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## On LP-Sasakian Manifolds satisfying Certain Curvature Tensors

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### Abstract

The object of the present paper is to study some curvature condition on LP-Sasakian manifolds which satisfy  $P\widetilde{W}_2 = 0$ ,  $\widetilde{W}_2\cdot\widetilde{W}_2 = 0$ ,  $L\widetilde{W}_2 = 0$ ,  $\widetilde{W}_2\cdot L = 0$ ,  $W_2\cdot\widetilde{W}_2 = 0$  and  $\widetilde{W}_2\cdot R = 0$ .

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### 1. Introduction

In 1989, K. Matsumoto introduced the notion of LP-Sasakian manifold. Then I. Mihai and R. Rosca [3] introduced the same notion independently and they obtained several results on this manifold. LP-Sasakian manifold have also been studied by K. Matsumoto and I. Mihai [6], B. Prasad [1], U. C. De, K. Matsumoto and A. A. Shaikh [9], M. Tafdar and A. Bhattacharyya [7], Venkatesha and C. S. Bagewadi [10], Shaikh, Prakasha and Ahmad [11], Prasad and Haseeb [12], Berman [13] and others.

### 2. Preliminaries

A differentiable manifold  $M^n$  of dimension  $n$  is called LP-Sasakian manifold Matsumoto [4, 6], if it admits a (1,1) tensor field  $\phi$ , a contravariant vector field  $\xi$ , a covariant vector field  $\eta$  and a Lorentzian metric  $g$  which satisfy

$$\eta(\xi) = -1, \quad (2.1)$$

$$\phi^2(X) = X + \eta(X)\xi, \quad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.3)$$

$$g(X, \xi) = \eta(X), \quad D_X \xi = \phi X, \quad (2.4)$$

$$(D_X\eta)(Y) = g(\phi X, Y), \quad (2.5)$$

Let us put

$$F(X, Y) = g(\phi X, Y). \quad (2.6)$$

Then the tensor field  $F$  of  $(0,2)$  type is symmetric i.e.

$$F(X, Y) = F(Y, X), \quad (2.7)$$

$$(D_X\phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \quad (2.8)$$

where  $D$  denotes the operator of covariant differentiation with respect to the Lorentzian metric  $g$ .

It can easily be seen that in an LP-Sasakian manifold the following relation hold:

$$\phi\xi = 0, \quad \eta(\phi X) = 0. \quad (2.9)$$

Also, an LP-Sasakian manifold  $M^n$  are said to be Einstein and  $\eta$ -Einstein if its Ricci tensor  $S$  is of the form

$$\begin{aligned} S(X, Y) &= a'g(X, Y) \\ \text{and } S(X, Y) &= a'g(X, Y) + b'\eta(X)\eta(Y), \end{aligned} \quad (2.10)$$

for any vector field  $X, Y$  where  $a', b'$  are function on  $M^n$ . Further on such a LP-Sasakian manifold with  $(\phi, \eta, \xi, g)$  structure the following relation holds (Matsumoto and Mihai [6] and De, Matsumoto and Shaikh [9])

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (2.11)$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.12)$$

$$R(\xi, X)\xi = X + \eta(X), \quad (2.13)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (2.14)$$

$$S(X, \xi) = (n-1)\eta(X), \quad (2.15)$$

$$Q\xi = (n-1)\xi \quad (2.16)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y). \quad (2.17)$$

The notion of pseudo- $W_2$  curvature tensor  $\widetilde{W}_2$  was given by (Prasad and Maurya [2])

$$\begin{aligned} \widetilde{W}_2(X, Y)Z &= aR(X, Y)Z + b[g(Y, Z)QX - g(X, Z)QY] - \\ &\quad \frac{r}{n} \left[ \frac{a}{n-1} + b \right] [g(Y, Z)X - g(X, Z)Y]. \end{aligned} \quad (2.18)$$

If  $a = 1$ ,  $b = -\frac{1}{n-1}$ , then (2.18) takes the form

$$\begin{aligned}\widetilde{W}_2(X, Y)Z &= R(X, Y)Z - \frac{1}{n-1}[g(Y, Z)QX - g(X, Z)QY] \\ &= W_2(X, Y)Z.\end{aligned}\quad (2.19)$$

Thus  $W_2$ -curvature is particular case of  $\widetilde{W}_2$ -curvature, where  $a$  and  $b$  are real constant and  $R, Q$  and  $r$  are the Riemannian curvature tensor of type (1,3), the Ricci operator defined by  $g(QX, Y) = S(X, Y)$  and scalar curvature respectively. Prasad et. al [15] and Kumar [14] extended this notation on LP-Sasakian manifold and LP-Sasakian manifold with coefficient  $\alpha$ .

On the other hand, the projective curvature tensor  $P$  and the concircular tensor  $L$  in a Riemannian manifold  $(M^n, g)$  defined by (Mishra [8])

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y], \quad (2.20)$$

and

$$L(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \quad (2.21)$$

Putting  $\xi$  for  $X$  in (2.18), (2.19) (2.20) and (2.21) and using (2.14),(2.15) and (2.16), we have

$$\widetilde{W}_2(\xi, Y)Z = k[g(Y, Z)\xi - \eta(Z)Y] + b[(n-1)g(Y, Z)\xi - \eta(Z)QY], \quad (2.22)$$

$$\text{where } k = a - \frac{r}{n} \left[ \frac{a}{n-1} + b \right].$$

$$W_2(\xi, Y)Z = [g(Y, Z)\xi - \eta(Z)Y] - \frac{1}{n-1}[(n-1)g(Y, Z)\xi - \eta(Z)QY], \quad (2.23)$$

$$P(\xi, Y)Z = g(Y, Z)\xi - \frac{1}{n-1}S(Y, Z)\xi, \quad (2.24)$$

$$L(\xi, Y)Z = \left[ 1 - \frac{r}{n(n-1)} \right] [g(Y, Z)\xi - \eta(Z)Y]. \quad (2.25)$$

Now, we define  $P(X, Y).\widetilde{W}_2$ ,  $L(X, Y).\widetilde{W}_2$ ,  $\widetilde{W}_2(X, Y).\widetilde{W}_2$ ,  $\widetilde{W}_2(X, Y).L$ ,  $W_2(X, Y).\widetilde{W}_2$  and  $\widetilde{W}_2(X, Y).R$  as follows:

$$\begin{aligned}(P(X, Y).\widetilde{W}_2)(U, V)W &= P(X, Y)\widetilde{W}_2(U, V)W - \widetilde{W}_2(P(X, Y)U, V)W - \\ &\quad \widetilde{W}_2(U, P(X, Y)V)W - \widetilde{W}_2(U, V)P(X, Y)W,\end{aligned}\quad (2.26)$$

$$\begin{aligned}(L(X, Y).\widetilde{W}_2)(U, V)W &= L(X, Y)\widetilde{W}_2(U, V)W - \widetilde{W}_2(L(X, Y)U, V)W - \\ &\quad \widetilde{W}_2(U, L(X, Y)V)W - \widetilde{W}_2(U, V)L(X, Y)W,\end{aligned}\quad (2.27)$$

$$\begin{aligned}
(\widetilde{W}_2(X, Y) \cdot \widetilde{W}_2)(U, V)W = & \widetilde{W}_2(X, Y) \widetilde{W}_2(U, V)W - \widetilde{W}_2(\widetilde{W}_2(X, Y)U, V)W - \\
& \widetilde{W}_2(U, \widetilde{W}_2(X, Y)V)W - \widetilde{W}_2(U, V) \widetilde{W}_2(X, Y)W,
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
(W_2(X, Y) \cdot \widetilde{W}_2)(U, V)W = & W_2(X, Y) \widetilde{W}_2(U, V)W - \widetilde{W}_2(W_2(X, Y)U, V)W - \\
& \widetilde{W}_2(U, W_2(X, Y)V)W - \widetilde{W}_2(U, V)W_2(X, Y)W,
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
(\widetilde{W}_2(X, Y) \cdot L)(U, V)W = & \widetilde{W}_2(X, Y)L(U, V)W - L(\widetilde{W}_2(X, Y)U, V)W - \\
& L(U, \widetilde{W}_2(X, Y)V)W - L(U, V) \widetilde{W}_2(X, Y)W,
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
(\widetilde{W}_2(X, Y) \cdot R)(U, V)W = & \widetilde{W}_2(X, Y)R(U, V)W - R(\widetilde{W}_2(X, Y)U, V)W - \\
& R(U, \widetilde{W}_2(X, Y)V)W - R(U, V) \widetilde{W}_2(X, Y)W.
\end{aligned} \tag{2.31}$$

### 3. LP-Sasakian manifold satisfying $P(\xi, Y) \cdot \widetilde{W}_2$

In this section we consider a LP-Sasakian manifold  $M^n$  satisfying the condition

$$P(\xi, Y) \cdot \widetilde{W}_2 = 0. \tag{3.1}$$

From (2.26), we have

$$\begin{aligned}
(P(\xi, Y) \cdot \widetilde{W}_2)(U, V)W = & P(\xi, Y) \widetilde{W}_2(U, V)W - \widetilde{W}_2(P(\xi, Y)U, V)W - \\
& \widetilde{W}_2(U, P(\xi, Y)V)W - \widetilde{W}_2(U, V)P(\xi, Y)W,
\end{aligned}$$

Using (3.1), we have

$$\begin{aligned}
& P(\xi, Y) \widetilde{W}_2(U, V)W - \widetilde{W}_2(P(\xi, Y)U, V)W - \\
& \widetilde{W}_2(U, P(\xi, Y)V)W - \widetilde{W}_2(U, V)P(\xi, Y)W = 0.
\end{aligned} \tag{3.2}$$

Taking the inner product with  $X$  and using (2.24) and (3.2), we get

$$\begin{aligned}
& [g(Y, \widetilde{W}_2(U, V)W)\eta(X) - g(Y, U)g(\widetilde{W}_2(\xi, V)W, X) - \\
& g(Y, V)g(\widetilde{W}_2(U, \xi)W, X) - g(Y, W)g(\widetilde{W}_2(U, V)\xi, X)] - \\
& \frac{1}{n-1}[S(Y, \widetilde{W}_2(U, V)W)\eta(X) - S(Y, U)g(\widetilde{W}_2(\xi, V)W, X) \\
& - S(Y, V)g(\widetilde{W}_2(U, \xi)W, X) - S(Y, W)g(\widetilde{W}_2(U, V)\xi, X)] = 0.
\end{aligned} \tag{3.3}$$

Taking  $\xi$  for  $U$  in (3.3) and using (2.4) and (2.15), we have

$$\begin{aligned} & [g(Y, \widetilde{W}_2(\xi, V)W)\eta(X) - g(Y, W)g(\widetilde{W}_2(\xi, V)\xi, X)] - \\ & \frac{1}{n-1}[S(Y, \widetilde{W}_2(\xi, V)W)\eta(X) - S(Y, W)g(\widetilde{W}_2(\xi, V)\xi, X)] = 0, \end{aligned} \quad (3.4)$$

Using (2.23) in (3.4), we have

$$\begin{aligned} & -k[\{g(Y, V)\eta(W)\eta(X) + g(Y, W)\eta(V)\eta(X) + g(Y, W)g(V, X)\} \\ & - \frac{1}{n-1}\{S(Y, V)\eta(W)\eta(X) + S(Y, W)\eta(V)\eta(X) + S(Y, W)g(V, X)\}] \\ & - b(n-1)[\{S(Y, V)\eta(W)\eta(X) + g(Y, W)\eta(V)\eta(X) + g(Y, W)S(V, X)\} \\ & - \frac{1}{n-1}\{S(QY, V)\eta(W)\eta(X) + S(Y, W)\eta(V)\eta(X) + S(Y, W)g(V, X)\}] = 0, \end{aligned} \quad (3.5)$$

where  $S(QY, Z) = S^2(Y, Z)$ .

Putting  $e_i$  for  $X$  and  $W$  in (3.5),  $\{e_i\}$  is an orthonormal basis of the tangent space at a point of the manifold and taking summation over  $e_i$ ,  $1 \leq i \leq n$ , we get

$$[r - n(n-1)][b(n-1)S(V, X) + kg(V, X) + \{k + b(n-1)\}\eta(V)\eta(X)] = 0. \quad (3.6)$$

From (3.6), we have

$$\begin{aligned} & [r - n(n-1)] = 0, \quad \text{and} \quad b(n-1)S(V, X) + kg(V, X) \\ & + \{k + b(n-1)\}\eta(V)\eta(X) = 0, \end{aligned} \quad (3.7)$$

respectively. In the case of  $r - n(n-1) \neq 0$ , if  $b = 0$ , then from (3.7), we get

$$a[r - n(n-1)] = 0.$$

But  $r - n(n-1) \neq 0$ , we get  $a = 0$ .

This is the contradiction. Thus we find  $b \neq 0$ .

In view of (3.7), we have

$$\begin{aligned} S(V, X) &= \left[ \frac{a}{b(1-n)} + \frac{r}{n} \left( \frac{a}{b(n-1)^2} \right) \right] g(V, X) + \\ & \left[ \frac{a}{b(1-n)} + \frac{r}{n} \left( \frac{a}{b(n-1)^2} \right) + 1 \right] \eta(V)\eta(X), \end{aligned}$$

which is an  $\eta$ -Einstein manifold.

Hence we have the following theorem:

**Theorem 3.1.** Let  $M^n$  be an  $n$ -dimensional ( $n > 2$ ) LP-Sasakian manifold satisfying the condition  $P(\xi, Y) \cdot \widetilde{W}_2 = 0$ . Then  $M^n$  is a part of

- (i)  $r = n(n - 1)$ , that is, scalar curvature is constant or
- (ii) an  $\eta$ -Einstein manifold, provided,  $a \neq 0$ , and  $b \neq 0$ .

#### 4. LP-Sasakian manifold satisfying $\widetilde{W}_2(\xi, Y) \cdot \widetilde{W}_2$

In this section we consider a LP-Sasakian manifold  $M^n$  satisfying the condition

$$\widetilde{W}_2(\xi, Y) \cdot \widetilde{W}_2 = 0. \quad (4.1)$$

From (2.28), we have

$$\begin{aligned} (\widetilde{W}_2(\xi, Y) \cdot \widetilde{W}_2)(U, V)W &= \widetilde{W}_2(\xi, Y) \widetilde{W}_2(U, V)W - \widetilde{W}_2(\widetilde{W}_2(\xi, Y)U, V)W - \\ &\quad \widetilde{W}_2(U, \widetilde{W}_2(\xi, Y)V)W - \widetilde{W}_2(U, V) \widetilde{W}_2(\xi, Y)W, \end{aligned}$$

Using (4.1), we have

$$\begin{aligned} \widetilde{W}_2(\xi, Y) \widetilde{W}_2(U, V)W - \widetilde{W}_2(\widetilde{W}_2(\xi, Y)U, V)W - \\ \widetilde{W}_2(U, \widetilde{W}_2(\xi, Y)V)W - \widetilde{W}_2(U, V) \widetilde{W}_2(\xi, Y)W = 0. \end{aligned} \quad (4.2)$$

Taking the inner product with  $X$  and putting  $\xi$  for  $U$  in (4.2), we have

$$\begin{aligned} g(\widetilde{W}_2(\xi, Y) \widetilde{W}_2(\xi, V)W, X) - g(\widetilde{W}_2(\widetilde{W}_2(\xi, Y)\xi, V)W, X) - \\ g(\widetilde{W}_2(\xi, \widetilde{W}_2(\xi, Y)V)W, X) - g(\widetilde{W}_2(\xi, V) \widetilde{W}_2(\xi, Y)W, X) = 0. \end{aligned} \quad (4.3)$$

Putting  $e_i$  for  $X$  in (4.3),  $\{e_i\}$  is an orthonormal basis of the tangent space at a point of the manifold and taking summation over  $e_i$ ,  $1 \leq i \leq n$ , we get

$$\begin{aligned} g(\widetilde{W}_2(\xi, Y) \widetilde{W}_2(\xi, e_i)W, e_i) - g(\widetilde{W}_2(\widetilde{W}_2(\xi, Y)\xi, e_i)W, e_i) - \\ g(\widetilde{W}_2(\xi, \widetilde{W}_2(\xi, Y)e_i)W, e_i) - g(\widetilde{W}_2(\xi, e_i) \widetilde{W}_2(\xi, Y)W, e_i) = 0. \end{aligned} \quad (4.4)$$

In view of (2.18), (2.22) and (4.4), we have

$$\begin{aligned} (a - b) \left[ bS(QY, W) - \left( k - \frac{rb}{n} \right) S(Y, W) \right] \\ + (n - 1) [k \{a + b(n - 2)\} - b^2 \{r - (n - 1)\}] g(Y, W) = 0. \end{aligned} \quad (4.5)$$

If  $b = 0$  and  $k = a \left[ 1 - \frac{r}{n(n-1)} \right]$ , then view of (4.5), we have

$$a \left[ \frac{r}{n(n-1)} - 1 \right] [S(Y, W) - (n - 1)g(Y, W)] = 0.$$

If  $a \neq 0$ , then

$$S(Y, W) = (n-1)g(Y, W), \quad \text{or} \quad r = n(n-1).$$

Hence we have the following theorem:

**Theorem 4.1.** Let  $M^n$  be an  $n$ -dimensional ( $n > 2$ ) LP-Sasakian manifold satisfying the condition  $\widetilde{W}_2(\xi, Y) \cdot \widetilde{W}_2 = 0$ . Then we get

- (i) if  $b = 0$ , then  $M^n$  is an Einstein manifold,
- (ii) if  $b \neq 0$ , then we get

$$\begin{aligned} & \left[ bS(QY, W) - \left( \frac{k}{b} - \frac{r}{n} \right) S(Y, W) \right] - \frac{n-1}{(a-b)b} \\ & [k \{a + b(n-2)\} - b^2 \{r - (n-1)\}] g(Y, W) = 0. \end{aligned}$$

## 5. LP-Sasakian manifold satisfying $L(\xi, Y) \cdot \widetilde{W}_2$

In this section we consider a LP-Sasakian manifold  $M^n$  satisfying the condition

$$L(\xi, Y) \cdot \widetilde{W}_2 = 0. \quad (5.1)$$

From (2.27), we have

$$\begin{aligned} (L(\xi, Y) \cdot \widetilde{W}_2)(U, V)W &= L(\xi, Y) \widetilde{W}_2(U, V)W - \widetilde{W}_2(L(\xi, Y)U, V)W - \\ & \quad \widetilde{W}_2(U, L(\xi, Y)V)W - \widetilde{W}_2(U, V)L(\xi, Y)W. \end{aligned}$$

Using (5.1), we have

$$\begin{aligned} & L(\xi, Y) \widetilde{W}_2(U, V)W - \widetilde{W}_2(L(\xi, Y)U, V)W - \\ & \quad \widetilde{W}_2(U, L(\xi, Y)V)W - \widetilde{W}_2(U, V)L(\xi, Y)W = 0. \end{aligned} \quad (5.2)$$

Taking the inner product with  $X$  in (5.2) and using (2.25), we have

$$\begin{aligned} & \left[ 1 - \frac{r}{n(n-1)} \right] [g(Y, \widetilde{W}_2(U, V)W)\eta(X) - \eta(\widetilde{W}_2(U, V)W)g(X, Y) \\ & \quad - g(Y, U)g(\widetilde{W}_2(\xi, V)W, X) + \eta(U)g(\widetilde{W}_2(Y, V)W, X) - \\ & \quad g(Y, V)g(\widetilde{W}_2(u, \xi)W, X) + \eta(V)g(\widetilde{W}_2(U, Y)W, X) - \\ & \quad g(\widetilde{W}_2(U, V)\xi, X)g(Y, W) + g(\widetilde{W}_2(U, V)Y, X)\eta(W)] = 0. \end{aligned} \quad (5.3)$$

Again from (2.25), we have  $r \neq n(n-1)$ . Thus from (5.3), we have

$$\begin{aligned}
 & g(Y, \widetilde{W}_2(U, V)W)\eta(X) - \eta(\widetilde{W}_2(U, V)W)g(X, Y) \\
 & - g(Y, U)g(\widetilde{W}_2(\xi, V)W, X) + \eta(U)g(\widetilde{W}_2(Y, V)W, X) - \\
 & g(Y, V)g(\widetilde{W}_2(U, \xi)W, X) + \eta(V)g(\widetilde{W}_2(U, Y)W, X) - \\
 & g(\widetilde{W}_2(U, V)\xi, X)g(Y, W) + g(\widetilde{W}_2(U, V)Y, X)\eta(W) = 0.
 \end{aligned} \tag{5.4}$$

Taking  $\xi$  for  $U$  in (5.4) and using (2.15), we have

$$\begin{aligned}
 & [g(V, W)g(X, Y) - g(Y, W)g(V, X)] + b[(n-1)g(V, W)g(X, Y) - \\
 & S(Y, V)\eta(W)\eta(X) + (n-1)g(X, Y)\eta(V)\eta(W) + S(V, X)\eta(Y)\eta(W) \\
 & - S(X, Y)\eta(V)\eta(W) - g(Y, W)S(V, X) + (n-1)g(V, Y)\eta(W)\eta(X) \\
 & - S(V, X)\eta(Y)\eta(W)] - a'R(Y, V, W, X) - b[S(X, Y)g(V, W) \\
 & - S(V, X)g(Y, W)] + \frac{r}{n} \left( \frac{a}{n-1} + b \right) [g(X, Y)g(V, W) \\
 & - g(V, X)g(Y, W)] = 0.
 \end{aligned} \tag{5.5}$$

Putting  $e_i$  for  $Y$  and  $W$  in (5.5),  $\{e_i\}$  is an orthonormal basis of the tangent space at a point of the manifold and taking summation over  $e_i$ ,  $1 \leq i \leq n$ , we get

$$\begin{aligned}
 (a-b)S(V, W) = & [a(n-1) + bn(n-1) - br]g(V, W) + \\
 & b[n(n-1) - r]\eta(V)\eta(W).
 \end{aligned} \tag{5.6}$$

Hence we have the following theorem :

**Theorem 5.1.** An  $n$ -dimensional ( $n > 2$ ) LP-Sasakian manifold  $M^n$  satisfying the condition  $L(\xi, Y) \cdot \widetilde{W}_2 = 0$  is an  $\eta$ -Einstein manifold, provided  $a - b \neq 0$  and  $r \neq n(n-1)$ .

## 6. LP-Sasakian manifold satisfying $\widetilde{W}_2(\xi, Y) \cdot L$

In this section we consider a LP-Sasakian manifold  $M^n$  satisfying the condition

$$\widetilde{W}_2(\xi, Y) \cdot L = 0. \tag{6.1}$$

From (2.30), we have

$$\begin{aligned}
 (\widetilde{W}_2(\xi, Y) \cdot L)(U, V)W = & \widetilde{W}_2(\xi, Y)L(U, V)W - L(\widetilde{W}_2(\xi, Y)U, V)W - \\
 & L(U, \widetilde{W}_2(\xi, Y)V)W - L(U, V)\widetilde{W}_2(\xi, Y)W.
 \end{aligned}$$

Using (6.1), we have

$$\begin{aligned} & \widetilde{W}_2(\xi, Y)L(U, V)W - L(\widetilde{W}_2(\xi, Y)U, V)W - \\ & L(U, \widetilde{W}_2(\xi, Y)V)W - L(U, V)\widetilde{W}_2(\xi, Y)W = 0. \end{aligned} \quad (6.2)$$

Taking the inner product with  $X$  in (6.2) and using (2.22), we have

$$\begin{aligned} & (\widetilde{W}_2(\xi, Y)L(U, V)W, X) - g(L(\widetilde{W}_2(\xi, Y)U, V)W, X) - \\ & g(L(U, \widetilde{W}_2(\xi, Y)V)W, X) - g(L(U, V)\widetilde{W}_2(\xi, Y)W, X) = 0. \end{aligned} \quad (6.3)$$

Putting  $\xi$  for  $U$  in (6.3) and using (2.4), (2.1) and (2.25), we have

$$\begin{aligned} & \left[1 - \frac{r}{n(n-1)}\right][k+b][g(V, W)g(X, Y) - g(Y, W)g(V, X) + \\ & 2g(V, W)S(X, Y) - (n-1)g(Y, V)\eta(W)\eta(X) + \eta(V)\eta(X)S(Y, W) - \\ & (n-1)g(Y, W)\eta(V)\eta(X) - (n-1)g(Y, W)g(V, X) + \eta(W)\eta(X)S(V, Y)] - \\ & k\left[g(R(Y, V)W, X) - \frac{r}{(n(n-1))}\{g(V, W)g(X, Y) - g(Y, W)g(V, X)\}\right] - \\ & b\left[g(R(QY, V)W, X) - \frac{r}{(n(n-1))}\{g(V, W)S(X, Y) - S(Y, W)g(V, X)\}\right] = 0. \end{aligned} \quad (6.4)$$

Putting  $e_i$  for  $X$  and  $V$  in (6.4),  $\{e_i\}$  is an orthonormal basis of the tangent space at a point of the manifold and taking summation over  $e_i$ ,  $1 \leq i \leq n$ , we get

$$\begin{aligned} & bS(QY, W) - \left[\frac{br}{n-1} - k - b\right]S(Y, W) \\ & - (n-1)\left[k + (n-1)b - \frac{br}{n}\right]g(Y, W) = 0, \end{aligned}$$

when  $b = 0$ , then above equation can be written as

$$k[S(Y, W) - (n-1)g(Y, W)] = 0,$$

which means that  $k[r - n(n-1)] = 0$ . From (2.25), we find  $r \neq n(n-1)$ . Thus we get  $k = 0$ ; namely  $a = 0$ . This is the contradiction. Therefore we get  $b \neq 0$  and

$$\begin{aligned} & S(QY, W) = \left[\frac{r}{n-1} - \frac{k}{b} - 1\right]S(Y, W) - \\ & (n-1)\left[\frac{r}{n} - \frac{k}{b} - (n-1)\right]g(Y, W). \end{aligned} \quad (6.5)$$

Hence we have the following theorem:

**Theorem 6.1.** In an  $n$ -dimensional ( $n > 2$ ) LP-Sasakian manifold  $M^n$  satisfying the condition  $\widetilde{W}_2(\xi, Y).L = 0$  holds on  $M^n$ , then the equation (6.5) is satisfied on  $M^n$ , provided  $a \neq 0$  and  $b \neq 0$ .

## 7. LP-Sasakian manifold satisfying $W_2(\xi, Y).\widetilde{W}_2$

In this section we consider a LP-Sasakian manifold  $M^n$  satisfying the condition

$$W_2(\xi, Y).\widetilde{W}_2 = 0 \quad (7.1)$$

From (2.29), we have

$$\begin{aligned} (W_2(\xi, Y).\widetilde{W}_2)(U, V)W = & W_2(\xi, Y)\widetilde{W}_2(U, V)W - \widetilde{W}_2(W_2(\xi, Y)U, V)W - \\ & \widetilde{W}_2(U, W_2(\xi, Y)V)W - \widetilde{W}_2(U, V)W_2(\xi, Y)W, \end{aligned}$$

Using (7.1), we have

$$\begin{aligned} & W_2(\xi, Y)\widetilde{W}_2(U, V)W - \widetilde{W}_2(W_2(\xi, Y)U, V)W - \\ & \widetilde{W}_2(U, W_2(\xi, Y)V)W - \widetilde{W}_2(U, V)W_2(\xi, Y)W = 0. \end{aligned} \quad (7.2)$$

Taking the inner product with  $X$  and putting  $\xi$  for  $U$ , we have

$$\begin{aligned} & g(W_2(\xi, Y)\widetilde{W}_2(\xi, V)W, X) - g(\widetilde{W}_2(W_2(\xi, Y)\xi, V)W, X) - \\ & g(\widetilde{W}_2(\xi, W_2(\xi, Y)V)W, X) - W(\widetilde{W}_2(\xi, V)W_2(\xi, Y)W, X) = 0. \end{aligned} \quad (7.3)$$

Putting  $e_i$  for  $X$  and  $V$  in (7.3),  $\{e_i\}$  is an orthonormal basis of the tangent space at a point of the manifold and taking summation over  $e_i$ ,  $1 \leq i \leq n$ , we get

$$(a - b) \left[ S(QY, W) - \left\{ \frac{r}{n} + (n - 1) \right\} S(Y, W) + \frac{r}{n(n - 1)} g(Y, W) \right] = 0.$$

Hence we have the following theorem:

**Theorem 7.1.** In an  $n$ -dimensional ( $n > 2$ ) LP-Sasakian manifold satisfying the condition  $W_2(\xi, Y).\widetilde{W}_2 = 0$ , then we get

- (i) if  $a - b = 0$ , or
- (ii) if  $a - b \neq 0$ , then

$$S(QY, W) = \left\{ \frac{r}{n} + (n - 1) \right\} S(Y, W) - \frac{r}{n(1 - n)} g(Y, W).$$

### 8. LP-Sasakian manifold satisfying $\widetilde{W}_2(\xi, Y).R$

In this section we consider a LP-Sasakian manifold  $M^n$  satisfying the condition

$$\widetilde{W}_2(\xi, Y).R = 0. \quad (8.1)$$

From (2.31) and (8.1), we have

$$\begin{aligned} (\widetilde{W}_2(\xi, Y).R)(U, V)W &= \widetilde{W}_2(\xi, Y)R(U, V)W - L(\widetilde{W}_2(\xi, Y)U, V)W - \\ &\quad R(U, \widetilde{W}_2(\xi, Y)V)W - R(U, V)\widetilde{W}_2(\xi, Y)W, \end{aligned} \quad (8.2)$$

Taking the inner product with  $X$  and putting  $\xi$  for  $U$  in (8.2), we have

$$\begin{aligned} k[g(V, W)g(X, Y) - g(R(Y, V)W, X) - g(Y, W)g(V, X) - g(V, X)(W)\eta(Y)] \\ + b[g(V, W)S(X, Y) - (n-1)\eta(W)\eta(X)g(Y, V) - g(R(QY, V)W, X) \\ + \eta(V)(X)S(Y, W) - (n-1)\eta(V)(X)g(Y, W) \\ - (n-1)g(Y, W)g(V, X) + \eta(W)\eta(X)S(V, Y)] = 0. \end{aligned} \quad (8.3)$$

Putting  $e_i$  for  $X$  and  $V$  in (8.3),  $\{e_i\}$  is an orthonormal basis of the tangent space at a point of the manifold and taking summation over  $e_i$ ,  $1 \leq i \leq n$ , we get

$$bS(QY, W) + kS(Y, W) - (n-1)[k - (n-1)]g(Y, W) = 0. \quad (8.4)$$

If  $b = 0$ , then equation (8.4) will be

$$a[r - n(n-1)][S(Y, W) - (n-1)g(Y, W)] = 0,$$

Thus, we have

$$S(Y, W) = (n-1)g(Y, W).$$

Hence we have the following theorem:

**Theorem 8.1.** In an  $n$ -dimensional ( $n > 2$ ) LP-Sasakian manifold  $M^n$  satisfying the condition  $\widetilde{W}_2(\xi, Y).R = 0$ , then we get

- (i) if  $b = 0$ , then the manifold is Einstein manifold, provided  $a \neq 0$ .
- (ii) if  $b \neq 0$ , then

$$S(QY, W) = -\frac{k}{b}S(Y, W) + (n-1)[k - (n-1)]g(Y, W).$$

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