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The L-dual of a Generalized m-Kropina Space

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Abstract

In 1987, R. Miron introduced the concept of L-duality between Cartan spaces and Finsler spaces ([5]) . The geometry of higher order Finsler spaces were sudied in ([1] , [8]) . The theory of higher order Lagrange and Hamilton spaces were discussed in ([6] , [7] , [9]) . Some special problems concerning the L-duality and classes of Finsler spaces were studied in ([3] , [13]) . In ([2] , [10] , [11]) the L-duals of Randers, Kropina and Matsumoto space were introduced. The L-dual of an (α, β) Finsler space was introduced in [12] .In this paper we give the L-dual of a generalized m-Kropina Space.

Keywords and Phrases : Generalized m-Kropina space, Finsler space, Cartan space, the duality between Finsler and Cartan spaces.
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1. Introduction

Let $F^n = (M, F)$ be an n-dimensional Finsler space. The fundamental function $F(x, y)$ is called an (α, β)-metric if F is homogeneous function of α and β of degree one, where $\alpha^2 = a(y, y) = a_{ij}y^i y^j, y = y^i \frac{\partial}{\partial x^i} |_x \in T_x M$ is Riemannian metric, and $\beta = b_i(x) y^i$ is a one-form on $\widetilde{T}M = TM \setminus \{0\}$.

A Finsler space with fundamental function [4]:

$$F(x, y) = \alpha(x, y) + \beta(x, y) \tag{1.1}$$

is called a Randers space.

A Finsler space having the fundamental function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\beta(x, y)} \tag{1.2}$$

is called a Kropina space.

A Finsler space with fundamental function:

$$F(x,y) = \frac{\alpha^2(x,y)}{\alpha(x,y) - \beta(x,y)} \tag{1.3}$$

is called a Matsumoto space.

The generalized (α, β)-metric:

$$F = \frac{\alpha^{m+1}(x,y)}{\beta^m(x,y)}, \qquad (m \neq 0, -1) \tag{1.4}$$

is called a generalized m-Kropina metric and the Finsler space equipped with this metric is called a generalized m-Kropina space.

Definition 1. A Cartan space C^n is a pair (M, H) which consists of a real n-dimensional $C^\infty-$manifold M and a Hamiltonian function $H : T^x M \diagdown \{0\} \longrightarrow R$, where $(T^x M, \pi^x, M)$ is the cotangent bundle of M such that $H(x,p)$ has the properties as follows [2]:

1. It is two homogeneous with respect to $p_i \, (i, j, k, ..., = 1, ..., n)$.
2. The tensor field $g^{ij}(x,p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}$ is nondegenerate.

Let $C^n = (M, K)$ be an n-dimensional Cartan space having the fundamental function $K(x,p)$. We also consider Cartan spaces having the metric functions of the following forms:

$$K(x,p) = \sqrt{a^{ij}(x) p_i p_j} + b^i(x) p_i \tag{1.5}$$

or

$$K(x,p) = \frac{a^{ij}(x) p_i p_j}{b^i(x) p_i} \tag{1.6}$$

with $a_{ij} a^{jk} = \delta_i^k$ and we will again call these spaces Randers and, respectively, Kropina spaces on the cotangent bundle $T^* M$.

Definition 2. A regular Lagrangian (Hamiltonian) on a domain $D \subset TM$ $(D^* \subset T^* M)$ is a real smooth function $L : D \longmapsto \mathbb{R}$ $(H : D^* \longmapsto \mathbb{R})$ such that the matrix with entries

$g_{ab}(x,y) := \dot{\partial}_a \dot{\partial}_b L(x,y)$ $\quad \left(g^{*ab}(x,p) := \partial^{\cdot a} \partial^{\cdot b} H(x,p)\right)$ is everywhere nondegenerate on D(D*)[2].

A Lagrange (Hamilton) manifold is a pair (M, L) ((M, H)), where M is a smooth manifold and L(H) is regular Lagrangian(Hamiltonian) on D(D*).

Example 3. (1.) Every Finsler space $F^n = (M, F(x, y))$ is a Lagrange manifold with $L = \frac{1}{2}F^2$. Every Cartan space $C^n = (M, \overline{F}(x, p))$ is a Hamilton manifold with $H = \frac{1}{2}\overline{F}^2$. (Here $\overline{F}$ is positively 1-homogeneous in p_i and the tensor $\overline{g}^{ab} = \frac{1}{2}\partial^{\cdot a}\partial^{\cdot b}\overline{F}^2$ is nondegenerate).

(2.) (M, L) and (M, H) with

$L\,(x, y) = \frac{1}{2}a_{ij}\,(x)\,y^i y^j + b_i\,(x)\,y^i + c(x)$ and $H\,(x, p) = \frac{1}{2}\overline{a}^{ij}\,(x)\,p_i p_j + \overline{b}^i\,(x)\,p_i + \overline{c}(x)$

are Lagrange and Hamilton manifolds respectively. (Here $a_{ij}\,(x)\,, \overline{a}^{ij}\,(x)$ are the fundamental tensors of Riemannian manifold, b$_i$ are components of covector field, $\overline{b}^i$ are the components of a vector field, c and $\overline{c}$ are smooth functions on M).

Let $L\,(x, y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H\,(x, p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$.As we know [10] if L is a differentiable map, we can consider the fiber derivative of L, locally given by diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$:

$$\varphi\,(x, y) = \left(x^i, \dot{\partial}_\alpha L\,(x, y)\right) \tag{1.7}$$

which is called the Legendre transformation. We can define, in this case, the function $H : U^* \longmapsto \mathbb{R}$:

$$H\,(x, p) = p_a y^a - L\,(x, y)\,, \tag{1.8}$$

where $y = (y^\alpha)$ is the solution of the equations:

$$p_a = \dot{\partial}_\alpha L\,(x, y)\,. \tag{1.9}$$

In the same manner, the fiber derivative is given locally by:

$$\varphi\,(x, p) = \left(x^i, \dot{\partial}^a H\,(x, p)\right), \tag{1.10}$$

φ is a diffeomorphism between same open sets $U^* \subset D^*$ and $U \subset D$ and we can consider the function $L : U \longmapsto R$:

$$L\,(x, y) = p_a y^a - H\,(x, p)\,, \tag{1.11}$$

where $p = (p_a)$ is the solution of the equations:

$$y^a = \overset{.\,a}{\partial} H(x,p). \tag{1.12}$$

The Hamiltonian given by (1.8) is called the Legendre transformation of the Lagrangian L and the Lagrangian given by (1.11) is called the Legendre transformation of the Hamiltonian H.

If (M,K) is a Cartan space, then (M,H) is a Hamiltonian Manifold [10], where $H(x,p) = \frac{1}{2}K^2(x,p)$ is 2-homogeneous on a domain of T^*M. So, we get the following transformation of H on U:

$$L(x,y) = p_a y^a - H(x,p) = H(x,p) \tag{1.13}$$

1.1 Theorem [10]

The scalar field given by (1.13) is a positively 2-homogeneous regular Lagrangian on U.

Therefore, we get Finsler metric F of U, so that

$$L(x,y) = \frac{1}{2}F^2 \tag{1.14}$$

Thus, for the Cartan space (M,K) we always can locally associate a Finsler space (M,F) which will be called the L-dual of a Cartan space $\left(M, C_{|U^*}\right)$ vice-versa, we can associate, locally, a Cartan space to every Finsler space which will be called the L-dual of a Finsler space $\left(M, C_{|U}\right)$.

2. The (α,β) Finsler-(α^*,β^*) Cartan L-duality

2.1 Theorem ([2], [10])

Let (M,F) be a Randers space and $b = \left(a_{ij}b^ib^j\right)^{\frac{1}{2}}$ the Riemannian length of b_i. Then:

1. If $b^2 = 1$, the L-dual of (M,F) is a Kropina space on T^*M with:

$$H(x,p) = \frac{1}{2}\left(\frac{a^{ij}p_ip_j}{2b^ip_i}\right)^2 \tag{2.1}$$

2. If $b^2 \neq 1$, the L-dual of (M,F) is a Randers space on T^*M with:

$$H(x,p) = \frac{1}{2}\left(\sqrt{\overline{a}^{ij}p_ip_j} \pm \overline{b}^i p_i\right)^2, \tag{2.2}$$

where

$$\overline{a}^{ij} = \frac{1}{1-b^2}a^{ij} + \frac{1}{(1-b^2)^2}b^ib^j; \qquad \overline{b}^i = \frac{1}{1-b^2}b^i$$

(in (2.2) '$-$' corresponds to $b^2 \prec 1$ and '+' corresponds to $b^2 \succ 1$).

2.2 Theorem ([2], [10])

The L-dual of a Kropoina space is a Randera space on T^*M with the Hamiltonian:

$$H(x,p) = \frac{1}{2}\left(\sqrt{\overline{a}^{ij}p_ip_j} \pm \overline{b}^i p_i\right)^2, \tag{2.3}$$

where

$$\overline{a}^{ij} = \frac{b^2}{4}a^{ij}; \qquad \overline{b}^i = \frac{1}{2}b^i$$

in (2.3) '$-$' corresponds to $\beta \prec 0$ and '+' corresponds to $\beta \succ 0$).

In [2] $\alpha^* = \left(a^{ij}(x)\, p_ip_j\right)^{\frac{1}{2}}, \beta^* = b^i(x)\, p_i$, where $a^{ij}(x)$ are the reciprocal components of a_{ij} and $b^i(x)$ are the components of the vector field on M, $b^i(x) = a^{ij}(x)\, b_j(x)$. We can consider the metric functions $K = \alpha^* + \beta^*$ (Randers metric on T^*M) or $K = \frac{\alpha^{*2}}{\beta^*}$ (Kropina metric on T^*M) defined on a domain $D^* \subset T^*M$. So, we can easily rewrite the previous theorems:

2.3 Theorem

Let (M, F) be a Randers space and $b = \left(a_{ij}b^ib^j\right)^{\frac{1}{2}}$ the Riemannian length of b_i. Then:

1. If $b^2 = 1$, the L-dual of (M, F) is a Kropina space on T^*M with:

$$H(x,p) = \frac{1}{2}\left(\frac{\alpha^{*2}}{2\beta^*}\right)^2 \tag{2.4}$$

2. If $b^2 \neq 1$, the L-dual of (M, F) is a Randers space on T^*M with:

$$H(x,p) = \frac{1}{2}\left(\alpha^* \pm \beta^*\right)^2, \tag{2.5}$$

with $\alpha^* = \left(\overline{a}^{ij}(x)\; p_ip_j\right)^{\frac{1}{2}}, \qquad \beta^* = \overline{b}^i(x)\, p_i,$

where

$$\overline{a}^{ij} = \frac{1}{1-b^2}a^{ij} + \frac{1}{(1-b^2)^2}b^i b^j; \qquad \overline{b}^i = \frac{1}{1-b^2}b^i$$

(in (2.5) '$-$' corresponds to $b^2 \prec 1$ and '$+$' corresponds to $b^2 \succ 1$).

2.4 Theorem

The L-dual of a Kropoina space is a Randers space on T^*M with the Hamiltonian:

$$H(x,p) = \frac{1}{2}(\alpha^* \pm \beta^*)^2, \tag{2.6}$$

where

$$\overline{a}^{ij} = \frac{b^2}{4}a^{ij}; \qquad \overline{b}^i = \frac{1}{2}b^i$$

(in (2.6) '$-$' corresponds to $\beta \prec 0$ and '$+$' corresponds to $\beta \succ 0$).

3. The L-dual of a Generalized m-Kropina Space

In this case we have $F = \alpha^{m+1}(x,y).\beta^{-m}(x,y)\ (m \neq 0,-1)$. We put $\alpha^2 = a_{ij}y^i y^j = y_i y^i, \beta = b_i y^i, \beta^* = b^i p_i, p^i = a^{ij}p_j, \alpha^{*2} = p_i p^i = a^{ij}p_i p_j$.

We have

$$p_i = \frac{1}{2}\dot{\partial}_i F^2 = \frac{1}{2}\dot{\partial}_i \left(\frac{\alpha^{m+1}(x,y)}{\beta^m(x,y)}\right)^2 = F\dot{\partial}_i\left(\frac{\alpha^{m+1}}{\beta^m}\right)$$

which on simplification gives

$$p_i = F\left[\frac{(m+1)F}{\alpha^2}y_i - \frac{mF}{\beta}b_i\right] \tag{3.1}$$

Contracting (3.1) by p^i and b^i respectively, we get

$$\alpha^{*2} = F\left[\frac{(m+1)F^3}{\alpha^2} - \frac{mF}{\beta}\beta^*\right] \tag{3.2}$$

and

$$\beta^* = F\left[\frac{(m+1)F}{\alpha^2}\beta - \frac{mF}{\beta}b^2\right]. \tag{3.3}$$

In [13], for a Finsler (α,β)-metric F on a manifold M, there is a positive function $\phi = \phi(s)$ on $(-b_0; b_0)$ with $\phi(0) = 1$ and $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, where $\alpha^2 = a_{ij}y^i y^j = y_i y^i$, and $\beta = b_i y^i$ with $\|\beta\|_x \prec b_0, \forall x \epsilon M$ and ϕ satisfies $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) \succ 0, (|\, s \,| \leq b_0)$.

A generalized m-Kropina metric is a special (α, β)-metric with $\phi = \frac{1}{s^m}$. Using Shen's [14] notation $s = \frac{\beta}{\alpha}$, (3.2) and (3.3) become:

$$\alpha^{*2} = F\left[\frac{(m+1)\,F}{s^{2m}} - \frac{m}{s^{m+1}}\beta^*\right] \tag{3.4}$$

and

$$\beta^* = F\left[\frac{(m+1)}{s^{m-1}} - \frac{m}{s^{m+1}}b^2\right]. \tag{3.5}$$

Putting $s^m = t$, so that $s = t^{\frac{1}{m}}$ in (3.4) and (3.5), we get

$$\alpha^{*2} = \frac{(m+1)\,F^2}{t^2} - \frac{mF}{t^{\frac{m+1}{m}}}\beta^* \tag{3.6}$$

and

$$\beta^* = \frac{(m+1)\,F}{t^{\frac{m-1}{m}}} - \frac{mF}{t^{\frac{m+1}{m}}}b^2. \tag{3.7}$$

From (3.7), we get

$$\beta^* t^2 = F\left[(m+1)\,t^{\frac{m+1}{m}} - mt^{\frac{m-1}{m}}b^2\right]. \tag{3.8}$$

For $b^2 = 1$, from (3.8), we get

$$F = \frac{\beta^* t}{(m+1)\,t^{\frac{1}{m}} - mt^{-\frac{1}{m}}}. \tag{3.9}$$

From (3.6) and (3.9), we get

$$\alpha^{*2}(m+1)^2\,t^{\frac{4}{m}} - \left\{2m\,(m+1)\,\alpha^{*2} - \left(m^2-1\right)\beta^{*2}\right\}t^{\frac{2}{m}} + m^2\left(\alpha^{*2} - \beta^{*2}\right) = 0 \tag{3.10}$$

or

$$\alpha^{*2}(m+1)^2\,s^4 - \left\{2m\,(m+1)\,\alpha^{*2} - \left(m^2-1\right)\beta^{*2}\right\}s^2 + m^2\left(\alpha^{*2} - \beta^{*2}\right) = 0. \tag{3.11}$$

Solving (3.11), we get

$$s = \sqrt{\frac{c \pm \beta^*\sqrt{d}}{e}}, \tag{3.12}$$

where

$$c = 2m\alpha^{*2} - (m-1)\,\beta^{*2},$$
$$d = 4m\alpha^{*2} - (m-1)^2\,\beta^{*2},$$
$$e = 2\,(m+1)\,\alpha^{*2}.$$

Using (3.12) in (3.9), we get

$$F=\frac{\beta^{*}\left(\frac{c\pm\beta^{*}\sqrt{d}}{e}\right)^{\frac{m+1}{2}}}{(m+1)\left(\frac{c\pm\beta^{*}\sqrt{d}}{e}\right)-m}. \tag{3.13}$$

Hence $H(x,p)=\frac{1}{2}F^{2}$ is given by

$$H(x,p)=\frac{1}{2}\frac{\beta^{*2}\left(\frac{c\pm\beta^{*}\sqrt{d}}{e}\right)^{m+1}}{\left\{(m+1)\left(\frac{c\pm\beta^{*}\sqrt{d}}{e}\right)-m\right\}^{2}}. \tag{3.14}$$

Putting $\beta^{*}=b^{i}p_{i},\alpha^{*}=\left(a^{ij}(x)\,p_{i}p_{j}\right)^{\frac{1}{2}}$ in (3.14), we get

$$H(x,p)=\frac{1}{2}\frac{\left(b^{i}p_{i}\right)^{2}\left(\frac{c\pm\left(b^{i}p_{i}\right)\sqrt{d}}{e}\right)^{m+1}}{\left\{(m+1)\left(\frac{c\pm\left(b^{i}p_{i}\right)\sqrt{d}}{e}\right)-m\right\}^{2}}. \tag{3.15}$$

Next, we find $H(x,p)$ for $b^{2}\neq 1$. From (3.8), we have

$$F=\frac{\beta^{*}t}{(m+1)\,t^{\frac{1}{m}}-mt^{-\frac{1}{m}}b^{2}}. \tag{3.16}$$

Using (3.16) in (3.6), we get

$$\alpha^{*2}(m+1)^{2}\,t^{\frac{4}{m}}-\left\{2m(m+1)\,b^{2}\alpha^{*2}-\left(m^{2}-1\right)\beta^{*2}\right\}t^{\frac{2}{m}}+m^{2}b^{2}\left(\alpha^{*2}b^{2}+\beta^{*2}\right)=0 \tag{3.17}$$

or

$$\alpha^{*2}(m+1)^{2}\,s^{4}-\left\{2m(m+1)\,b^{2}\alpha^{*2}-\left(m^{2}-1\right)\beta^{*2}\right\}s^{2}+m^{2}b^{2}\left(\alpha^{*2}b^{2}+\beta^{*2}\right)=0. \tag{3.18}$$

Solving (3.18), we get

$$s=\sqrt{\frac{f\pm\beta^{*}\sqrt{g}}{e}}, \tag{3.19}$$

where

$$f=2mb^{2}\alpha^{*2}-(m-1)\,\beta^{*2},$$
$$g=4mb^{2}\alpha^{*2}+(m-1)^{2}\,\beta^{*2},$$
$$e=2\,(m+1)\,\alpha^{*2}.$$

Using (3.19) in (3.16), we get

$$F=\frac{\beta^*\left(\frac{f\pm\beta^*\sqrt{g}}{e}\right)^{\frac{m+1}{2}}}{(m+1)\left(\frac{f\pm\beta^*\sqrt{g}}{e}\right)-mb^2}. \tag{3.20}$$

Hence $H(x,p)=\frac{1}{2}F^2$ is given by

$$H(x,p)=\frac{1}{2}\frac{\beta^{*2}\left(\frac{f\pm\beta^*\sqrt{g}}{e}\right)^{m+1}}{\left\{(m+1)\left(\frac{f\pm\beta^*\sqrt{g}}{e}\right)-mb^2\right\}^2}. \tag{3.21}$$

Putting $\beta^*=b^ip_i, \alpha^*=\left(a^{ij}(x)\,p_ip_j\right)^{\frac{1}{2}}$ in (3.21), we get

$$H(x,p)=\frac{1}{2}\frac{\left(b^ip_i\right)^2\left(\frac{f\pm\left(b^ip_i\right)\sqrt{g}}{e}\right)^{m+1}}{\left\{(m+1)\left(\frac{f\pm(b^ip_i)\sqrt{g}}{e}\right)-mb^2\right\}^2}. \tag{3.22}$$

Hence we have the following:

3.1 Theorem

Let (M,F) be a generalized m-Kropina space and $b=\left(a_{ij}b^ib^j\right)^{\frac{1}{2}}$ the Riemannian length of b_i. Then:

1. If $b^2=1$, the L-dual of (M,F) is a space on T^*M having the fundamental function:

$$H(x,p)=\frac{1}{2}\frac{\beta^{*2}\left(\frac{c\pm\beta^*\sqrt{d}}{e}\right)^{m+1}}{\left\{(m+1)\left(\frac{c\pm\beta^*\sqrt{d}}{e}\right)-m\right\}^2},$$

where

$$c=2m\alpha^{*2}-(m-1)\,\beta^{*2},$$
$$d=4m\alpha^{*2}-(m-1)^2\,\beta^{*2},$$
$$e=2\,(m+1)\,\alpha^{*2}.$$

2. If $b^2\neq 1$, the L-dual of (M,F) is a space on T^*M having the fundamental function:

$$H\left(x,p\right)=\frac{1}{2}\frac{\beta^{*2}\left(\frac{f\pm\beta^{*}\sqrt{g}}{e}\right)^{m+1}}{\left\{\left(m+1\right)\left(\frac{f\pm\beta^{*}\sqrt{g}}{e}\right)-mb^{2}\right\}^{2}},$$

where

$$f=2mb^{2}\alpha^{*2}-\left(m-1\right)\beta^{*2},$$
$$g=4mb^{2}\alpha^{*2}+\left(m-1\right)^{2}\beta^{*2},$$
$$e=2\left(m+1\right)\alpha^{*2}.$$

Using $\beta^{*}=b^{i}p_{i},\alpha^{*}=\left(a^{ij}\left(x\right)p_{ii}p_{j}\right)^{\frac{1}{2}}$, the above theorem can be rewritten as:

3.2 Theorem

Let (M,F) be a generalized m-Kropina space and $b=\left(a_{ij}b^{i}b^{j}\right)^{\frac{1}{2}}$ the Riemannian length of b_{i}. Then:

1. If $b^{2}=1$, the L-dual of (M,F) is a space on $T^{*}M$ having the fundamental function:

$$H\left(x,p\right)=\frac{1}{2}\frac{\left(b^{i}p_{i}\right)^{2}\left(\frac{c\pm\left(b^{i}p_{i}\right)\sqrt{d}}{e}\right)^{m+1}}{\left\{\left(m+1\right)\left(\frac{c\pm\left(b^{i}p_{i}\right)\sqrt{d}}{e}\right)-m\right\}^{2}},$$

where

$$c=2m\alpha^{*2}-\left(m-1\right)\beta^{*2},$$
$$d=4m\alpha^{*2}-\left(m-1\right)^{2}\beta^{*2},$$
$$e=2\left(m+1\right)\alpha^{*2}.$$

2. If $b^{2}\neq1$, the L-dual of (M,F) is a space on $T^{*}M$ having the fundamental function:

$$H\left(x,p\right)=\frac{1}{2}\frac{\left(b^{i}p_{i}\right)^{2}\left(\frac{f\pm\left(b^{i}p_{i}\right)\sqrt{g}}{e}\right)^{m+1}}{\left\{\left(m+1\right)\left(\frac{f\pm\left(b^{i}p_{i}\right)\sqrt{g}}{e}\right)-mb^{2}\right\}^{2}},$$

where

$$f=2mb^{2}\alpha^{*2}-\left(m-1\right)\beta^{*2},$$
$$g=4mb^{2}\alpha^{*2}+\left(m-1\right)^{2}\beta^{*2},$$
$$e=2\left(m+1\right)\alpha^{*2}.$$

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