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# Black Holes in Non-stationary de Sitter Space with Variable $\Lambda(u)$

**Ngangbam Ishwarchandra† and Ng. Ibohal§**

†Department of Physics,
National Institute of Technology, Agartala-799046, India
§Department of Mathematics,
Manipur University, Imphal - 795003, Manipur, India
E-mail: †ngishwarchandra@gmail.com, §ngibohal@associates.iucaa.in and §ngibohal@gmail.com
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## Abstract

In this paper we discuss a class of non-stationary solutions of Einstein's field equations based on the non-stationary de Sitter space-time. These solutions include Schwarz-schild-de Sitter and Vaidya-de Sitter black holes with a cosmological variable $\Lambda(u)$. Schwarzschild-de Sitter solution with variable $\Lambda(u)$ is regarded as a generalization of Schwarzschild-de Sitter solution with constant $\Lambda$. Vaidya-de Sitter black hole with variable $\Lambda(u)$ is also a generalization of the radiating Vaidya black hole embedded into the stationary de Sitter space with constant $\Lambda$. It is shown the interaction of the Vaidya null fluid with the non-stationary de Sitter field expressing in an energy-momentum tensor. The energy-momentum tensor of the embedded de Sitter black holes satisfies the energy conservation law. The energy conditions (like weak, strong and dominant conditions) for the energy-momentum tensor are also studied. The physical properties of the time-like vector fields for both the embedded solutions are discussed. It is also found that the space-time geometry of Schwarzschild-de Sitter and Vaidya-de Sitter solution with variable $\Lambda(u)$ are type D in the Petrov classifications of space-times. We also discuss the surface gravity, temperature and entropy of the space-time on the cosmological black hole horizons. It is also suggested that the modified Einstein's field equations associated with a variable cosmological $\Lambda(u)$ will take the form $R_{ab}-(1/2)Rg_{ab}+\Lambda(u)g_{ab}=-K\{T_{ab}+T_{ab}^{(\mathrm{NS})}\}$ for any type of matter field distribution $T_{ab}$.

**Keywords :** Schwarzschild-de Sitter black hole, de Sitter solution, Vaidya-de Sitter black hole, energy conditions, surface gravity.

## 1. Introduction

The original de Sitter cosmological model is *conformally* flat space-time with *constant curvature* [1] and is also a *non-rotating* and *stationary* solution. Therefore, the non-rotating stationary de Sitter model is a solution of Einstein's field equations for an empty space with constant curvature, whereas the *rotating* stationary de Sitter model proposed in [2] is a solution for *non-empty* space with *non-constant* curvature. Because of the stationary and non-rotating properties of the original de Sitter space, the non-rotating Schwarzschild black hole with constant mass can embed to produce Schwarzschild-de Sitter cosmological black hole with two event horizons - one for black hole and other for cosmological [3]. Similarly, the rotating stationary de Sitter cosmological universe [2] can conveniently embed into the rotating stationary Kerr-Newman solution to produce rotating Kerr-Newman-de Sitter cosmological black hole with constant cosmological term. This Kerr-Newman-de Sitter black hole metric can be expressed in terms of Kerr-Schild ansatz with different backgrounds. The expressibility of an embedded black hole in different Kerr-Schild ansatze means that, it is always true to talk about either Kerr-Newman black hole embedded into the rotating de Sitter space as Kerr-Newman-de Sitter or the rotating de Sitter space into Kerr-Newman black hole as rotating de Sitter-Kerr-Newman black hole - geometrically both are the same. That is, physically one may not be able to predict which space starts first to embed into what space. One thing we found from the study of Hawking's radiation of Kerr-Newman-de Sitter black hole [4], is that, there is no effect on the cosmological constant $\Lambda$ during the evaporation process of electrical radiation. The cosmological constant $\Lambda$ always remains unaffected in Einstein's field equations during Hawking's radiation process. That is, unless some external forces apply to remove the cosmological term $\Lambda$ from the space-time geometry, it continues to exist along with the electrically radiating objects, rotating or non-rotating. This means that it might have started to embed from the very early stage of the embedded black hole, and should continue to embed forever. It is noted that the Kerr-Newman-de Sitter black hole proposed in [2] is different from the one obtained by Carter [5] in the terms involving cosmological constant.

The black hole embedded into de Sitter space plays an important role in classical general relativity that the cosmological constant is found present in the inflationary scenario of the early universe in a stage where the universe is geometrically similar to the original de Sitter space [6]. Also embedded black holes can avoid the direct formation of negative mass naked singularities during Hawking's black hole evaporation process [4]. Here our aim is to study black holes embedded into the non-stationary de Sitter space with variable $\Lambda(u)$ and find the effect of variable $\Lambda(u)$.

In this paper, Section 2 deals with a brief introduction of the Einstein's field equations associated with the mass function $\hat{M}(u,r)$ in Newman-Penrose (NP) formalism [7] for future references. We obtain the energy conditions for a general energy-momentum tensor field. The field equations in NP formalism help in deriving the line-element of non-stationary de Sitter space-time having variable cosmological $\Lambda(u)$ in Section 3. Sections 4 and 5 develop the derivations of embedded Schwarzschild-de Sitter and Vaidya-de Sitter black holes with variable $\Lambda(u)$ based on the power series expansion of the mass function [8] respectively. We show that the time-like vector field of an observer in the Vaidya-de Sitter space is expanding, accelerating, shearing but non-zero twist. We also obtain the temperatures proportional to the surface gravities of the embedded space-times. The paper is concluded in Section 6 with reasonable remarks and physical interpretations of the solutions presented here.

## 2. Field equations in Newman-Penrose formalism

We consider a line-element of a general canonical metric in Eddington-Finkestein coordinate systems $\{u,r,\theta,\phi\}$

$$ds^2 = \Big\{1 - \frac{2}{r}\hat{M}(u,r)\Big\}du^2 + 2du\,dr - r^2 d\Omega^2, \tag{2.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the line-element on the unit two-sphere, and $\hat{M}(u,r)$ is referred to as the mass function and related to the gravitational fields within a given range of radius $r$. Here $u$ is the retarded time coordinate $u = t - r$.

Here $\ell_a$, $n_a$ and $m_a$ are given as follows

$$\begin{aligned} &\ell_a = \delta^1_a, \quad n_a = \frac{1}{2}\Big\{1 - 2r\hat{M}(u,r)\Big\}\,\delta^1_a + \delta^2_a, \\ &m_a = -\frac{r}{\sqrt{2}}\,\Big\{\delta^3_a + i\sin\theta\,\delta^4_a\Big\}, \end{aligned} \tag{2.2}$$

where $\ell_a$, $n_a$ are real null vectors and $m_a$ is complex having its conjugate $\bar{m}_a$ with the normalization conditions $\ell_a n^a = 1 = -m_a\bar{m}^a$ and other inner products are zero.

The NP spin coefficients for the non-stationary metric (2.1) can be obtained from those of the general rotating metric in [2, 9] by setting rotational parameter $a = 0$ and are given as:

$$\begin{aligned}
&\kappa^* = \sigma = \lambda = \epsilon = \pi = \tau = \nu = 0,\\
&\rho^* = -\frac{1}{r}, \quad \beta = -\alpha = \frac{1}{2\sqrt{2}r}\cot\theta,\\
&\mu^* = -\frac{1}{2r}\Big\{1 - \frac{2}{r}\hat{M}(u,r)\Big\},\\
&\gamma = \frac{1}{2r^2}\Big\{\hat{M}(u,r) - r\hat{M}(u,r)_r\Big\}.
\end{aligned} \tag{2.3}$$

The Ricci scalars for the metric are found as

$$\begin{aligned}
&\phi_{00} = \phi_{01} = \phi_{10} = \phi_{20} = \phi_{02} = \phi_{12} = \phi_{21} = \phi_{22} = 0\\
&\phi_{11} = \frac{1}{4r^2}\Big\{2\hat{M}(u,r)_{,r} - r\,\hat{M}(u,r)_{,r\,r}\Big\}\\
&\phi_{22} = -\frac{1}{r^2}\,\hat{M}(u,r)_{,u}\\
&\Lambda^* = \frac{1}{12r^2}\Big\{2\hat{M}(u,r)_{,r} + r\,\hat{M}(u,r)_{,r\,r}\Big\}.
\end{aligned} \tag{2.4}$$

where $\Lambda^*$ is the Ricci scalar ($\Lambda^* \equiv \frac{1}{24}g^{ab}\,R_{ab}$). The Weyl scalars are as follows

$$\begin{aligned}
&\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0\\
&\psi_2 = \frac{1}{r^3}\Big\{-\hat{M}(u,r) + \frac{2r}{3}\,\hat{M}(u,r)_{,r} - \frac{r^2}{6}\hat{M}(u,r)_{,rr}\Big\}.
\end{aligned} \tag{2.5}$$

From the Einstein's field equations $R_{ab} - (1/2)Rg_{ab} = -KT_{ab}$ associated with the line-element (2.1), we find an energy-momentum tensor describing the matter distribution in the gravitational field as

$$T_{ab} = \mu\ell_a\ell_b + 2\,\rho\,\ell_{(a}\,n_{b)} + 2\,p\,m_{(a}\bar{m}_{b)}, \tag{2.6}$$

where the quantities are found as

$$\begin{aligned}
&\mu = -\frac{2}{Kr^2}\hat{M}(u,r)_{,u},\\
&\rho = \frac{2}{Kr^2}\hat{M}(u,r)_{,r}, \quad p = -\frac{2}{Kr}\hat{M}(u,r)_{,rr}
\end{aligned} \tag{2.7}$$

with the universal constant $K = 8\pi G/c^4$. These quantities are obtained from the relations with the Ricci scalars (2.4) $K\,\mu = 2\,\phi_{22}$, $K\,\rho = 2\,\phi_{11} + 6\,\Lambda$, and $K\,p = 2\,\phi_{11} - 6\,\Lambda$.

It is to emphasize that the energy-momentum tensor (2.6) does not describe a perfect fluid, i.e. for a perfect fluid, one has $T^{(\rm pf)}_{ab} = (\rho + p)u_a u_b - p\,g_{ab}$ with a unit time-like vector $u_a$ and its trace $T^{(\rm pf)} = \rho - 3p$, which is different from the one given in [2, 9].

**Energy conditions:** The energy-momentum tensor (2.6) is a general form of gravitational fields. For the analysis of the energy conditions of the energy-momentum tensor, we shall introduce an orthonormal tetrad with a unit time-like $u^a$ and three unit space-like vector fields $v^a$, $w^a$, $z^a$ using the null tetrad vectors (2.2) such as

$$\begin{aligned} u_a &= \frac{1}{\sqrt{2}}(\ell_a + n_a), \;\; v_a = \frac{1}{\sqrt{2}}(\ell_a - n_a), \\ w_a &= \frac{1}{\sqrt{2}}(m_a + \overline{m}_a), \;\; z_a = -\frac{i}{\sqrt{2}}(m_a - \overline{m}_a), \end{aligned} \tag{2.8}$$

with the normalization conditions $u_a u^a = 1$, $v_a v^a = w_a w^a = z_a z^a = -1$ and other inner products being zero. Then the metric tensor takes the form

$$g_{ab} = u_a u_b - v_a v_b - w_a w_b - z_a z_b. \tag{2.9}$$

Now we shall consider a non-space like vector fields for an observer

$$U_a = \hat{\alpha} u_a + \hat{\beta} v_a + \hat{\gamma} w_a + \hat{\delta} z_a, \tag{2.10}$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\delta}$ are arbitrary constants [2, 18], subjected to the condition that

$$U^a U_a = \hat{\alpha}^2 - \hat{\beta}^2 - \hat{\gamma}^2 - \hat{\delta}^2 \geq 0. \tag{2.11}$$

Then the energy-momentum tensor (2.6) can be written in terms of the orthonormal tetrad vectors given in (2.8) as

$$T_{ab} = \mu \ell_a \ell_b + (\rho + p)(u_a u_b - v_a v_b) - p g_{ab}. \tag{2.12}$$

Now $T_{ab} U^a U^b$ will represent the energy density as measured by the observer with the tangent vector $U^a$ (2.10). It is to emphasize that this $T_{ab}$ is a general energy-momentum tensor of non-stationary space-times [2]. It includes those of electromagnetic field, monopole, de Sitter cosmological model etc. However, it does not include a perfect fluid with unit time-like vector $u_a$. Hence, the energy-momentum tensor (2.12) is different from the one of a perfect fluid $T^{(\rm pf)}_{ab} = (\rho + p) u_a \, u_b - p \, g_{ab}$ with unit time-like vector $u_a$, and it is convenient to introduce all the energy conditions for $T_{ab}$ for future use:

(a) *Weak energy condition*: The energy momentum tensor obeys the inequality $T_{ab} U^a U^b \geq 0$ for any future directed time-like vector $U^a$ which implies that

$$\mu \geq 0, \quad \rho \geq 0, \quad \rho + p \geq 0. \tag{2.13}$$

(b) *Strong energy condition*: The Ricci tensor for $T_{ab}$ satisfies the inequality $R_{ab}\,U^a U^b \geq 0$ for any time-like vector $U^a$, i.e. $T_{ab}U^a U^b \geq (1/2)T$, which yields

$$\mu \geq 0, \quad p \geq 0, \quad \rho + p \geq 0. \tag{2.14}$$

(c) *Dominant energy condition*: For any future directed time-like vector $U^a$, $T_{ab}U^b$ should be a future directed non-space like (time-like or null) vector field. This condition is equivalent to

$$\mu \geq 0, \quad \rho^2 \geq 0, \quad \rho^2 - p^2 \geq 0. \tag{2.15}$$

It is noted that the strong energy condition does not imply the weak energy condition.

From (2.7) we observe that there is no straightforward way for solving the non-linear Einstein's field equations with the mass function $\hat{M}(u,r)$ to generate exact solutions of physical interest. In order to have a meaningful physical interpretation of the line-element (2.1) we have to consider some certain assumptions on the mass function $\hat{M}(u,r)$ as the line-element having the energy-momentum tensor (2.6) with the quantities (2.7) has no reasonable interpretation to regard it as exact solution of Einstein's field equations. In order to obtain the physically meaningful line-element, we have to assume the mass function $\hat{M}(u,r)$ or rather to restrict it in some forms. Thus we consider, without lose of generality, the mass function in the form of power series expansion as follows:

$$\hat{M}(u,r) \equiv \sum_{n=-\infty}^{+\infty} q_n(u)\, r^n. \tag{2.16}$$

Here $q_n(u)$ is referred to Wang-Wu function [9] of the retarded time coordinate. The u-constant surfaces are null cones open to the future and the r-constant is null coordinate. The retarded time coordinate is used to evaluate the radiating (or outgoing) energy-momentum tensor around the astronomical body [11].

For instance, when $\hat{M}(u,r)$ sets to a constant $M$ for $n = 0$, it is the Schwarzschild solution with $\mu = \rho = p = 0$. The Vaidya null radiating solution can be obtained, when one assumes the mass function to be $\hat{M}(u,r) = M(u)$, leading to the condition $\rho = p = 0$ in (2.7). This type of assumption on the mass function $\hat{M}(u,r)$ turns out to be the case of $n = 0$ in the power series expansion (2.16). However, to generate charged solutions like Reissner-Nordstrom and Vaidya-Bonnor, we have to use the combination $n = 0$ and $n = -1$ together in the power series expansion, that will provides the mass function $\hat{M}(u,r) = M(u) - e^2(u)r^{-1}$ for Vaidya-Bonnor solution. When $M(u)$ and $e(u)$ become constant with $\hat{M}(u,r) = M - e^2r^{-1}$, it gives the Reissner-Nordstrom solution. Therefore, the mass function $\hat{M}(u,r)$ can, without loss of generality, be expressed in the powers of $r$ as in (2.16). The above line-element (2.1) with the mass function (2.16) includes most of the known solutions of Einstein's field equations, that can be seen with the identifications of the index $n(=-1,0,1,3)$, depending on the system (rotating or non-rotating) mentioned in the introduction above.

For future reference, we note down the role of the power $n$ in the expansion series for the known spherically axisymmetric solutions as in [2], [8] and [9]:

(i) $n = 0$ corresponds to the term containing mass of the vacuum Kerr family solutions such as Schwarzschild, Kerr;
(ii) $n = -1$ is equivalent to the charged term of Kerr family such as Reissner-Nordstrom, Kerr-Newman;
(iii) $n = 1$ furnishes the term of the global monopole solution;
(iv) $n = 3$ provides the de Sitter cosmological models, rotating and non-rotating [12];
(v) $n = 2$ produces dark energy solution having equation of state parameter $w = -1/2$ [10].

These values of $n$ are conveniently used for generating stationary non-rotating and rotating solutions. Here in this paper for deriving black holes in non-stationary de Sitter space we will combine two values of power $n$ (i.e. $n = 0$ and $n = 3$) in both Sections 4 and 5 below. Similar combinations of the values of $n$ for obtaining embedded exact solutions, rotating or non-rotating, may be seen in [2].

## 3. Non-stationary de Sitter solution

In this section we shall show the derivation of a non-stationary de Sitter metric [12] with a cosmological term of variable $\Lambda(u)$ by using the power series expansion (2.8). In view of this, we consider the Wang-Wu function as

$$q_n(u) = \begin{cases} \Lambda(u)/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 0, \end{cases} \tag{3.1}$$

such that the mass function (2.16) becomes

$$\hat{M}(u,r) = \frac{1}{6}r^3\Lambda(u). \tag{3.2}$$

Then using this mass function, the line element (2.1) becomes the form, describing a *non-stationary* de Sitter line-element with cosmological function $\Lambda(u)$ in the null coordinates $(u, r, \theta, \phi)$ as

$$ds^2 = \left\{1 - \frac{1}{3}r^2\Lambda(u)\right\} du^2 + 2du\, dr - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{3.3}$$

Here $\Lambda(u)$ is an arbitrary non-increasing function of the retarded time coordinate $u$. The non-stationary de Sitter metric (3.3) has singularity horizons at $r = \pm\{3\Lambda^{(-1)}(u)\}^{1/2}$ [12]. From Einstein's field equations $R_{ab} - (1/2)Rg_{ab} = -KT_{ab}$, we find that the above metric possesses an energy-momentum tensor as

$$KT_{ab} = -\frac{1}{3}r\Lambda(u)_{,u}\ell_a\ell_b + \Lambda(u)g_{ab} \tag{3.4}$$

where $\ell_a = \delta^1_a$ is a null vector and the universal constant $K = 8\pi G/c^4$. The trace of the tensor (3.4) is given by $KT = 4\Lambda(u)$. Here it is to mention that the energy-momentum tensor (3.4) involves a Vaidya-like null radiation term $-\frac{1}{3}r\Lambda(u)_{,u}\ell_a\ell_b$, which arises from the *non-stationary state of motion* of an observer traveling in the *non-stationary* de Sitter universe (3.3). This non-stationary part of the energy-momentum tensor (3.4) contributes the nature of null radiating matters present in (3.3) whose energy-momentum tensor will vanish when $r \to 0$, and has zero trace. However, it still maintains the non-stationary status that $\Lambda(u) \neq$ constant, showing that the space-time of the observer is naturally time dependent even at $r \to 0$. Let us denote it as $T^{(\mathrm{NS})}_{ab}$, where 'NS' stands for *non-stationary* as it arises from the *non-stationary* state of the universe.

Using the energy-momentum tenor (3.4) we could write Einstein's field equations as follows

$$R_{ab} - \frac{1}{2}R\, g_{ab} + \Lambda(u)g_{ab} = -T^{(\mathrm{NS})}_{ab}, \tag{3.5}$$

It is noted that the right side of the equation (3.5) does not involve the universal constant $K$. The derivation of this non-stationary de Sitter cosmological universe (3.3) is in agreement with the original stationary de Sitter model [1] when $\Lambda(u)$ takes a constant value in (3.4).

Now expressing the metric tensor in terms of complex null tetrad vectors [7] $g_{ab} = 2\ell_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}$, we can write the energy-momentum tensor (3.4) as follows

$$T_{ab} = \mu\,\ell_a\ell_b + 2\,\rho\,\ell_{(a}n_{b)} + 2\,p\,m_{(a}\overline{m}_{b)}, \tag{3.6}$$

where $\rho$ and $p$ are the density and pressure of the non-stationary de Sitter models respectively and are found as:

$$\rho = -p = \frac{\Lambda(u)}{K}, \qquad \mu = -\frac{r}{3K}\Lambda(u)_{,u}. \tag{3.7}$$

The trace of energy-momentum tensor (3.6) is given as

$$T \equiv g^{ab}T_{ab} = 2(\rho - p) = \frac{4}{K}\Lambda(u). \tag{3.8}$$

Here it is observed that $\rho - p > 0$ for the non-stationary de Sitter model. From (3.8) we find the equation of state

$$w = \frac{p}{\rho} = -1 \tag{3.9}$$

with the negative pressure of variable $\Lambda(u)$. This shows the fact that the non-stationary de Sitter solution (3.3) is in agreement with the cosmological constant $\Lambda$ de Sitter model possessing the equation of state $w = -1$ in the dark energy scenario [13, 14, 15], when $\Lambda(u)$ takes a constant value $\Lambda$. The energy-momentum tensor (3.6) satisfies the energy conservation equation

$$T^{ab}{}_{;b} = 0. \tag{3.10}$$

The energy-momentum tensor (3.6) with $\mu$, $\rho$ and $p$ given in (3.7) is the same form of (2.6) with the coefficients in (2.7). So the tensor of the form (2.6) is the general form of energy-momentum tensor whose coefficients $\mu$, $\rho$ and $p$ may vary depending on the choice of the matter present in a particular space-time.

The metric (3.3) has an apparent singularity at $r = \pm\{3\Lambda^{(-1)}(u)\}^{1/2}$. The root $r_+ = 3^{1/2}\Lambda(u)^{-1/2}$ corresponds to a cosmological horizon at $r_+ = 3^{1/2}\Lambda(u)^{-1/2}$. According to Carter [5] and York [16], we introduce a scalar $\kappa$ defined by the relation $n^b\nabla_b n^a = \kappa\, n^a$, where the null vector $n^a$ is parameterized by the coordinate $u$, such that $d/du = n^a\nabla_a$. On the horizon $r = r_+$, the scalar $\kappa$ is referred to the surface gravity of the de Sitter model and is obtained as

$$\kappa = 3^{-1/2}\Lambda(u)^{1/2}.$$

The entropy $\mathcal{S}$ on the horizon is related with the area $\mathcal{A}$ of the horizon as $\mathcal{S} = \mathcal{A}/4$ and is obtained as

$$\mathcal{S} = 3\pi\Lambda(u)^{-1}.$$

It is also found that the non-stationary de Sitter space-time is conformally flat $C_{abcd} = 0$, i.e. all the Weyl scalars are vanished

$$\psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0. \tag{3.11}$$

The Kretschmann scalar for non-rotating de Sitter model (3.3) takes the form

$$\mathcal{K} \equiv R_{abcd}R^{abcd} = \frac{8}{3}\Lambda(u)^2, \tag{3.12}$$

which does not involves any derivative term of $\Lambda(u)$, and will not change its value at $r \to 0$ and $r \to \infty$. The above Kretschmann scalar will become the one of original de Sitter model when $\Lambda(u)$ takes a constant value $\Lambda(u)$. That is, though the energy-momentum tensor (3.4) for non-stationary model is found different from the one of stationary de Sitter solution, the forms of Kretschmann scalar for both *non-stationary* (3.3) and *stationary* models [1] have similar structures with a difference in nature of the cosmological function $\Lambda(u)$.

**de Sitter solution as dark energy**: The stationary de Sitter with a cosmological constant $\Lambda$ is a solution of Einstein's field equations for an non-empty space $T_{ab} \neq 0$ possessing a ratio of pressure ($p = -\Lambda/K$) to the density ($\rho = \Lambda/K$) as an equation of state parameter $w = p/\rho = -1$. This negative parameter $w = -1$ is the most important characteristic property of de Sitter space to be considered as a candidate of dark energy solution of Einstein's field equations [13, 14, 15] and references there in. We have seen from (3.9) above that the non-stationary de Sitter solution with variable $\Lambda(u)$ admits the equation of state parameter $w = p/\rho = -1$ associated with the pressure ($p = -\Lambda(u)/K$) and the density ($\rho = \Lambda(u)/K$). Also due to the negative pressure, the energy-momentum tensor (3.6) of the non-stationary de Sitter solution violates the strong energy condition (2.14) above. This violation of strong energy condition implies the repulsive gravitational field of the non-stationary de Sitter solution. The space-time geometry of de Sitter solution with parameter $w = -1$ (either stationary or non-stationary) is conformally flat (3.11).

Here it will be better to introduce other dark energy solution of mass $m$ possessing negative pressure $p = -(2/Kr)m$ and density $\rho = (4/Kr)m$, and having an equation state parameter $w = -1/2$ with minus sign [10]. The energy-momentum tensor of this dark energy solution also violates the strong energy condition (2.14) leading to a repulsive gravitational field in the space-time geometry. The dark energy solution with equation state parameter $w = -1/2$ (either stationary or non-stationary) of Ibohal et al. [10] is conformally flat. It is emphasized that the equations of state parameter $w = -1/2$ for the dark energy solution is belonged to the range $-1 < w < 0$ focussed for the best fit with cosmological observations [14] and references there in as $-1 < -1/2 < 0$. In general we may summarize the properties of dark energy solutions of Einstein's field equations that *the energy-momentum tensor of a space-time geometry (i)* *possesses a negative pressure, (ii) has an equation of state with minus sign, (iii) violates the strong energy condition.* It is to mention that the energy-momentum of a matter field satisfying the strong energy condition indicates the attractive gravitational field of the matter.

## 4. Schwarzschild black hole in non-stationary de Sitter space

In this section we propose the derivation of an exact solution describing the Schwarzschild-de Sitter solution with the variable $\Lambda(u)$ of the above section. which may be treated as the Schwarzschild black hole in the non-stationary de Sitter background with variable $\Lambda(u)$. It is emphasized that the Schwarzschild solution embedded into the *constant* $\Lambda$ de Sitter space is known [3]; however, we have not seen published the Schwarzschild solution embedded into the *variable* $\Lambda(u)$ de Sitter space. For deriving the Schwarzschild-de Sitter solution with $\Lambda(u)$ we consider the Wang-Wu functions $q(u)$ in (2.16) as follows:

$$q_n(u) = \begin{cases} M, & \text{when } n = 0 \\ \Lambda(u)/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 0, 3, \end{cases} \tag{4.1}$$

such that the mass function takes the form

$$M(u, r) = M + \frac{1}{6} r^3 \Lambda(u). \tag{4.2}$$

The function $M(u, r)$ is the combination of two values of $n$ ($n = 0$ and $n = 3$) together to get one embedded solution. Now using this mass function in the metric (2.1), we obtain a non-stationary metric, describing the Schwarzschild metric embedded into the *non-stationary* de Sitter space to produce Schwarzschild-de Sitter solution with $\Lambda(u)$ as

$$ds^2 = \left\{1 - \frac{2M}{r} - \frac{1}{3} r^2 \Lambda(u)\right\} du^2 + 2du\, dr - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{4.3}$$

where $M$ is the mass of the Schwarzschild black hole and $\Lambda(u)$ denotes the de Sitter cosmological function of retarded time coordinate $u$. When we set the function $\Lambda(u)$ to be a constant $\Lambda$, the line element (4.3) will become the Schwarzschild-de Sitter space-time with cosmological constant [3]. When the Schwarzschild mass vanishes $M = 0$, and function $\Lambda(u)$ is set to be a constant $\Lambda$, the line element will recover the original de Sitter solution [1].

The complex null vectors for the Schwarzschild-de Sitter solution can be found as follows:

$$\ell_a = \delta^1_a, \quad n_a = \frac{1}{2\, r^2} \Delta\, \delta^1_a + \delta^2_a,$$

$$m_a = -\frac{r}{\sqrt{2}}\left\{\delta^3_a + i\sin\theta\,\delta^4_a\right\} \tag{4.4}$$

where $\Delta = r^2 - 2rM - \Lambda(u)\,r^4/3$. Here $\ell_a$, $n_a$ are real null vectors and $m_a$ is complex with the normalization conditions $\ell_a n^a = 1 = -m_a\bar{m}^a$ and the other inner products of the null vectors are zero. From Einstein's field equations $R_{ab} - (1/2)Rg_{ab} = -KT_{ab}$, we find the energy-momentum tensor describing the matter field for the Schwarzschild-de Sitter space-time (4.3) as

$$T_{ab} \;=\; \mu\,\ell_a\ell_b + 2\,\rho\,\ell_{(a}n_{b)} + 2\,p\,m_{(a}\bar{m}_{b)}, \tag{4.5}$$

which is the same as that of non-stationary de Sitter solution with the quantities (3.7)

$$\begin{aligned}\rho &= -\,p = \frac{\Lambda(u)}{K},\\ \mu &= -\frac{r}{3K}\Lambda(u)_{,u}.\end{aligned} \tag{4.6}$$

The trace of energy momentum tensor $T_{ab}$ is given in (3.8). This is because the Schwarzschild solution is a vacuum space-time. Here it is observed that $\rho - p > 0$ for the non-stationary de Sitter model and the trace $T$ does not involve the Schwarzschild mass $M$, showing the vacuum space-time (4.3). The Ricci scalar $\Lambda^*$ $(\equiv \frac{1}{24}g^{ab}\,R_{ab})$, takes the form $\Lambda^* = \frac{1}{6}\Lambda(u)$ describing the existence of matter field in Schwarzschild-de Sitter space-time (4.3). The energy-momentum tensor for the solution (4.3) may be written in the following decomposition form as:

$$T_{ab} = T^{(\rm NS)}_{ab} + T^{(\rm dS)}_{ab}, \tag{4.7}$$

where the $T^{(\rm NS)}_{ab}$ the non-stationary contribution of de Sitter field $\Lambda(u)$ associated with the derivative term $\Lambda(u)_{,u}$ and $T^{(\rm dS)}_{ab}$ the cosmological de Sitter matter are given, respectively

$$T^{(\rm NS)}_{ab} = \mu^{(\rm NS)}\ell_a\ell_b, \tag{4.8}$$

$$T^{(\rm dS)}_{ab} = 2\rho^{(\rm dS)}\ell_{(a}n_{b)} + 2p^{(\rm dS)}\,m_{(a}\bar{m}_{b)}\}, \tag{4.9}$$

where the coefficients are given by

$$\begin{aligned}\mu^{(\rm NS)} &= -\frac{r}{3K}\Lambda(u)_{,u}\\ \rho^{(\rm dS)} &= -\,p^{(\rm dS)} = \frac{\Lambda(u)}{K}.\end{aligned} \tag{4.10}$$

Here $\mu^{(\rm NS)}$ the non-stationary null density associated with the derivative of $\Lambda(u)$, $\rho^{(\rm dS)}$ and $p^{(\rm dS)}$ are the density and the pressure of de Sitter matter. When the function $\Lambda(u)$ becomes constant, it will provide $T_{ab}^{(\mathrm{NS})} = 0$, then the space-time will be that of the Schwarzschild-de Sitter with constant $\Lambda$. If $M = 0$, $\Lambda(u) \neq$ constant then the remaining space-time (4.3) will be the non-stationary de Sitter model with variable $\Lambda(u)$ (3.3). At that time the $T_{ab}^{(\mathrm{NS})}$ and $T_{ab}^{(\mathrm{dS})}$ will exist indicating the non-stationary de Sitter matter distribution; and for constant $\Lambda$, $T_{ab}^{(\mathrm{NS})} = 0$, then the energy-momentum tensor (4.7) will reduce to $T_{ab} = T_{ab}^{(\mathrm{dS})} = \Lambda g_{ab}$ for the well-known de Sitter model with constant $\Lambda$.

For the metric (4.3) the energy-momentum tensor (4.7) can be written in the form of Guth's modification of $T_{ab} \rightarrow T_{ab} + \Lambda g_{ab}$ for early inflation of the universe [6] as

$$T_{ab} = T_{ab}^{(\mathrm{NS})} + \Lambda(u) g_{ab}, \tag{4.11}$$

where $g_{ab}$ is the Schwarzschild-Sitter metric tensor. This may represent the inflation of vacuum Schwarzschild space in the non-stationary de Sitter space with cosmological variable $\Lambda(u)$.

Here we shall justify the nature of the embedded solution in the form of Kerr-Schild ansatze in different backgrounds. The Schwarzschild-de Sitter metric can be expressed in Kerr-Schild ansatz on the non-stationary de Sitter background

$$g_{ab}^{(\mathrm{SchdS})} = g_{ab}^{(\mathrm{dS})} + 2Q(u,r)\ell_a \ell_b \tag{4.12}$$

where $Q(u,r) = -M\, r^{-1}$. Here, $g_{ab}^{(\mathrm{dS})}$ is the non-stationary de Sitter metric (3) and $\ell_a$ is geodesic, shear free, expanding and zero twist null vector for both $g_{ab}^{(\mathrm{dS})}$ as well as $g_{ab}^{(\mathrm{SchdS})}$. The above Kerr-Schild form can also be recast on the Schwarzschild background as

$$g_{ab}^{(\mathrm{SchdS})} = g_{ab}^{(\mathrm{Sch})} + 2Q(u,r)\ell_a \ell_b \tag{4.13}$$

where $Q(u,r) = -\Lambda(u) r^2/6$. These two Kerr-Schild forms (4.12) and (4.13) show the fact that the non-stationary Schwarzschild-de Sitter space-time (4.3) with variable $\Lambda(u)$ is a solution of Einstein's field equations. They establish the structure of embedded black hole that either "the Schwarzschild black hole is embedded into the non-stationary de Sitter cosmological universe to produce Schwarzschild-de Sitter black hole" or the non-stationary de Sitter universe is embedded into the Schwarzschild black hole to obtain the de Sitter-Schwarzschild black hole – both nomenclature possess the same geometrical meaning. That is, we cannot physically predict which space started first to embed into another [2].

**Surface gravity**: The metric (4.3) will describe a cosmological black hole with the horizons at the values of $r$ for which the polynomial equation $\Delta = r^2 - 2rM - \Lambda(u)\, r^4/3 = 0$ has three roots $r_1$, $r_2$, and $r_3(= \bar{r}_2)$. The explicit roots are given as

$$r_1 = -\frac{1}{(3Q)^{\frac{1}{3}}} - \frac{1}{\Lambda(u)}(3Q)^{\frac{1}{3}}$$
$$r_2 = \frac{1}{2\Lambda(u)}\Big[(1+i\sqrt{3})\Lambda(u)(3Q)^{-\frac{1}{3}} + (1-i\sqrt{3})(3Q)^{\frac{1}{3}}\Big], \tag{4.14}$$

where $Q = \Lambda^2(u)\{M + (1/3)\Lambda(u)^{(-1/2)}\sqrt{9\Lambda(u)M^2 - 1}\}$. These roots satisfy the following relation

$$(r-r_1)(r-r_2)(r-r_3) = -\frac{3}{\Lambda(u)}\Big\{r - 2M - \frac{\Lambda(u)}{3}\, r^3\Big\} \tag{4.15}$$

Here we are interested only the real root $r_1$ which may describe the horizon of the Schwarzschild-de Sitter cosmological black hole, as the complex roots have less physical interpretation.

The surface gravity $\kappa$ of a horizon is defined by the relation $n^b\nabla_b n^a = \kappa n^a$, where the null vector $n^a$ in (3.4) above is parameterized by the coordinate $u$, such that $d/du = n^b\nabla_b$ [5], [16]. $\nabla_b$ is the covariant derivative. The surface gravities at $r = r_i, (i = 1, 2, 3)$ are as

$$\kappa = \frac{1}{r^2}\left\{M - r + \frac{\Lambda(u)r^3}{6}\right\}\Big|_{r=r_i}, \tag{4.16}$$

and the entropy of the horizon is found as,

$$\mathcal{S} = \pi\, r^2\Big|_{r=r_i}. \tag{4.17}$$

Here we consider a case of extreme Schwarzschild-de Sitter black hole having the mass function $M = \pm(1/3)\Lambda(u)^{(-1/2)}$, if $\Lambda(u) > 0$ for a particular constant value of $u$. This implies that the real root $r_1$ take the values $r_1 = -2\Lambda(u)^{(-1/2)}$ at that point, and the two complex roots are coincided $r_2 = r_3 = \Lambda(u)^{(-1/2)}$. The surface gravity on the cosmological black hole horizon $r = r_1$ takes the form $\kappa = -(3/4)\Lambda(u)^{(-1/2)}$. However, it is vanished at $r = r_2 = r_3$. Then we obtain the Hawking's temperature of the cosmological black hole horizon at $r = r_1$ from the relation $\hat{T} = \kappa/2\pi$ as

$$\hat{T} = -\frac{3}{8\pi}\Lambda(u)^{(-\frac{1}{2})} \tag{4.18}$$

for a particular constant value of $u$. The temperature associated with the real root $r_1$ in (4.14) will never vanish for the existence of variable $\Lambda(u)$ in the space-time geometry of the Schwarzschild-de Sitter black hole. The condition $-(1/3)\Lambda(u)^{(-1/2)} \leq M \leq +(1/3)\Lambda(u)^{(-1/2)}$ for some constant value of $u$ with the Schwarzschild mass $M$ may be the non-stationary generalization of the condition $-(1/3)\Lambda^{(-1/2)} \leq M \leq +(1/3)\Lambda^{(-1/2)}$ of the Schwarzschild-de Sitter black hole with constant mass and constant $\Lambda$ [3].

The Schwarzschild-de Sitter metric (4.3) describes a non-stationary embedded spherically symmetric solution whose Weyl curvature tensor is type $D$

$$\psi_2 \equiv -C_{pqrs}\ell^p m^q \overline{m}^r n^s = -Mr^{-3} \tag{4.19}$$

in Petrov classification possessing a repeated null direction $\ell_a$ which is geodesic, shear free, expanding and zero-twist, as other Weyl scalars are vanished $\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$. It is observed that the variable $\Lambda(u)$ does not involve in the expression of $\psi_2$ above, showing the conformally flat character of non-stationary de Sitter space [12]. The Kretschmann scalar for Schwarzschild-de Sitter model (4.3) takes the form

$$R_{abcd}R^{abcd} = \frac{48}{r^6}M^2 + \frac{8}{3}\Lambda^2(u), \tag{4.20}$$

This invariant does not diverge at the origin.

## 5. Vaidya black hole in non-stationary de Sitter solution

We shall discuss the derivation of the *Vaidya-de Sitter* solution with a non-stationary variable $\Lambda(u)$, which may be treated as the non-stationary Vaidya-de Sitter black hole or the Vaidya black hole on the non-stationary de Sitter background with variable $\Lambda(u)$. It is emphasized that the Vaidya solution embedded into the *constant* $\Lambda$ de Sitter space is known [17, 18]. For deriving the Vaidya-de Sitter solution with $\Lambda(u)$ we consider the Wang-Wu functions $q(u)$ in (2.16) as follows:

$$q_n(u) = \begin{cases} M(u), & \text{when } n = 0 \\ \Lambda(u)/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 0, 3, \end{cases} \tag{5.1}$$

such that the mass function (2.16) takes the form

$$M(u,r) = M(u) + \frac{1}{6}r^3\Lambda(u). \tag{5.2}$$

Then using this mass function in the metric (2.1), we obtain a non-stationary metric, describing the Vaidya metric embedded into the *non-stationary* de Sitter model to produce Vaidya-de Sitter solution with variable cosmological function $\Lambda(u)$ as

$$ds^2 = \left\{1 - \frac{2M(u)}{r} - \frac{1}{3}r^2\Lambda(u)\right\} du^2 + 2du\,dr - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{5.3}$$

where $M(u)$ is the mass of the Vaidya black hole and $\Lambda(u)$ denotes the de Sitter cosmological function of retarded time coordinate $u$. When we set the function $\Lambda(u)$ to be a constant $\Lambda$, the line element (5.3) will become the Vaidya-de Sitter space-time with cosmological constant. When the Vaidya mass vanishes as $M(u) = 0$, and function $\Lambda(u)$ to be a constant $\Lambda$, the line element will recover the original de Sitter solution.

The complex null vectors for the Vaidya-de Sitter solution can be chosen as follows:

$$\begin{aligned} \ell_a &= \delta^1_a, \quad n_a = \frac{1}{2\,r^2}\Delta\,\delta^1_a + \delta^2_a, \\ m_a &= -\frac{r}{\sqrt{2}}\left\{\delta^3_a + i\sin\theta\,\delta^4_a\right\} \end{aligned} \tag{5.4}$$

where $\Delta = r^2 - 2rM(u) - \Lambda(u)\,r^4/3$. Here $\ell_a$, $n_a$ are real null vectors and $m_a$ is complex with the normalization conditions $\ell_a n^a = 1 = -m_a\bar{m}^a$ and the other inner products of the null vectors are zero as before. From Einstein's field equations we find the energy-momentum tensor describing the matter field for the non-stationary space-time (5.3) as

$$T_{ab} = \mu\,\ell_a\ell_b + 2\,\rho\,\ell_{(a}n_{b)} + 2\,p\,m_{(a}\bar{m}_{b)}, \tag{5.5}$$

where the coefficients $\rho$, $p$ and $\mu$ are the density, the pressure and the null density, respectively and are given below:

$$\begin{aligned} \rho &= -\,p = \frac{\Lambda(u)}{K}, \\ \mu &= -\frac{1}{K\,r^2}M(u)_{,u} - \frac{r}{3K}\Lambda(u)_{,u}. \end{aligned} \tag{5.6}$$

The trace of energy momentum tensor $T_{ab}$ (5.5) is found as

$$T = \frac{4}{K}\Lambda(u). \tag{5.7}$$

Here it is observed that $\rho - p > 0$ for the non-stationary Vaidya-de Sitter solution and the trace $T$ does not involve the Vaidya mass $M(u)$, showing that Vaidya solution is the null fluid distribution of the space-time (5.3). The Ricci scalar $\Lambda^*$ ($\equiv \frac{1}{24}g^{ab}\,R_{ab}$), describing matter field takes the form $\Lambda^* = \frac{1}{6}\Lambda(u)$. The energy-momentum tensor (5.5) satisfies the energy conservation equations, written in

NP formalism [12]

$$T^{ab}{}_{;b} = 0. \tag{5.8}$$

As in general relativity the physical properties of a space-time geometry are determined by the nature of the matter distribution in the space, we must express the energy-momentum tensor (5.5) in such a way that one should be able to understand it easily. Thus, the total energy-momentum tensor for the solution (5.3) may be written in the following decomposition form as:

$$T_{ab} = T^{(\rm V)}_{ab} + T^{(\rm NS)}_{ab} + T^{(\rm dS)}_{ab}, \tag{5.9}$$

where the $T^{(\rm V)}_{ab}$ being for the Vaidya null radiating fluid, $T^{(\rm NS)}_{ab}$ the non-stationary contribution of de Sitter field $\Lambda(u)$ associated with the derivative term $\Lambda(u)_{,u}$. and $T^{(\rm dS)}_{ab}$ the cosmological de Sitter matter are given, respectively

$$T^{(\rm V)}_{ab} = \mu^{(\rm V)}\ell_a\ell_b, \tag{5.10}$$

$$T^{(\rm NS)}_{ab} = \mu^{(\rm NS)}\ell_a\ell_b, \tag{5.11}$$

$$T^{(\rm dS)}_{ab} = 2\rho^{(\rm dS)}\ell_{(a}n_{b)} + 2p^{(\rm dS)}\, m_{(a}\bar{m}_{b)}\}, \tag{5.12}$$

where the coefficients are given by

$$\begin{aligned}
&\mu^{(\rm V)} = -\frac{1}{K\,r^2}M(u)_{,u}, \quad \mu^{(\rm NS)} = -\frac{r}{3K}\Lambda(u)_{,u} \\
&\rho^{(\rm dS)} = -\,p^{(\rm dS)} = \frac{\Lambda(u)}{K}
\end{aligned} \tag{5.13}$$

where $\mu^{(\rm V)}$ is the null density for the Vaidya null fluid $T^{(\rm V)}_{ab}$, $\mu^{(\rm NS)}$ the non-stationary null density associated with the derivative of $\Lambda(u)$, $\rho^{(\rm dS)}$ and $p^{(\rm dS)}$ are the density and the pressure of de Sitter matter. When the function $\Lambda(u)$ becomes constant, it will provide $T^{(\rm NS)}_{ab} = 0$, then the space-time will be that of the Vaidya-de Sitter with constant $\Lambda$. If $M(u) = 0$, $\Lambda(u) \neq$ constant we have $T^{(\rm V)}_{ab} = 0$, then the remaining space-time (5.3) will be the non-stationary de Sitter model with variable $\Lambda(u)$. At that stage the $T^{(\rm NS)}_{ab}$ and $T^{(\rm dS)}_{ab}$ will exist indicating the non-stationary de Sitter matter distribution; and for constant $\Lambda$, $T^{(\rm NS)}_{ab} = 0$, then the total energy-momentum tensor (5.9) will reduce to $T_{ab} = T^{(\rm dS)}_{ab} = \Lambda g_{ab}$ for the well-known de Sitter model with constant $\Lambda$.

For the metric (5.3) the energy-momentum tensor (5.9) can be written in the form of Guth's modification of $T_{ab} \to T_{ab} + \Lambda g_{ab}$ [6] for early inflation of the universe as

$$T_{ab} = T^{(\rm V)}_{ab} + T^{(\rm NS)}_{ab} + \Lambda(u)g_{ab}, \tag{5.14}$$

where $g_{ab}$ is the Vaidya-Sitter metric tensor. This shows the inflation of radiating Vaidya space in the non-stationary de Sitter space with cosmological variable $\Lambda(u)$.

Here we shall show that the Vaidya-Sitter metric is an embedded solution by writing in the form of Kerr-Schild ansatze in different backgrounds. The Vaidya-de Sitter metric can be expressed in Kerr-Schild ansatz

$$g_{ab}^{(\mathrm{VdS})} = g_{ab}^{(\mathrm{dS})} + 2Q(u,r,\theta)\ell_a\ell_b \tag{5.15}$$

where $Q(u,r) = -M(u)\,r^{-1}$. Here, $g_{ab}^{(\mathrm{dS})}$ is the non-stationary de Sitter metric and $\ell_a$ is geodesic, shear free, expanding and zero twist null vector for both $g_{ab}^{(\mathrm{dS})}$ as well as $g_{ab}^{(\mathrm{VdS})}$. The above Kerr-Schild form can also be recast on the Vaidya background as

$$g_{ab}^{\mathrm{VdS}} = g_{ab}^{\mathrm{V}} + 2Q(u,r,\theta)\ell_a\ell_b \tag{5.16}$$

where $Q(u,r) = -\Lambda(u)r^2/6$. These two Kerr-Schild forms (5.15) and (5.16) support the fact that the non-stationary Vaidya-de Sitter space-time (5.3) with variable $\Lambda(u)$ is a solution of Einstein's field equations. They establish the structure of embedded black hole that either "the null radiating Vaidya black hole is embedded into the non-stationary de Sitter cosmological universe to produce Vaidya-de Sitter black hole" or the non-stationary de Sitter universe is embedded into the Vaidya black hole to obtain the de Sitter-Vaidya black hole – both nomenclature possess the same geometrical meaning. That, it is not physically possible to predict which space started first to embed into another.

However, in the context of combination of exact solutions, it is worth to mention the fact that the two metrics $g_{ab}^{(\mathrm{V})}$ and $g_{ab}^{(\mathrm{dS})}$ cannot be added to obtain $g_{ab}^{(\mathrm{VdS})}$ as

$$g_{ab}^{(\mathrm{VdS})} \neq \frac{1}{2}\Big\{g_{ab}^{(\mathrm{V})} + g_{ab}^{(\mathrm{dS})}\Big\}.$$

This indicates that it is not possible to derive a new embedded solution by adding two physically known solutions in general relativity [2].

**Surface gravity**: The metric (5.3) will describe a cosmological black hole with the horizons at the values of $r$ for which the polynomial equation $\Delta = r^2 - 2rM(u) - \Lambda(u)\,r^4/3 = 0$ has three roots $r_1$, $r_2$, and $r_3(=\bar{r}_2)$. The explicit roots are given as

$$r_1 = -\frac{1}{(3Q)^{\frac{1}{3}}} - \frac{1}{\Lambda(u)}(3Q)^{\frac{1}{3}}$$

$$r_2 = \frac{1}{2\Lambda(u)}\Big[(1+i\surd 3)\Lambda(u)(3Q)^{-\frac{1}{3}} + (1-i\surd 3)(3Q)^{\frac{1}{3}}\Big], \tag{5.17}$$

where

$$Q = \Lambda^2(u)\{M(u) + (1/3)\Lambda(u)^{(-1/2)}\sqrt{9\Lambda(u)M^2(u)-1}\}. \tag{5.18}$$

These roots satisfy the following relation

$$(r-r_1)(r-r_2)(r-r_3) = -\frac{3}{\Lambda(u)}\Big\{r - 2M(u) - \frac{\Lambda(u)}{3}\,r^3\Big\}.$$

Here we are interested only the real root $r_1$ which may describe the horizon of the Vaidya-de Sitter cosmological black hole, as the complex roots have less physical interpretation.

The surface gravity $\kappa$ of a horizon is defined by the relation $n^b\nabla_b n^a = \kappa n^a$, where the null vector $n^a$ in (5.4) above is parameterized by the coordinate $u$, such that $d/du = n^b\nabla_b$ [5], [16]. $\nabla_b$ is the covariant derivative. The surface gravities at $r = r_i, (i = 1, 2, 3)$ are as

$$\kappa = \frac{1}{r^2}\left\{M(u) - r + \frac{\Lambda(u)r^3}{6}\right\}\Big|_{r=r_i}. \tag{5.19}$$

Consequently, the temperatures of the horizons $r = r_i$ proportional to the surface gravities can be obtained from the Bekenstein-Hawking relation $\hat{T} = \kappa/2\pi$. Then the entropies of the horizons are found from the area-entropy relation $S = A/4$ are found as,

$$\mathcal{S} = \pi\, r^2\Big|_{r=r_i}. \tag{5.20}$$

**Extreme black hole case**: Here we shall consider a case of extreme Vaidya-de Sitter black hole from (5.18) having the mass function

$$M(u) = \pm(1/3)\Lambda(u)^{(-1/2)},$$

if $\Lambda(u) > 0$ . This implies that the real root $r_1$ take the values $r_1 = -2\Lambda(u)^{(-1/2)}$, and the two complex roots are coincided $r_2 = r_3 = \Lambda(u)^{(-1/2)}$. The surface gravity on the cosmological black hole horizon $r = r_1$ takes the form $\kappa = -(3/4)\Lambda(u)^{(-1/2)}$. However, it is vanished at $r = r_2 = r_3$. Then we obtain the Hawking's temperature of the cosmological black hole horizon at $r = r_1$ from the relation $\hat{T} = \kappa/2\pi$ as

$$\hat{T} = -\frac{3}{8\pi}\Lambda(u)^{(-\frac{1}{2})}. \tag{5.21}$$

The temperature associated with the real root $r_1$ in (5.17) will never vanish as long as the de Sitter $\Lambda(u)$ exists in the space-time geometry of the Vaidya-de

Sitter black hole. The condition $-(1/3)\Lambda(u)^{(-1/2)} \leq M(u) \leq +(1/3)\Lambda(u)^{(-1/2)}$ of the Vaidya mass $m(u)$ is the non-stationary generalization of the condition $-(1/3)\Lambda^{(-1/2)} \leq M \leq +(1/3)\Lambda^{(-1/2)}$ of the Schwarzschild-de Sitter black hole with constant mass and $\Lambda$ [3].

Here, we find that the pressure $p$ with the minus sign given in (5.13) does not satisfy the strong energy condition (2.14). This violation of the strong energy condition is due to the negative pressure, and leads to a repulsive gravitational force of the matter field in the space-time (5.3). The violation indicates different properties of the energy-momentum tensor (5.5) or (5.9) with (5.13) from those of the *ordinary matter* fields, like perfect fluid, electromagnetic field etc., having positive pressures. In particular, it is to note that this strong energy condition is satisfied by the energy-momentum tensor of electromagnetic field with $\rho = p = e^2/(Kr^4)$ of Reissner-Nordstrom space-time.

**Properties of the time-like vector**: The motion of a matter field distribution is determined by the nature of the time-like vector field $u_a = \frac{1}{\sqrt{2}}(\ell_a + n_a)$ associated with energy-momentum tensor (5.5). It is found that the fluid flow of the Vaidya-de Sitter model having variable $\Lambda(u)$ is expanding ($\Theta = u^a{}_{;a} \neq 0$), accelerating ($\dot{u}_a = u_{a;b}u^b \neq 0$), shearing $\sigma_{ab} \neq 0$ and zero twist ($w_{ab} = 0$).

$$\Theta = \frac{1}{\sqrt{2}r^2}\Big\{r + M(u) + \frac{2}{3}r^3\Lambda(u)\Big\} \tag{5.22}$$

$$\dot{u}_a = \frac{1}{\sqrt{2}r^2}\Big\{M(u) - \frac{1}{3}r^3\Lambda(u)\Big\}v_a \tag{5.23}$$

$$\sigma_{ab} = -\frac{1}{3\sqrt{2}r^2}\Big\{r + 4M(u) - \frac{1}{3}r^3\Lambda(u)\Big\}\Big(v_a v_b - m_{(a}\bar{m}_{b)}\Big), \tag{5.24}$$

where $v_a = \frac{1}{\sqrt{2}}(\ell_a - n_a)$ is a space-like vector field $v^a v_a = -1$. From these we observe that both the mass $M(u)$ and the variable $\Lambda(u)$ are appeared in all the three equations. When $M(u) = 0$, the remaining equations will be for non-stationary de Sitter space, whereas, if $\Lambda(u) = 0$, these will be for Vaidya radiating black hole. This means that the time-like observer in the Vaidya space will have a four velocity vector field $u_a$ which is expanding, accelerating, shearing but zero-twist for $\Lambda(u) = 0$. This is also true for the observer in non-stationary de Sitter space, when $M(u) = 0$.

The Vaidya-de Sitter metric (5.3) describes a non-stationary embedded spherically symmetric solution whose Weyl curvature tensor is type $D$

$$\psi_2 \equiv -C_{pqrs}\ell^p m^q \overline{m}^r n^s = -M(u)r^{-3}. \tag{5.25}$$

in Petrov classification possessing a repeated null direction $\ell_a$ which is geodesic, shear free, expanding and zero-twist, as other Weyl scalars are vanished $\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$. It is observed that the variable $\Lambda(u)$ does not involve in the expression of $\psi_2$ above, showing the conformally flat character of non-stationary de Sitter space [2] in the embedded Vaidya-de Sitter space. The Kretschmann scalar for Vaidya-de Sitter model (5.3) takes the form

$$R_{abcd}R^{abcd} = \frac{48}{r^6}M^2(u) + \frac{8}{3}\Lambda^2(u), \tag{5.26}$$

This invariant does not diverge when $r \to 0$ as the $\Lambda(u)$ term will remain for the embedded solution (5.3). This scalar does not involve any derivative terms of $M(u)$ and $\Lambda(u)$, and will not affect its form when the mass $M(u)$ and the variable $\Lambda(u)$ become the constant.

## 6. Conclusion

Here we have investigated the non-stationary de Sitter solution with variable $\Lambda(u)$ from the field equations, expressed in Newman-Penrose formalism. The dynamical evolution of variable $\Lambda(u)$ in Einstein's field equations is seen as $T^{(\mathrm{NS})}_{ab} = -\frac{1}{3}r\Lambda(u)_{,u}\ell_a\ell_b$ in (3.5) and also in (5.9). This term will always exist for the non-stationary de Sitter solution with variable $\Lambda(u)$. However, this will vanish for the stationary de Sitter solution with constant $\Lambda$. This is to say that a direct replacement of constant $\Lambda$ by a variable $\Lambda(u)$ in Einstein's field equations $R_{ab} - (1/2)Rg_{ab} + \Lambda g_{ab} = -KT_{ab}$ as $R_{ab} - (1/2)Rg_{ab} + \Lambda(u)g_{ab} = -KT_{ab}$ will not, in general, satisfy the energy conservation equations. Therefore, in order to study a cosmological variable $\Lambda(u)$ problem, we needs to introduce the extra term $T^{(\mathrm{NS})}_{ab}$ in the field equations for any type of matter field distribution $T_{ab}$ as

$$R_{ab} - (1/2)Rg_{ab} + \Lambda(u)g_{ab} = -K\{T_{ab} + T^{(\mathrm{NS})}_{ab}\}, \tag{6.1}$$

where

$$T^{(\mathrm{NS})}_{ab} = -\frac{1}{3}r\Lambda(u)_{,u}\ell_a\ell_b \tag{6.2}$$

and $\ell^a$ is a null vector associated with the matric tensor $g_{ab}\ell^a\ell^b = 0$. These equations (6.1) may be treated as *the modified Einstein's field equations* associated with a variable cosmological $\Lambda(u)$. For example, the energy-momentum tensors $T_{ab}$ for electromagnetic field in Reissner-Nordstrom and Vaidya-Bonnor black holes embedded into the non-stationary de Sitter space with a variable $\Lambda(u)$ will satisfy the modified field equations (6.1) [2, 9, 12]. It is also shown that, according to Guth's modification [6] of energy-momentum tensor in (5.14) above, the Vaidya black hole might have inflated into the non-stationary de Sitter space with variable $\Lambda(u)$ in the early stage of the universe. It is worth to mention that, when $r \to 0$ at the origin, $T_{ab}^{(\mathrm{NS})}$ will be vanished without disturbing the non-stationary status of variable $\Lambda(u)$.

We discuss exact solutions of Einstein field equations for the black holes embedded in a non-stationary de Sitter background with variable $\Lambda(u)$. These solutions with variable $\Lambda(u)$ may be regarded as the generalizations of Schwarzschild-de Sitter [3] and Vaidya-de Sitter solutions with constant $\Lambda$ [17]. The Vaidya-de Sitter solution with constant $\Lambda$ is also a generalized form of the Schwarzschild-de Sitter solution with constant mass $M$ and constant $\Lambda$. The Schwarzschild-de Sitter solution is interpreted as a black hole in asymptotically de Sitter space [3]. Thus, in this regard, the Schwarzschild-de Sitter and Vaidya-de Sitter solutions with variable $\Lambda(u)$ may be interpreted as black holes in asymptotically non-stationary de Sitter space. We have also seen that in the extreme case of Vaidya-de Sitter black hole, the Vaidya mass $M(u)$ has the limit $-(1/3)\Lambda(u)^{(-1/2)} \leq M(u) \leq +(1/3)\Lambda(u)^{(-1/2)}$ for $\Lambda(u) > 0$ [20] as mentioned above. This Vaidya mass limit is also regarded as the generalization of the Schwarzschild mass $M$ limit $-(1/3)\Lambda(u)^{(-1/2)} \leq M \leq +(1/3)\Lambda(u)^{(-1/2)}$ for a particular value of $u$ of the Schwarzschild-de Sitter with variable $\Lambda(u)$. These two limits are straightforward generalizations of Schwarzschild-de Sitter case $-(1/3)\Lambda^{(-1/2)} \leq M \leq +(1/3)\Lambda^{(-1/2)}$ for constant mass $M$ and $\Lambda(>0)$ [3].

The energy conditions (2.13)–(2.15) are general conditions for any non-stationary space-time having the energy-momentum tensor of the type (2.6). Particularly, the energy-momentum tensors of non-stationary de Sitter solution (3.6), for Schwarzschild-de Sitter solution (4.5) and for Vaidya-de Sitter solution (5.5) have the same forms of conditions. But, because of the negative pressures (3.7) or (4.6) and (5.13), black holes in non-stationary de Sitter space do not satisfy the strong energy condition (2.14). It is also worth to mention that the above energy conditions are applicable for the energy-momentum tensor of non-stationary dark energy solution with equation of state parameter $w = -1/2$ proposed by Ibohal et al. [10]. The de Sitter solutions, stationary or non-stationary, are conformally flat. That the variable $\Lambda(u)$ does not disturb the property of Petrov type $D$ of Schwarzschild and Vaidya space-times in (4.19) and (5.25). In view of the energy-momentum tensor (3.6), the non-stationary de Sitter solution with variable $\Lambda(u)$ is stronger version than the stationary de Sitter with constant $\Lambda$, since the constant $\Lambda$ cannot produce the dynamical evolution term $T_{ab}^{(\mathrm{NS})}$.

It is expected that all known results of Schwarzschild-de Sitter solution as well as Vaidya-de Sitter solution with constant $\Lambda$, done by other authors, may be extended with the variable $\Lambda(u)$ of the non-stationary embedded black holes using the results in Section 4 and 5.

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