countable topological space is modeled on Radon measure space $(\mathbb{R}^n, \tau, \Sigma, \mu_R)$ and if $\exists$ a measurable homeomorphism ϕ between $U \subset (M, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathbb{R}^n, \tau, \Sigma, \mu_R)$ then $(M, \tau_1, \Sigma_1, \mu_{R_1})$ is a Radon measure manifold.

Definition 2.7. Radon measure on measurable regular space

Let $(\mathbb{R}^n, \tau, \Sigma)$ be a locally compact measurable Hausdorff regular space. A Radon measure μ_R on locally compact measurable Hausdorff regular space is defined as follows:

$\forall\, p \in \bar{A}$ and F_σ-set F in $(\mathbb{R}^n, \tau, \Sigma)$, $\exists$ G_δ-sets $\bar{A} \supset A$ and $\bar{B} \supset B \in \Sigma$, $\ni p \in A$, $F \subset \bar{B}, p \notin F$ and $\bar{A} \cap \bar{B} = \emptyset$ satisfying the Radon measure conditions:

I. For $p \in \bar{A}$,

(i) $p \in \bar{A} \subset (\mathbb{R}^n, \tau, \Sigma)$, $\mu_R(\bar{A}) < \infty$;

(ii) For any Borel subset $\bar{A} \supset A$,

$\mu_R(\bar{A}) = \sup\{\mu_R(E_i) : \forall\, i \in I; E_i \subset \bar{A}; E_i \text{ compact and measurable}\}$,

(iii) For any Borel subset $\bar{A} \supset A$,

$\mu_R(\bar{A}) = \inf\{\mu_R(O_i) : \forall\, i \in I; O_i \supseteq \bar{A}; O_i \in \mathcal{O} \text{ open and measurable}\}$,

where $\mathcal{O}$ is the family of Borel open subsets of $(\mathbb{R}^n, \tau, \Sigma, \mu)$.

II. For $F \subset B \subseteq \bar{B}$,

(i) $\forall q \in F \subset (\mathbb{R}^n, \tau, \Sigma)$; $\mu_R(\bar{B}) < \infty$;

(ii) $\mu_R(\bar{B}) = \sup\{\mu_R(F_i)\colon \in I; F_i \subset \bar{B}; F_i \text{ compact and measurable}\}$,

(iii) For any Borel subset $\bar{A} \supset A$

$\mu_R(\bar{B}) = \inf\{\mu_R(O_i) : \forall\, i \in I; O_i \supseteq \bar{B}; O_i \in \mathcal{O} \text{ open and measurable}\}$,

where $\mathcal{O}$ is the family of Borel open subsets of $(\mathbb{R}^n, \tau, \Sigma, \mu)$.

Definition 2.8. Radon e-regularity property

A locally compact measurable regular topological space is said to satisfy a Radon e-regularity property, if it admits a Radon measure μ_R.

Definition 2.9. Locally compact measurable normal space
The space $(\mathbb{R}^n, \tau, \Sigma)$ is called a locally compact measurable normal topological space, if given any pair of disjoint F_σ-sets E and $F \in \Sigma$ in $(\mathbb{R}^n, \tau, \Sigma)$, there exists a pair of disjoint G_δ-sets $\bar{A}$ and $\bar{B}$ such that $E \subset \bar{A}$ and $F \subset \bar{B}$.

Definition 2.10. Radon measure on locally compact measurable normal topological space
Let $(\mathbb{R}^n, \tau, \Sigma)$ be a locally compact measurable normal topological space. A Radon measure μ_R on $(\mathbb{R}^n, \tau, \Sigma)$ is defined as follows:
For any pair of disjoint Borel closed subsets $\bar{E} \supset E$ and $\bar{F} \supset F$ in $(\mathbb{R}^n, \tau, \Sigma)$, $\exists$ Borel open subsets $A \subset \bar{A}$ and $B \subset \bar{B} \in \Sigma, \ni E \subset \bar{A}$ and $F \subset \bar{B}$ and $\bar{A} \cap \bar{B} = \emptyset$ satisfying the Radon measure conditions:
I. For $E \subset \bar{A}$ in $(\mathbb{R}^n, \tau, \Sigma)$,
(i) $\mu_R(\bar{A}) < \infty$,
(ii) For any Borel compact subset $\bar{A}$ in $(\mathbb{R}^n, \tau, \Sigma)$,
$\mu_R(\bar{A}) = sup\{\mu(E_i); \forall\ i \in I : E_i \subseteq \bar{A} : E_i$ compact and measurable$\}$,
(iii) For any Borel compact subset $\bar{B} \supset B$,
$\mu_R(\bar{A}) = \inf\{\mu_R(O_i) : \forall\ i \in I; O_i \supseteq \bar{A}; O_i \in \mathcal{O}$ open and measurable$\}$,
where $\mathcal{O}$ is the family of Borel open subsets of $(\mathbb{R}^n, \tau, \Sigma, \mu)$.
II. For $F \subset \bar{B}$ in $(\mathbb{R}^n, \tau, \Sigma)$,
(i) $\mu_R(\bar{B}) < \infty$,
(ii) For any Borel compact subset $\bar{B}$ in $(\mathbb{R}^n, \tau, \Sigma)$,
$\mu_R(\bar{B}) = sup\{\mu(F_i); i \in I : F_i \subseteq \bar{B} : F_i$ compact and measurable$\}$,
(iii) For any Borel subset $\bar{A} \supset A$,
$\mu_R(\bar{B}) = \inf\{\mu_R(O_i) : \forall\ i \in I; O_i \supseteq \bar{B}; O_i \in \mathcal{O}$ open and measurable$\}$,
where $\mathcal{O}$ is the family of Borel open subsets of $(\mathbb{R}^n, \tau, \Sigma, \mu)$.

Definition 2.11. Radon e-normality property
A locally compact measurable normal topological space is said to satisfy a Radon e-normality property, if it admits a Radon measure.

Definition 2.12. Radon measure - invariant condition on measure manifold
For any two Radon measure atlases, say, $\mathcal{A}_1$ and $\mathcal{A}_2 \in A^k(M) \subset (M, \tau, \Sigma, \mu)$, $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$ on (M, τ, Σ, μ).

Definition 2.13. Radon measure structure - invariance
Suppose $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds. Let $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be a measurable homeomorphism then F is Radon measure structure-invariant if $\mathcal{A}_1 \sim \mathcal{A}_2$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$

with Radon measure invariant condition $\mu_{R_1}(\mathcal{A}_1) = \mu_{R_1}(\mathcal{A}_2)$ then $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ with Radon measure invariant condition $\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2))$.

3. Main Results

In this paper the concept of Radon Measure Manifold is studied in terms of Radon measure charts and Radon measure atlases with some extended topological properties that remain invariant under measurable homeomorphism and Radon measure structure-invariant map.

Definition 3.1. Radon Measure conditions on a measurable chart
A Radon measure on a measurable chart (U, ϕ) of a measure manifold (M, τ, Σ, μ) is a positive Borel measure

$$\mu : B \longrightarrow [0, \infty]$$

which is finite on compact Borel subsets and is inner regular in the sense that for every Borel charts $(U, \phi) \subset (M, \tau, \Sigma, \mu)$ we have
(i) $\mu_R(U) = sup\{\mu_R(K) : K \subseteq U; K \in \mathcal{K}\}$ (3)
where $\mathcal{K}$ denote the family of all compact Borel subsets and μ_R is outer regular on a family $\mathcal{F}$ of Borel charts if for every $(U, \phi) \subset (M, \tau, \Sigma, \mu)$ we have,
(ii) $\mu_R(U) = inf\{\mu_R(\mathcal{O}) : O \supseteq U; O \in \mathcal{O}\}$ (4)
where $\mathcal{O}$ denote the family of all open Borel subsets.

Definition 3.2. Radon measure chart
A measurable chart $((U, \tau_{/U}, \Sigma_{/U}), \phi)$ of (M, τ, Σ) equipped with a Radon measure $\mu_{R_{/U}}$ satisfying the Radon measure conditions (3) and (4) is called a Radon measure chart denoted by $((U, \tau_{/U}, \Sigma_{/U}, \mu_{R_{/U}}), \phi)$.
Since $\cup_{i=1}^{\infty}(U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_R/U_i) = (M, \tau, \Sigma, \mu)$, we can measure measurable manifold
(M, τ, Σ) by Radon measure.

Definition 3.3. Radon measure conditions on measurable atlas
A measurable atlas $\mathcal{A}_i$ is Radon measurable if it satisfies the following Radon measure conditions:
(i) Let $\mathcal{F} = \{\cup_{i=I}(U_i, \phi_i)\}$ be a family of all Radon measure charts of (M, τ, Σ, μ_R). A Radon measure μ_R of a measurable atlas $\mathcal{A}_i$ is a positive Borel measure

$$\mu : \mathcal{B} \longrightarrow [0, \infty]$$

which is finite on measurable compact charts (U_i, ϕ_i) and is inner regular in the sense that for every measurable atlas $\mathcal{A}_i$, we have

$\mu_R(\mathcal{A}_i) = \sup\{\mu_R(U_i)\colon \forall\, i \in I;\, U_i \subseteq \mathcal{A}_i;\, \forall\, U_i \in \mathcal{F}\},$ (5)
(ii) Also μ_R is outer regular on a family $\mathcal{O}$ of measurable charts if, for every measurable atlas $\mathcal{A}_i \in A^k(\mathcal{M})$ we have,
$\mu_R(\mathcal{A}_i) = inf\{\mu_R(\mathcal{O}) : \mathcal{O} \supseteq \mathcal{A}_i, \mathcal{O} \in \mathcal{M}\}.$ (6)

Definition 3.4. Radon measure atlas
By an $\mathbb{R}^n$-Radon measure atlas of class $C^k (k \geq 1)$ on a measurable manifold (M, τ, Σ), we mean a countable collection $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$ of n-dimensional Radon measure charts $((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{R_{/U_i}}), \phi_{i_{/U_i}})$ for all $i \in I$ on (M, τ, Σ, μ_R) satisfying the following conditions:
($\mathbf{a_1}$) $\cup_{i \in I}(U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{R_{/U_i}}) = (M, \tau, \Sigma, \mu_R)$. That is, the countable union of all Radon measure charts in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$ cover (M, τ, Σ, μ_R).

($\mathbf{a_2}$) For any pair of Radon measure charts $((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{R_{/U_i}}), \phi_{i_{/U_i}})$ and $((U_j, \tau_{/U_j}, \Sigma_{/U_j}, \mu_{R1_{/U_j}}), \phi_{j_{/U_j}})$ in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$, the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are :
(1) differentiable maps of class C^k $(k \geq 1)$
i.e.,$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j) \subseteq (\mathbb{R}^n, \tau_1, \Sigma_1, \mu_{R_1})$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j) \subseteq (\mathbb{R}^n, \tau_1, \Sigma_1, \mu_{R_1})$ are differentiable maps of class C^k $(k \geq 1)$.
(2) Radon measurable:
Transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are Radon measurable functions if,
a) any Borel subset $K \subseteq \phi_i(U_i \cap U_j)$ is Radon measurable in $(\mathbb{R}^n, \tau_1, \Sigma_1, \mu_{R_1})$, then $(\phi_i \circ \phi_j^{-1})^{-1}(K) \in \phi_j(U_i \cap U_j)$ is also Radon measurable.
b) $\phi_j \circ \phi_i^{-1}$ is Radon measurable if $S \subseteq \phi_j(U_i \cap U_j)$ is Radon measurable in $(\mathbb{R}^n, \tau_1, \Sigma_1, \mu_{R_1})$, then $(\phi_j \circ \phi_i^{-1})^{-1}(S) \in \phi_i(U_i \cap U_j)$ is also Radon measurable,

($\mathbf{a_3}$) Any two measure atlases $(\mathcal{A}_1, \tau_{/\mathcal{A}_1}, \Sigma_{/\mathcal{A}_1}, \mu_{/\mathcal{A}_1})$,$(\mathcal{A}_2, \tau_{/\mathcal{A}_2}, \Sigma_{/\mathcal{A}_2}, \mu_{/\mathcal{A}_2})$ are compatible on (M, τ, Σ, μ_R) satisfying the two equivalence relations:
i) $\mathcal{A}_1 \sim \mathcal{A}_2$, iff $\mathcal{A}_1 \cup \mathcal{A}_2 \in \mathcal{A}^k(M)$,
ii) $\mathcal{A}_1 \sim \mathcal{A}_2$, iff $\mu_R(\mathcal{A}_1) = k\mu_R(\mathcal{A}_2) \in \mathcal{A}^k(M)$, where $k = 1, 2, ...$

Definition 3.5. Radon Measure Manifold
A Radon measure space (M, τ, Σ, μ_R) with differentiable structure of class C^k and a Radon measure structure induced by μ_R on (M, τ, Σ, μ_R) is called a Radon measure manifold of class C^k.
That is, i) $\mathcal{A}_1 \sim \mathcal{A}_2$, iff $\mathcal{A}_1 \cup \mathcal{A}_2 \in \mathcal{A}^k(M)$ and
ii) $\mathcal{A}_1 \sim \mathcal{A}_2$, iff $\mu_R(\mathcal{A}_1) = k\mu_R(\mathcal{A}_2) \in \mathcal{A}^k(M)$, where $k = 1, 2, ...$

By using Theorem 3.3 [11], we show that e-regularity property remains invariant under measurable homeomorphism and Radon measure structure - invariant map
$F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds.

Note 3.1. Let $A = \{p \in \bar{U_i}, E \subseteq \bar{U_j}\colon \forall\ i, j \in I : E$ is compact Borel subset$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $\mathcal{A}^k(M) \subseteq (M, \tau, \Sigma, \mu_R)$.

Theorem 3.1. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant map. If extended regular property holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant map.
We show that if extended regular property, say, P_1 holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_1 also holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.
Suppose $A_1, A_2 \in \mathcal{A}^k(M) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ be two Radon measure subsets of $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.
Let P_1 holds $\mu_{R_1} - a.e.$ on $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1}) : A_1 \sim A_2$ iff $\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$.
Now since $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is measurable homeomorphism and Radon measure structure-invariant map, according to definition 2.10, for every $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $\mu_{R_1}(A_1) > 0$, $\mu_{R_1}(A_2) > 0 \ \exists\ F(A_1), F(A_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$,
$\mu_{R_2}(F(A_1)) > 0$, $\mu_{R_2}(F(A_2)) > 0 : A_1 \sim A_2 \Leftrightarrow F(A_1) \sim F(A_2)$ then $\mu_{R_1}(A_1) = \mu_{R_1}(A_2) \Leftrightarrow \mu_{R_2}(F(A_1)) = \mu_{R_2}(F(A_2))$.
Therefore, extended regular property holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Theorem 3.2. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be metrizable Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $G : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be C^∞ measurable homeomorphism and Radon measure structure-invariant maps. Then extended-regular property is invariant under the composition map $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Proof. Let $(M_1,\tau_1,\Sigma_1,\mu_{R_1})$, $(M_2,\tau_2,\Sigma_2,\mu_{R_2})$ and $(M_3,\tau_3,\Sigma_3,\mu_{R_3})$ be Radon measure manifolds.
Let $F:(M_1,\tau_1,\Sigma_1,\mu_{R_1})\longrightarrow(M_2,\tau_2,\Sigma_2,\mu_{R_2})$ and $G:(M_2,\tau_2,\Sigma_2,\mu_{R_2})\longrightarrow(M_3,\tau_3,\Sigma_3,\mu_{R_3})$ be C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.
We show that if extended-regular property, say, P_1 holds $\mu_{R_1}-a.e.$ on non-empty Borel sets A_1, $A_2\subset(M_1,\tau_1,\Sigma_1,\mu_{R_1})$ then P_1 also holds $\mu_{R_3}-a.e.$ on $(M_3,\tau_3,\Sigma_3,\mu_{R_3})$ under measurable homeomorphism and Radon measure structure-invariant map $G\circ F$.
Let the extended-regular property P_1 holds $\mu_{R_1}-a.e.$ on the non empty Borel sets A_1 and $A_2\subset(M_1,\tau_1,\Sigma_1,\mu_{R_1})$:$A_1\sim A_2$ if and only if $\mu_{R_1}(A_1)=\mu_{R_1}(A_2)$ by definition 2.12.
By above Theorem, if $F:(M_1,\tau_1,\Sigma_1,\mu_{R_1})\longrightarrow(M_2,\tau_2,\Sigma_2,\mu_{R_2})$ is C^∞ measurable homeomorphism and Radon measure structure - invariant map, then for every A_1, $A_2\subset(M_1,\tau_1,\Sigma_1,\mu_{R_1})$,
$\mu_{R_1}(A_1)>0$, $\mu_{R_1}(A_2)>0$, $\exists\ F(A_1)$, $F(A_2)\subset(M_2,\tau_2,\Sigma_2,\mu_{R_2})$,
$\mu_{R_2}(F(A_1))>0$, $\mu_{R_2}(F(A_2))>0$:
$A_1\sim A_2\Rightarrow F(A_1)\sim F(A_2)$ with Radon measure structure condition:
$\mu_{R_1}(A_1)=\mu_{R_1}(A_2)\Rightarrow\mu_{R_2}(F(A_1))=\mu_{R_2}(F(A_2))$
according to definition 2.13. (i)
Similarly, we show that under $G:(M_2,\tau_2,\Sigma_2,\mu_{R_2})\longrightarrow(M_3,\tau_3,\Sigma_3,\mu_{R_3})$, if P_1 holds $\mu_{R_2}-a.e.$ on the non-empty Borel sets $F(A_1)$ and $F(A_2)\subset(M_2,\tau_2,\Sigma_2,\mu_{R_2})$ then P_1 also holds $\mu_{R_3}-a.e.$ on the non-empty Borel sets $(G\circ F)(A_1)$ and $(G\circ F)(A_2)\subset(M_3,\tau_3,\Sigma_3,\mu_{R_3})$ where,
$(G\circ F(A_1))=\{G\circ F(p))\in G\circ F(\bar{U}_i)),G\circ F(E_j))\subseteq G\circ F(\bar{U}_j):i,j\in I;G\circ F(E_j)$ is compact Borel subset$\}$ where $\mu_{R_3}(G\circ F(A_1))>0$ and $(G\circ F(A_2))=\{G\circ F(q)\in G\circ F(\bar{U}_i),G\circ F(E_j)\subseteq G\circ F(\bar{U}_j):i,j\in I;G\circ F(E_j)$ is compact Borel subset$\}$ where $\mu_{R_3}(G\circ F(A_2))>0$ on $(M_3,\tau_3,\Sigma_3,\mu_{R_3})$:$F(A_1)\sim F(A_2)$ $\Rightarrow(G\circ F)(A_1)\sim(G\circ F)(A_2)$ with the Radon measure condition:
$\mu_{R_2}(F(A_1))=\mu_{R_2}(F(A_2)\Rightarrow\mu_{R_3}(G\circ F(A_1))=\mu_{R_3}(G\circ F(A_2))$
according to definition 2.13. (ii)
This implies under $G\circ F:(M_1,\tau_1,\Sigma_1,\mu_{R_1})\longrightarrow(M_3,\tau_3,\Sigma_3,\mu_{R_3})$, if P_1 holds $\mu_{R_1}-a.e.$ on A_1, $A_2\subset(M_1,\tau_1,\Sigma_1,\mu_{R_1})$ then P_1 also holds $\mu_{R_3}-a.e.$ on $(G\circ F(A_1)),(G\circ F(A_2))\subset(M_3,\tau_3,\Sigma_3,\mu_{R_3})$:
$A_1\sim A_2\Rightarrow(G\circ F(A_1))\sim(G\circ F(A_2))$ with the Radon measure condition:
$\mu_{R_1}(A_1)=\mu_{R_1}(A_2)\Rightarrow\mu_{R_3}(G\circ F(A_1))=\mu_{R_3}(G\circ F(A_2))$. (iii)
Therefore, from (i), (ii) and (iii), if extended-regular property holds $\mu_{R_1}-a.e.$

on
$(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_3} - a.e.$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Corollary 3.1. Let $M_1, ..., M_n$ be Radon measure manifolds and $F_1 : M_1 \longrightarrow M_2, ..., F_n : M_{n-1} \longrightarrow M_n$ be measurable homeomorphisms and Radon measure structure-invariant maps. Then, if extended-regular property holds $\mu_{R_1} - a.e.$ on M_1 then it also holds $\mu_{R_n} - a.e.$ on M_n under the composition function of $F_1, ..., F_n$.

Remark 3.1. Using the above results, one can generate a new **category of Regular Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.

By using Theorem 3.4 [11], we show that e-normality property remains invariant under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds.

Note 3.2. Let $A = \{E_i \subseteq \bar{U}_i, E_j \subseteq \bar{U}_j$: $\forall\, i, j \in I : E_i$ and E_j are compact Borel subsets$\}$ where $\mu_R(A) > 0$. If $\mu_R(A) = 0$ then A represents the **dark region** of $\mathcal{A}^k(M) \subseteq (M, \tau, \Sigma, \mu_R)$.

Theorem 3.3. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant map. If extended-normal property holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1,$
$\mu_{R_1})$ then the property also holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be measurable homeomorphism and Radon measure structure-invariant map.
We show that if extended-normal property, say, P_2 holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1,$
$\mu_{R_1})$ then P_2 also holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.
Suppose $A_1, A_2 \in \mathcal{A}^k(M) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ be two Radon measure subsets of $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.
Let P_2 holds $\mu_{R_1} - a.e.$ on $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1}) :\, A_1 \sim A_2$ iff $\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$.
Now since $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is measurable homeomorphism and Radon measure structure-invariant map, according to definition 2.13,

for every $A_1, A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $\mu_{R_1}(A_1) > 0$, $\mu_{R_1}(A_2) > 0 \; \exists \; F(A_1), F(A_2)$ $\subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$,
$\mu_{R_2}(F(A_1)) > 0$, $\mu_{R_2}(F(A_2)) > 0$: $A_1 \sim A_2 \Leftrightarrow F(A_1) \sim F(A_2)$ then $\mu_{R_1}(A_1) = \mu_{R_1}(A_2) \Leftrightarrow \mu_{R_2}(F(A_1)) = \mu_{R_2}(F(A_2))$.
Therefore, extended-normal property holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Theorem 3.4. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be metrizable Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $G : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be C^∞ measurable homeomorphism and Radon measure structure-invariant maps. Then extended-normal property is invariant under the composition map $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Proof. Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be Radon measure manifolds.
Let $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $G : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.
We show that if extended-normal property, say, P_2 holds $\mu_{R_1} - a.e.$ on non-empty Borel sets A_1, $A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_2 also holds $\mu_{R_3} - a.e.$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under measurable homeomorphism and Radon measure structure-invariant map $G \circ F$.
Let the extended-normal property P_2 holds $\mu_{R_1} - a.e.$ on the non empty Borel sets A_1 and $A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$:$A_1 \sim A_2$ if and only if $\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$ by definition 2.12.
By above Theorem, if $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is C^∞ measurable homeomorphism and Radon measure structure - invariant map, then for every A_1, $A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$,
$\mu_{R_1}(A_1) > 0$, $\mu_{R_1}(A_2) > 0$, $\exists$ $F(A_1)$, $F(A_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$, $\mu_{R_2}(F(A_1)) > 0$, $\mu_{R_2}(F(A_2)) > 0$: $A_1 \sim A_2 \Rightarrow F(A_1) \sim F(A_2)$ with Radon measure structure condition:
$\mu_{R_1}(A_1) = \mu_{R_1}(A_2) \Rightarrow \mu_{R_2}(F(A_1)) = \mu_{R_2}(F(A_2))$
according to definition 2.13. (i)
Similarly, we show that under $G : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$, if P_2 holds $\mu_{R_2} - a.e.$ on the non-empty Borel sets $F(A_1)$ and $F(A_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then P_2 also holds $\mu_{R_3} - a.e.$ on the non-empty Borel sets $(G \circ F)(A_1)$ and $(G \circ F)(A_2) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ where,
$(G \circ F(A_1)) = \{G \circ F(E_i)) \subseteq G \circ F(\bar{U_i})), G \circ F(E_j)) \subseteq G \circ F(\bar{U_j}) : i, j \in I; G \circ F(E_i)$ and $G \circ F(E_j)$ are compact Borel subsets$\}$ where $\mu_{R_3}(G \circ F(A_1)) > 0$ and

$(G \circ F(A_2)) = \{G \circ F(E_i) \in G \circ F(\bar{U_i}), G \circ F(E_j) \subseteq G \circ F(\bar{U_j}) : i, j \in I; G \circ F(E_i)$ and $G \circ F(E_j)$ are compact Borel subsets$\}$ where $\mu_{R_3}(G \circ F(A_2)) > 0$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$:
$F(A_1) \sim F(A_2) \Rightarrow (G \circ F)(A_1) \sim (G \circ F)(A_2)$ with the Radon measure condition:
$\mu_{R_2}(F(A_1)) = \mu_{R_2}(F(A_2) \Rightarrow \mu_{R_3}(G \circ F(A_1)) = \mu_{R_3}(G \circ F(A_2))$
according to definition 2.10. (ii)
This implies under $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$, if P_2 holds $\mu_{R_1} - a.e.$ on A_1, $A_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_2 also holds $\mu_{R_3} - a.e.$ on $(G \circ F(A_1)), (G \circ F(A_2)) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$:
$A_1 \sim A_2 \Rightarrow (G \circ F(A_1)) \sim (G \circ F(A_2))$ with the Radon measure condition:
$\mu_{R_1}(A_1) = \mu_{R_1}(A_2) \Rightarrow \mu_{R_3}(G \circ F(A_1)) = \mu_{R_3}(G \circ F(A_2))$. (iii)
Therefore, from (i), (ii) and (iii), if extended-normal property holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1,$
$\mu_{R_1})$ then it also holds $\mu_{R_3} - a.e.$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Corollary 3.2. Let $M_1, ..., M_n$ be Radon measure manifolds and $F_1 : M_1 \longrightarrow M_2, ..., F_n : M_{n-1} \longrightarrow M_n$ be measurable homeomorphisms and Radon measure structure-invariant maps. Then, if extended-normal property holds $\mu_{R_1} - a.e.$ on M_1 then it also holds $\mu_{R_n} - a.e.$ on M_n under the composition function of $F_1, ..., F_n$.

Remark 3.2. Using the above results, one can generate a new **category of Normal Radon measure manifold** $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of C^∞ measurable homeomorphisms and Radon measure structure-invariant maps.

4. Conclusion

The study of topological properties on Radon measure manifold have rich applications in the field of Engineering Science and Life Science.

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