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Differential Geometry – The Key role to Physical Laws

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In every physical system, there is a background space on which it evolves. So there is a natural question "does a dynamical system is free to evolve in a given space or it is dictated by the symmetric structure of the space?" In the lecture, this question has been tried to be answered.

In every physical evolution there are two basic approaches to study the properties of the equations of motion, namely, (i) symmetry analysis of the evolution equations and (ii) the collineations (symmetries) of the background space in which it evolves. However, both the approaches have some similarities : (a) both the symmetries form a Lie algebra, (b) both the approaches can not determine the physical system or the space uniquely. So mathematically, it is reasonable to examine how the two algebras are related.

This issue has been addressed 100 years ago by Albert Einstein in course of formulating the general theory of relativity through the notion of equivalence principle – a free falling particle in a given gravitational field moves along the geodesic of the space. In the following, we shall address this issue more precisely from the differential geometric point of view.

The affinely parametrized geodesics in a Riemannian space are a set of homogeneous second order quasilinear ordinary differential equations (ODE) which are uniquely determined by the metric of the space. On the other hand, such a set of ODE can be (partially) characterized by its Lie point symmetry (LPS). Also a metric is characterized (not fully) by its collineations. So it is expected that the LPS of geodesics has some close association with the collineations of the metric.

A LPS of an ODE is a point transformation in the space of variables which preserves the solution set of the ODE. However, the LPS of geodesics in a Riemannian space can be considered as automorphisms which preserves the set of the solution curves. On the other hand from differential geometric point of view the point transformations preserving the geodesic equations are known as projective transformations. So it is expected that the projective algebra of the metric is related to the LPS of the geodesics.

The LPS of a system of ODE s is determined in two steps : at first the conditions which the components of the symmetry vector must satisfy and secondly one has to solve these conditions. But if one expresses the symmetry conditions in terms of collineations of the metric then LPS can be associated with the special projective group of the metric. Similarly the Noether point symmetry is associated with a sub-algebra of the special projection algebra of the space – the homothetic algebra. Also the conserved current related to Noether point symmetry also depends on the geometry of the space in which motion takes place. Therefore, it is reasonable to conclude that there is profound interrelation between geometry of space and the dynamics on it, *i.e.* the space is not a simple carrier of the motion but it is the major modulator of the evolution of the system. I shall conclude my talk by giving a brief overview about collineations of Riemannian space.

A collineation of a n-D Riemannian space is a vector field $\overrightarrow{X}$ which satisfies : $\mathcal{L}_{\overrightarrow{X}} A = B$. Here A is some index system of geometric object (not necessarily a tensor) and B is a tensor field having same indices as A. So for collineation of the metric we have $A = g_{ij}$. The vector field $\overrightarrow{X}$ is said to be Conformal Killing vector (CKV) if $\mathcal{L}_{\overrightarrow{X}} g_{ij} = 2\psi(x) g_{ij}$ with $\psi(x) = \frac{1}{n} X^i_{,i}$.

On the other hand, if $\psi_{,ij} = 0$, then the vector field $\overrightarrow{X}$ is called a special CKV (sp. CKV) while the vector field $\overrightarrow{X}$ is called a homothetic vector field (HV) if $\psi(x^k) =$ non-zero constant and it will be a Killing vector field (HV) if $\psi(x^k) = 0$.

Further, the set of all CKV s of the metric g_{ij} form a Lie algebra, termed as conformal algebra (CA) of the metric of the space. Similarly, the collection of all HV s and KV s form algebras known as homothetic algebra (HA) and Killing algebra (KA) respectively. It should be noted that both these algebras are sub-algebras of the conformal algebras *i.e.* KA $\subseteq$ HA $\subseteq$ CA . In a n-dimensional Riemannian space the maximum dimension of the conformal algebra is $\frac{1}{2}(n+1)(n+2)$ (with $n > 2$) and if it is attained then the space is called conformally flat. Similarly the maximum dimension of the Killing algebra is $\frac{1}{2}n(n+1)$. If a conformally flat space admits the maximum possible Killing vectors then the space is called a space of constant curvature.

Moreover, the vector field $\overrightarrow{X}$ is called affine collineations (AC) if it satisfies $\mathcal{L}_{\overrightarrow{X}}\Gamma^{\alpha}_{\beta\gamma} = 0$. An affine collineation carries a geodesic into a geodesic and preserves the affine parameter along each geodesic. Also the collection of all AC form a Lie algebra termed as affine algebra. In general for any vector field $\overrightarrow{\xi}$,

$$\mathcal{L}_{\overrightarrow{\xi}}\Gamma^{\alpha}_{\beta\gamma} = g^{\alpha\delta}\left[\left(\mathcal{L}_{\overrightarrow{\xi}}g_{\beta\delta}\right)_{;\gamma} + \left(\mathcal{L}_{\overrightarrow{\xi}}g_{\delta\gamma}\right)_{;\beta} - \left(\mathcal{L}_{\overrightarrow{\xi}}g_{\beta\gamma}\right)_{;\delta}\right]$$

Thus if $\overrightarrow{\xi}$ is a HV or a KV then $\mathcal{L}_{\overrightarrow{X}}\Gamma^{\alpha}_{\beta\gamma} = 0$ *i.e.* ξ is also an AC. Also homothetic algebra is a sub-algebra of the affine algebra. A n-dimensional space can have the maximal $n(n+1)$-dimensional affine algebra. If a space has maximal affine algebra then the space is flat. The collineation can be summarized in the following table :

Table : Collineations of a Riemannian space

Collineation $\mathcal{L}_{\overrightarrow{\xi}}A = B$	A	B
Killing vector	g_{ij}	0
Homothetic vector	g_{ij}	$2\psi g_{ij}$, $\psi_{,i} = 0$
Conformal Killing vector	g_{ij}	$2\psi g_{ij}$, $\psi_{,i} \neq 0$
Affine Collineation	Γ^{i}_{jk}	0

References

[1] Tramparlis, M. and Paliathanasis, A. : J. Phys. Conf. Ser., 621 (2015)012014 ; arXiv : 1501.06047.

[2] Paliathanasis, A. and Tramparlis, M. : J. Geom. Phys., 107 (2016) 45 ; arXiv : 1605.03447.

[3] Stephani, H. : Differential Equations : Their Solutions using symmetry, Camb. Univ. Press, Cam-bridge, England, (1989).

[4] Blumanands, G. and Kumei, S. : Symmetries and Differential Equations, Springer-Verlag, N. Y., (1989).