J. T. S.
Vol. 11 (2017), pp.49-55
https://doi.org/10.56424/jts.v11i01.10585

Bayesian Estimation of the Shape Parameters of Pareto Distribution using Linex Loss Function with type II Censoring

S. P. Singh

Department of Applied Sciences and Humanities
KIPM College of Engineering and Technology, GIDA, Gorakhpur, India
E-mail : satyap211@gmail.com
(Received: July 12, 2017)

Abstract

In this paper Bayes estimator of the shape parameter of the Pareto distribution have been obtained by taking quasi, inverted gamma and uniform prior distributions using the linex loss function. These estimators for compared with the corresponding Bayes estimators under squared error loss function.

Keywords and phrases : Squared error loss function, asymmetric loss function, prior distribution, diffuse and non-informative prior, posterior pdf and expectations, inverted gamma distribution.

1. Introduction

Let us consider the Pareto distribution whose probability density function is given by

$$f(x;\sigma,\,\theta)\;=\;\frac{1}{\theta\,\sigma}\left(\frac{x}{\sigma}\right)^{-\left(\frac{1}{\theta}+1\right)};\;\theta>0,\quad x\geq\sigma \tag{1.1}$$

where σ and θ are scale and shape parameters, respectively. Let us suppose that n items having the life time distribution with pdf as (1.1) are put to life test experiment, without replacement, and the experiment is terminated as soon as $r(\leq n)$ items have failed. If $\underline{X}=(X_1,....,X_r)$ denote the random vector of the 'r' observations (life times) as obtained above. Then the joint probability density function of $\underline{X}$ is given by

$$f(\underline{x}|\theta)=\frac{n!}{(n-r)!}\left(\frac{1}{\sigma\,\theta}\right)^{r}e^{-\left(\frac{1}{\theta}+1\right)T_r} \tag{1.2}$$

where $T_r = \left[\sum_{i=1}^{r} \log\left(\frac{x_i}{\sigma}\right) + (n-r)\log\left(\frac{x_{(r)}}{\sigma}\right)\right]$. Thus, the maximum likelihood estimator (MLE) θ is given by

$$\hat{\theta} = \frac{T_r}{r}. \tag{1.3}$$

and the pdf of $\hat{\theta}$ is given by

$$f(\hat{\theta}) = \frac{\left(\frac{r}{\theta}\right)^r}{\Gamma(r)} \left(\hat{\theta}\right)^{r-1} e^{-r\,\hat{\theta}/\theta}, \qquad \hat{\theta} > 0. \tag{1.4}$$

The Bayes estimator $\hat{\theta}_L$ of θ is of course, is the optimal estimator relative to the chosen loss function L. A commonly used loss function is the squared error loss function (SELF), is symmetric and is given as

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2. \tag{1.5}$$

It is well known that the Bayes estimator under the above loss function, say $\hat{\theta}_S$, is the posterior mean. The squared error loss function (SELF) is often used due to the fact that it is symmetrical and also that it does not lead to complicated numerical methods for various calculations. Several authors, viz. Ferguson [3], Varian [5], Berger [2], Zellner [6] and Basu and Ebrahimi [1], to name a few, have recognized the inappropriateness of using symmetric loss function in several estimation problems. These authors have proposed different asymmetric loss functions e.g., Linex and many of it's variant forms.

Varian [5] introduced the following convex loss function known as LINEX (linear-exponential) loss function

$$L(\Delta) = be^{a\Delta} - c\Delta - b, \qquad a, c \neq 0,\ b > 0, \tag{1.6}$$

where $\Delta = \hat{\theta} - \theta$. It is clear that $L(0) = 0$ and the minimum occurs when $ab = c$, therefore, $L(\Delta)$ can be written as

$$L(\Delta) = b\left[\mathrm{e}^{\mathrm{a}\Delta} - a\Delta - 1\right], \qquad a \neq 0,\ b > 0, \tag{1.7}$$

where a and b are the parameters of the loss function may be defined as shape and scale respectively. This loss function has been considered by Zellner [6], Rojo [4]. Basu and Ebrahimi [1] considered the $L(\Delta)$ as

$$L(\Delta) = b\left[e^{a\Delta} - a\Delta - 1\right], \qquad a \neq 0,\ b > 0, \tag{1.8}$$

where

$$\Delta = \frac{\hat{\theta}}{\theta} - 1$$

and studied the Bayesian estimation under this asymmetric loss function for exponential lifetime distribution. This loss function is suitable for the situations where overestimation of θ is more costly than its underestimation.

Thus Bayes estimator under asymmetric loss $L(\Delta)$, i.e., $\hat{\theta}_A$ is the solution of the following equation

$$E_\pi\left[\frac{1}{\theta}\exp\left(\frac{a\hat{\theta}_A}{\theta}\right)\right] = e^a E_\pi\left(\frac{1}{\theta}\right). \tag{1.9}$$

In this paper, we have obtained Bayes estimator of θ using linex loss function, under three prior distribution, viz., quasi-density

$$g_1(\theta) = \frac{1}{\theta^d}; \qquad \theta > 0,\, d \geq 0 \tag{1.10}$$

here $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior; the inverted gamma distribution as natural conjugate with parameters α and $\beta(> 0)$ with p.d.f. given by

$$g_2(\theta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)}\,\theta^{-(\alpha+1)} \quad e^{-\beta/\theta} & ; \quad \theta > 0 \quad (\alpha,\,\beta) > 0 \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{1.11}$$

and uniform prior over $[\alpha, \beta]$ as

$$g_3(\theta) = \begin{cases} \frac{1}{\beta-\alpha} & ; \quad 0 < \alpha \leq \theta \leq \beta \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{1.12}$$

2. Bayes Estimator of θ under Quasi Prior $g_1(\theta)$

The posterior pdf of θ under $g_1(\theta)$, may be obtained, using (1.2)

$$f(\theta|\underline{x}) = \frac{T_r^{r+d-1}}{\Gamma(r+d-1)}\,\theta^{-(r+d)}\,e^{-(T_r/\theta)}; \quad \theta > 0,\; r+d > 1. \tag{2.1}$$

The Bayes estimator under squared error loss function is given by

$$\hat{\theta}_S = \frac{T_r}{(r+d-2)}; \quad r+d > 2. \tag{2.2}$$

Also, the Bayes estimator under linex loss function is obtained as

$$\hat{\theta}_A = \left(\frac{1-e^{-a/(r+d)}}{a}\right) T_r. \tag{2.3}$$

The Risk Functions

The risk function of the estimators $\hat{\theta}_S$ and $\hat{\theta}_A$, relative to squared error loss function are denoted by $R_S(\hat{\theta}_S)$ and $R_S(\hat{\theta}_A)$, respectively and those relative to linex $R_A(\hat{\theta}_S)$ and $R_A(\hat{\theta}_A)$, respectively are given by

$$R_S\left(\hat{\theta}_S\right) = \theta^2 \left[\frac{r(r+1)}{(r+d-1)^2} - \frac{2r}{(r+d-2)} + 1\right], \tag{2.4}$$

$$R_S\left(\hat{\theta}_A\right) = \theta^2 \left[\frac{r(r+1)}{a^2}\left(1-e^{-a/(r+d)}\right)^2 - \frac{2r}{a}\left(1-e^{-a/(r+d)}\right) + 1\right], \tag{2.5}$$

$$R_A\left(\hat{\theta}_S\right) = b\left[e^{-a}\left(1-\frac{a}{r+d-2}\right)^{-r} - \left(\frac{a\,r}{r+d-2}\right) + a - 1\right], \tag{2.6}$$

$$R_A\left(\hat{\theta}_A\right) = b\left[e^{-ad/(r+d)} - r\left(1-e^{-a/(r+d)}\right) + a - 1\right]. \tag{2.7}$$

3. Bayes Estimator of θ under Natural Conjugate Prior $g_2(\theta)$

The posterior pdf of θ under $g_2(\theta)$, using equation (1.2) comes out to be

$$f(\theta|\underline{x}) = \frac{(\beta+T_r)^{r+\alpha}}{\Gamma(r+\alpha)}\ \theta^{-(r+\alpha+1)}\ e^{-\frac{1}{\theta}(\beta+T_r)}, \tag{3.1}$$

Using equation (3.1), the Bayes estimator under SELF is given by

$$\hat{\theta}_S = \frac{\beta+T_r}{(r+\alpha-1)}. \tag{3.2}$$

The Bayes estimator under linex loss function $L(\Delta)$, using the value of $f(\theta|\underline{x})$ from equation (3.1) is the solution of equation (1.9) given by

$$\hat{\theta}_A = \left(\frac{1-e^{-a/(r+\alpha+1)}}{a}\right)(\beta+T_r). \tag{3.3}$$

The Risk Functions

The risk function of the estimators $\hat{\theta}_S$ and $\hat{\theta}_A$, relative to squared error loss function are given by

$$R_S\left(\hat{\theta}_S\right) = \theta^2\left[\left(\frac{r(r+1)+\frac{2r\beta}{\theta}+\frac{\beta^2}{\theta^2}}{(r+\alpha-1)^2}\right) - \frac{2\left(r+\frac{\beta}{\theta}\right)}{(r+\alpha-1)} + 1\right], \tag{3.4}$$

and

$$R_S\left(\hat{\theta}_A\right) = \theta^2\left[C^2\left(r(r+1)+\frac{2r\beta}{\theta}+\frac{\beta^2}{\theta^2}\right) - 2\,C\left(r+\frac{\beta}{\theta}\right) + 1\right], \tag{3.5}$$

where $C=\left(\frac{1-e^{-a/(r+\alpha+1)}}{a}\right)$.

The risk function of the estimator $\hat{\theta}_S$ and $\hat{\theta}_A$, relative to linex loss are given by

$$R_A\left(\hat{\theta}_S\right)=b\left[\left(e^{-a\left(1-\frac{\beta}{\theta(r+\alpha-1)}\right)}\right)\left(1-\frac{a}{r+\alpha-1}\right)^{-r}-\left(\frac{a\left(r+\frac{\beta}{\theta}\right)}{r+\alpha-1}\right)+a-1\right], \tag{3.6}$$

and

$$R_A\left(\hat{\theta}_A\right)=b\left[\left(e^{-a(\alpha+1)/(r+\alpha+1)}\right)\left(e^{\frac{\beta}{\theta}\left(1-e^{-a/(r+\alpha+1)}\right)}\right)-\left(1-e^{-a/(r+\alpha+1)}\right)\left(r+\frac{\beta}{\theta}\right)+a-1\right], \tag{3.7}$$

4. Bayes Estimator of θ under Uniform Prior $g_3(\theta)$

The posterior pdf under $g_3(\theta)$ may be obtained as

$$f(\theta\mid \underline{x})=\frac{T_r^{r-1}\,\theta^{-r}\,e^{-T_r/\theta}}{I_g\left(\frac{T_r}{\alpha},\,r-1\right)\;-\;I_g\left(\frac{T_r}{\beta},\,r-1\right)}, \tag{4.1}$$

where $I_g\left(x,n\right)=\int\limits_0^x e^{-t}\,t^{n-1}\,dt$ is the incomplete gamma function.

The Bayes estimator of θ under SELF is given by

$$\hat{\theta}_S=\left(\frac{I_g\left(\frac{T_r}{\alpha},\,r-2\right)-I_g\left(\frac{T_r}{\beta},\,r-2\right)}{I_g\left(\frac{T_r}{\alpha},\,r-1\right)-I_g\left(\frac{T_r}{\beta},\,r-1\right)}\right)\,T_r. \tag{4.2}$$

Using (1.6), the Bayes estimator of θ relative to linex loss function $\hat{\theta}_A$, where $\hat{\theta}_A$ is the solution of following equation

$$e^a\frac{I_g\left(\frac{T_r}{\alpha},r\right)-I_g\left(\frac{T_r}{\beta},\,r\right)}{I_g\left(\frac{T_r-a\hat{\theta}_A}{\alpha},\,r\right)-I_g\left(\frac{T_r-a\hat{\theta}_A}{\beta},\,r\right)}=\left(\frac{T_r}{T_r-a\hat{\theta}_A}\right)^r. \tag{4.3}$$

In this case risk functions cannot be obtained in a closed form.

5. Conclusion

It is evident from the equations (2.2), (2.3), (3.2), (3.3), (4.2) and (4.3) that Bayes estimators of the shape parameter of the Pareto distribution, under squared error, linex loss functions using quasi, inverted gamma and uniform

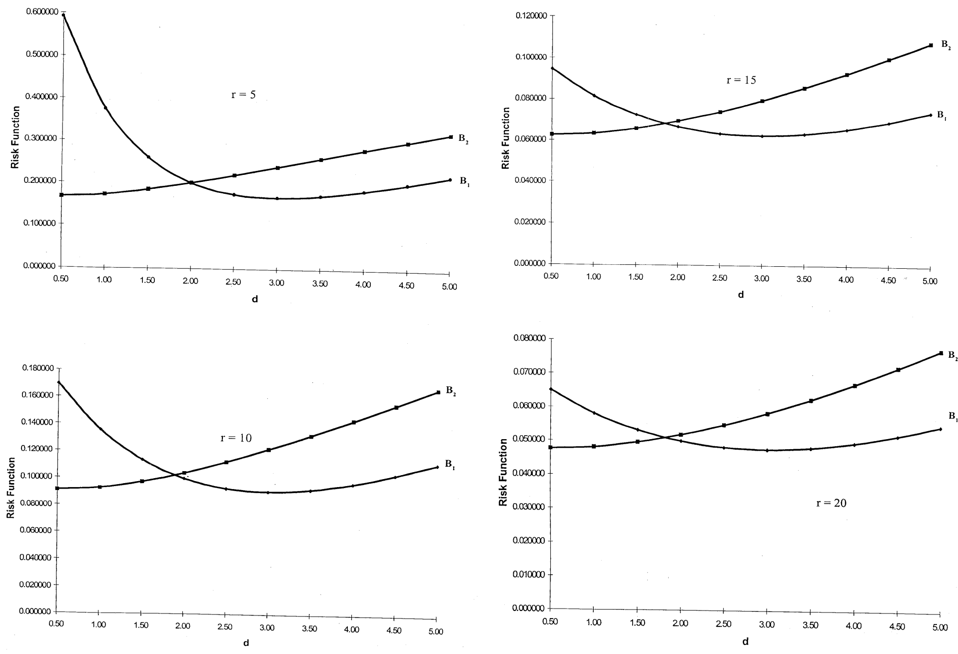

Fig. 1. Comparative Graph of Risk Function under SELF

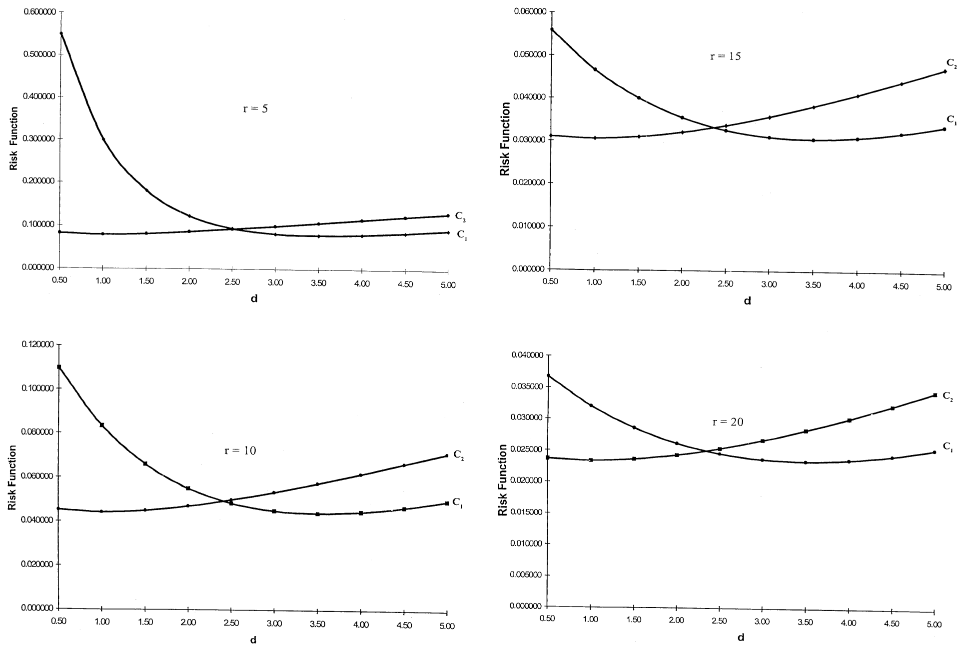

Fig. 2. Comparative Graph of Risk Function under Linex Loss Function

priors, have different expressions for their definitions. The Bayes estimators do depend upon the parameters of the prior distributions.

In figure-1 we have plotted the risk functions B_1 and B_2 of the Bayes estimators $\hat{\theta}_S$ and $\hat{\theta}_A$, respectively, under squared error loss function, as given in equation (2.4) and (2.5) for $a = 1$, $r = 5(5)20$ and $d = 0.5(0.5)5.0$.

In figure-2 we have plotted the risk functions C_1 and C_2 of the Bayes estimators $\hat{\theta}_S$ and $\hat{\theta}_A$, respectively, under linex loss function, as given in equation (2.6) and (2.7) for $a = 1$, $r = 5(5)20$ and $d = 0.5(0.5)5.0$.

From figure-1 and 2 it is clear that neither of the estimators uniformly dominates the other.

References

[1] Basu, A. and Ebrahimi, N. : Bayesian approach to life testing and reliability estimation using asymmetric loss function, J. Statist. Plan. Inf., 29 (1991), 21-31.

[2] Berger, James, O. : Statistical Decision Theory, Foundation, Concepts and Methods, Springer-Verlag, New York (1980).

[3] Ferguson, T. S. : Mathematical Statistics: A decision theoretic approach, Academic Press, New York, (1967).

[4] Rojo, J. : On admissibility of $cx + d$ with respect to the linex loss function, Common. Statist. Theory Methods, 16 (1987), 3745-3748.

[5] Varian, H. R. : A Bayesian approach to real state assessment in studies in Bayesian Econometrics and Statistics in honour of L. J. Savage, Eds. S. E. Feinberg and A. Zellner, Amsterdam, North Holland, (1975), 195-208.

[6] Zellner, A. : Bayes estimation and prediction using asymmetric loss functions, Jour. Amer. Statist. Assoc., 81 (1986), 446-451.