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# On Holomorphic Curvature of Complex Finsler with special $(\alpha, \beta)-$Metric

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**Abstract**

The notion of the holomorphic curvature for a Complex Finsler space $(M, F)$ is defined with respect to the Chern complex linear connection on the pull-back tangent bundle. This paper is about the fundamental metric tensor, inverse tensor and as a special approach of the pull-back bundle is devoted to obtain the Riemannian curvature and holomorphic curvature of Complex Finsler with special $(\alpha, \beta)$-metric.

**Key Words:** Complex Finsler space with $(\alpha, \beta)-$metric, Holomorphic flag curvature, Riemannian curvature, Ricci curvature.
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## 1. Introduction

The notion of holomorphic curvature of a complex Finsler spaces with respect to the Chern complex linear connection in breif Chern (c.l.c) as a connection in the holomorphic pull back tangent bundle $\pi^*(T'M)$. In [12], Nicolta Aldea has obtained the characterization of the holomorphic bisectional curvature and gave the generalization of the holomorphic curvature of the complex Finsler spaces which are called holomorphic flag curvature. After that in (2006) he devoted to obtain the characterization of holomorphic flag curvature.

In view point of reviews, our goal was to determine the conditions in which complex Finsler spaces with special $(\alpha, \beta)$- metric of holomorphic curvature.

## 2. Preliminaries

This section, includes the basic notions of Complex Finsler spaces.
An $\mathbb{R}$-Complex Finsler metric on $M$ is continuous function $F : TM \longrightarrow \mathbb{R}$ satisifying:
i. $L = F^2$ is a smooth on $T'\tilde{M}/0$;
ii. $F(z, \eta) \geq 0$,the equality holds if and only if $\eta = 0$;
iii. $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = |\lambda| F(z, \eta, \bar{z}, \bar{\eta}),\ forall\ \lambda \in \mathbb{R}$.

Let $M$ be a complex manifold, $dim_c M = n$ and $T'M$ the holomorphic tangent bundle in which as a complex manifold the local coordinates will be denoted by $(z^k, \eta^k)$. The complexified tangent bundle of $T'M$ is decompsed in $T_c(T^{'}M) = T'(T''M) \oplus T''(T'M)$.

Considering the restriction of the projection to $T^{'}\tilde{M} = T'M/0$, for pulling back of the holomorphic tangent bundle $T'M$ then it obtain a holomorphic tangent bundle $\pi' : \pi^*(T'M) \longrightarrow \widetilde{T'M}$, called the pull-back tangent bundle over the slit $T^{'}\tilde{M}$. We denote by $\{\frac{\partial}{\partial z^k}^*, \frac{\partial}{\partial \bar{z}^k}^*\}$, the local frame and by $\{dz^{*k}, d\bar{z}^{*k}\}$ the local frame and its dual.

Let $V(T'M) = ker\pi_* \subset T'(T'M)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial \eta^k}\}$. A complex nonlinear connection, breifly (c.n.c), determines a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, that is $T'(T'M) = H(T'M) \oplus V(T'M)$. The adapted frames is $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j}$, where $N^j_k(z, \eta)$ are the coefficients of the (c.n.c). Further we shall use the abbrevations $\delta_i = \frac{\delta}{\delta z^i}$, $\dot{\partial}_i = \frac{\partial}{\partial \eta^i}$, $\delta_{\bar{i}} = \frac{\delta}{\delta \bar{z}^i}$, $\dot{\partial}_i = \frac{\partial}{\partial \bar{\eta}^j}$, and their conjugates[7, 17, 4].
On $T'M$, let $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ be the fundamental metric tensor of a complex Finsler space $(M, L = F^2)$.

The isomorphism between $\pi^*(T'M)$ and $T'M$ induces an isomorphism of $\pi^*(T_C M)$ and $T_C M$. Thus, $g_{i\bar{j}}$ defines an Hermitian metric struture $G(z, \eta) = g_{j\bar{k}} d\bar{z}^{*j} \otimes d\bar{z}^{*k}$ on $\pi^*(T'M)$, with respect to the natural complex structure. Further, the Hermitian metric structure $G$ on $\pi^*(T'M)$ induces a Hermitian inner product $h(\xi, \gamma) := ReG(\chi, \bar{\gamma})$ and the angle $cos(\chi\gamma) = \frac{ReG(\chi, \bar{\gamma})}{\|\chi\| \|\bar{\gamma}\|}$, for any $\chi$, $\gamma$ the sections on $\pi^*(T'M)$, where $\|\chi\|^2 = \|\bar{\chi}\|^2 = G(\chi, \bar{\chi})$ (for details see in [10]).

On the other hand, $H(T'M)$ and $\pi^*(T'M)$ are isomorphic. Therefore, the structures on $\pi^*(T_C M)$ can be pulled-back to $H(T'M) \oplus \overline{H(T'M)}$. By this isomorphism the natural co-basis $dz^{*j}$ is identified with $dz^j$. In view of this constructions the pull-back tangent bundle $\pi^*(T'M)$ admits a unique complex linear connection $\nabla$, called the Chern (c.l.c), which is metric with respect to $G$ and of $(0,1)-$ type .

$$\omega^i_j(z,\eta) = L^i_{jk}(z,\eta)dz^k + C^i_{jk}(z,\eta)\delta\eta^k; \tag{2.1}$$

The Chern(c.l.c) on $\pi^*(T'M)$ determines the Chern-Finsler (c.n.c) on $(T'M)$, with the coefficients $N^i_k = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta z^k}\eta^j$, and its local coefficients of torsion and curvature are

$$T^i_{jk} := L^i_{jk} - L^i_{kj}; \tag{2.2}$$

$$R^i_{j\bar{h}k} := -\delta_{\bar{h}}L^i_{jk} - \delta_{\bar{h}}(N^l_k)C^i_{jl} \quad ; \sigma^i_{j\bar{h}k} := -\delta_{\bar{h}}C^i jk = \sigma^i_{k\bar{h}j} \quad ;$$

$$L^i_{jk} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta z^k};, \quad C^i_{jk} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial \eta^k}.$$

$$P^i_{j\bar{h}k} := -\dot{\partial}_{\bar{h}}L^i_{jk} - \dot{\partial}_{\bar{h}}(N^l_k)C^i_{jl}, \quad ; S^i_{j\bar{h}k} := -\dot{\partial}_{\bar{h}}C^i_{jk} = S^i_{k\bar{h}j}.$$

The Riemann type tensor

$$R(W,\bar{z},X,\bar{y}) := G(R(X,\bar{Y})W,\bar{Z}),$$

has properties:

$$R(W,\bar{Z},X,\bar{Y}) = W^i\bar{Z}^jX^k\bar{Y}^hR_{i\bar{j}k\bar{h}}; \quad R_{\bar{j}i\bar{h}k} := R^l_{i\bar{h}k}g_{l\bar{j}}; \tag{2.3}$$

$$R_{i\bar{j}k\bar{h}} = -R_{i\bar{j}\bar{h}k} = \overline{R_{\bar{j}i\bar{h}\bar{k}}} = R_{\bar{j}i\bar{h}k};$$

$$R^i_{j\bar{j}\bar{h}k} = R^i_{khj} \quad then \quad R_{i\bar{j}k\bar{h}} = R_{k\bar{j}k\bar{h}} = R_{k\bar{h}i\bar{j}}.$$

According to [7] the complex Finsler space $(M,F)$ is strongly Kähler if and only if $T^i_{jk} = 0$, Kähler if and only if $T^i_{jk}\eta^j = 0$ and weakly Kähler if and only if $g_{i\bar{l}}T^i_{jk}\eta^j\bar{\eta}^l = 0$. Note that for a complex Finsler metric which comes from a Hermitian metric on $M$, so-called purely Hermitian metric in [4].That is $g_{i\bar{j}} = g_{i\bar{j}}(z)$, the three nuances of Kähler spaces coinsider, in [1].
The holomorphic curvature of $F$ in direction $\eta$, with respect to the Chern (c.l.c),[7] is,

$$\kappa_F(z,\eta) := \frac{2R(\eta,\bar{\eta},\eta,\bar{\eta})}{G^2(\eta,\bar{\eta})} = \frac{2\bar{\eta}^j\eta^kR_{\bar{j}k}}{L^2(z,\eta)}, \tag{2.4}$$

where $\eta$ is viewed as local section of $\pi^*(T'M)$, that is $\eta := \eta^i\frac{\partial}{\partial z^i}$. Further on, we shall simply call it holomorphic curvature. It depends both on the position $z\epsilon M$ and the direction $\eta$.

**Definition 2.1.** The complex Finsler space $(M, F)$ is called generalized Einstein if $R_{\bar{j}k}$ is proportional to $t_{k\bar{j}}$, that is if there exists a real valued function $K(z, \eta)$, such that [14]

$$R_{\bar{j}k} = K(z, \eta)t_{k\bar{j}}, \tag{2.5}$$

where $R_{\bar{j}k} := R_{i\bar{j}k\bar{h}}\eta^i\bar{\eta}^h = -g_{l\bar{j}}\delta_{\bar{h}}(N^l_k)\bar{\eta}^h,\ t_{k\bar{j}} := L(z, \eta)g_{k\bar{j}} + \eta_k\bar{\eta}_j,\ \eta_k := \frac{\partial L}{\partial \eta^k}$, $\bar{\eta}_j; = \frac{\partial L}{\partial \bar{\eta}^j}$.

By finding the Chern(c.l.c) on $\pi_*(T'M)$ determines the Chern-Finsler on $T'M$,with the coefficient $N^i_k = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial z^k}\eta^j$ determines, we need to find the fundamental metric tensor followed by the invarients are given below:
Now from definition of Complex Finsler metric follows that $L$ is $(2, 0)$-homogeneous with respect to the real scalar $\lambda$ and is proved that the following identities are fuullfilled[6].

$$\frac{\partial L}{\partial \eta^i}\eta^i + \frac{\partial L}{\partial \bar{\eta}^i} = 2L \quad ; \quad g_{ij}\eta^i + g_{\bar{j}i}\bar{\eta}^i = \frac{\partial L}{\partial \eta^j}, \tag{2.6}$$

$$\frac{\partial g_{ik}}{\partial \eta^j}\eta^j + \frac{\partial g_{ij}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0 \quad ; \quad \frac{\partial g_{i\bar{k}}}{\partial \eta^j}\eta^j + \frac{g_{i\bar{k}}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0, \tag{2.7}$$

$$2Lg_{ij}\eta^i\eta^j + g_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2g_{i\bar{j}}\eta^i\bar{\eta}^j, \tag{2.8}$$

where,

$$g_{ij} = \frac{\partial^2 L}{\eta^i\eta^j} \quad ; \quad g_{i\bar{j}} = \frac{\partial^2 L}{\eta^i\bar{\eta}^j} \quad ; \quad g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\partial\bar{\eta}^i\partial\bar{\eta}^j}.$$

Here, to find the inverse of fundamental metric tensor $g_{i\bar{j}}$ we use the following proposition2.1.

**Proposition 2.1.** Suppose:
•$(Q_{ij})$ is a non-singular $n \times n$ complex matrix with inverse $Q^{ji}$;
•$C_i$ and $C_{\bar{i}} = \bar{C}_i, i = 1, ........n$ are complex numbers;
•$C^i := Q^{ji}C_j$ and its conjugates; $C^2 := C^iC_i = \bar{C}^iC_{\bar{i}}$; $H_{ij} := Q_{ij} \pm C_iC_j$.
Then,
i. $det(H_{ij}) = (1 \pm C^2)det(Q_{ij})$,
ii. whenever $(1 \pm C^2) \neq 0$, the matrix $(H_{ij})$ is invertible and in this case its inverse is $H^{ij} = Q^{ji} \pm \frac{1}{1 \pm C^2}C^iC^j$.

## 3. Fundamental metric tensor of Complex Finsler space with special $(\alpha, \beta)-$metrics

The $\mathbb{R}$-complex Finsler space produce the tensor fields $g_{ij}$ and $g_{i\bar{j}}$. The tensor field must $g_{i\bar{j}}$ be invertible in Hermitian geometry. These problems are

about to Hermitian $\mathbb{R}$-complex Finsler spaces, if $det(g_{i\bar{j}} \neq 0)$ and non-Hermitian $\mathbb{R}$-complex Finsler spaces, if $det(g_{ij} \neq 0)$. In this section, we determine the fundemental tensor of complex Square metric and inverse also.
Consider $\mathbb{R}$-complex Finsler space with special $(\alpha, \beta)$-metrics,

$$L(\alpha, \beta) = \left(\alpha + \frac{|\beta|^2}{\alpha}\right)^2, \tag{3.1}$$

then it follows that $F = \alpha + \frac{|\beta|^2}{\alpha}$.
Now, we find the following quantities of $F$.
From the equalities (2.6) and (2.7) with metric (3.1), we have

$$\alpha L_\alpha + \beta L_\beta = 2L,\ \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = L_\alpha, \tag{3.2}$$

$$\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta,\ \ \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L,$$

where

$$L_\alpha = \frac{\partial L}{\partial \alpha},\ L_\beta = \frac{\partial L}{\partial \beta},\ \ L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta},\ \ L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2},\ \ L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}. \tag{3.3}$$

$$L_\alpha = 2F\left(\frac{\alpha^2|\beta|^2}{\alpha^2}\right), \tag{3.4}$$

$$L_\beta = 4F\frac{|\beta|}{\alpha}, \tag{3.5}$$

$$L_{\alpha\alpha} = 2\left[\frac{\alpha^2 + |\beta|^2(2|\beta|^2)}{\alpha^4} + \left(\frac{\alpha^2 - |\beta|^2}{\alpha^4}\right)^2\right], \tag{3.6}$$

$$L_{\beta\beta} = 4\left[\frac{F}{\alpha} + \frac{2|\beta|^2}{\alpha^2}\right], \tag{3.7}$$

$$L_{\alpha\beta} = \frac{-8|\beta|^3}{\alpha^3}, \tag{3.8}$$

$$\alpha L_\alpha + |\beta| L_{|\beta|} = \alpha.2F\left(\frac{\alpha^2 - |\beta|^2}{\alpha^2}\right) + |\beta|\left(\frac{2F|\beta|}{\alpha}\right), \tag{3.9}$$

$$= 2F\left(\frac{\alpha^2 + |\beta|^2}{\alpha}\right) = 2F^2 = 2L, \tag{3.10}$$

$$\alpha L_{\alpha\alpha} + \beta L_{\alpha|\beta|} = 2\alpha\left[\frac{\alpha^2 + |\beta|^2}{\alpha^4}(2|\beta|^2)\right] + \left[|\beta|\left(\frac{-8|\beta|^3}{\alpha^3}\right)\right], \tag{3.11}$$

$$= 2F\left(\frac{\alpha^2 - |\beta|^2}{\alpha^2}\right) = L_\alpha. \tag{3.12}$$

We propose to determine the metric tensors of an $\mathbb{R}$-complex Finsler space using the following equalities:

$$g_{ij} = \frac{\partial^2 L(z,\eta,\bar{z},\lambda\bar{\eta})}{\partial\eta^i\partial\eta^j}, \quad g_{i\bar{j}} = \frac{\partial^2 L(z,\eta,\bar{z},\lambda\bar{\eta})}{\partial\eta^i\partial\bar{\eta}^j}.$$

Each of these being of interest in the following :
Consider,

$$\frac{\partial\alpha}{\partial\eta^i} = \frac{1}{2\alpha}(a_{ij}\eta^j + a_{i\bar{j}}\bar{\eta}^j) = \frac{1}{2\alpha l_i}, \quad \frac{\partial\beta}{\partial\eta^i} = \frac{1}{2}b_i.$$

$$\frac{\partial\alpha}{\partial\bar{\eta}^i} = \frac{1}{2\alpha}(a_{\bar{i}\bar{j}}\bar{\eta}^j + a_{i\bar{j}}\eta^j) = \frac{\partial\beta}{\partial\bar{\eta}^i} = \frac{1}{2}b_{\bar{i}},$$

where,

$$l_i = (a_{ij}\eta^j + a_{i\bar{j}}\eta^{\bar{j}}), \quad l_{\bar{j}} = a_{\bar{i}\bar{j}}\bar{\eta}^i + a_{i\bar{j}}\eta^i.$$

Then, we can find ,

$$l_i\eta^i + l_{\bar{j}}\bar{\eta}^j = 2\alpha^2.$$

We denote:

$$\eta^i = \tfrac{\partial L}{\partial\eta^i} = \tfrac{\partial}{\partial\eta^i}F^2 = 2F\tfrac{\partial}{\partial\eta^i}\left(\tfrac{\alpha^2}{\alpha-\beta}\right),$$

$$\eta_i = \rho_0 l_i + \rho_1 b_i,$$

where

$$\rho_0 = \frac{1}{2}\alpha^{-1}L_\alpha, \tag{3.13}$$

and

$$\rho_1 = \frac{1}{2}L_\beta. \tag{3.14}$$

Differentiating $\rho_0$ and $\rho_1$ with respect to $\eta^j$ and $\bar{\eta}^j$ respectively, which yields:

$$\frac{\partial\rho_0}{\partial\eta^j} = \rho_{-2}l_j + \rho_{-1}b_j,$$

and

$$\frac{\partial\rho_0}{\partial\bar{\eta}^j} = \rho_{-2}l_{\bar{j}} + \rho_{-1}b_{\bar{j}}.$$

Similarly,

$$\tfrac{\partial\rho_1}{\partial\eta^i} = \eta_{-1}l_i + \mu_0 b_i, \quad \tfrac{\partial\rho_1}{\partial\bar{\eta}^i} = \rho_{-1}l_{\bar{i}} + \mu_0 b_{\bar{i}},$$

where,

$$\rho_{-2}=\frac{\alpha L_{\alpha\alpha}-l_{\alpha}}{4\alpha^3},\quad \rho_{-1}=\frac{L_{\alpha\beta}}{4\alpha},\quad \mu_0=\frac{L_{\beta\beta}}{4}. \tag{3.15}$$

By direct computation using (3.13),(3.14),(3.15), we obtain the invarients of $\mathbb{R}$-complex Finsler space with special $(\alpha,\beta)$- metrics: $\rho_0,\ \rho_1,\ \rho_{-1},\ \rho_{-2}$ are given below:

$$\rho_0 = \frac{F}{\alpha}\left(\frac{\alpha^2-|\beta|^2}{\alpha^2}\right), \tag{3.16}$$

$$\rho_1 = \frac{2F|\beta|}{\alpha}, \tag{3.17}$$

$$\rho_{-2} = \left[\frac{\left(\alpha^2-|\beta|^2\right)^2-F\alpha(\alpha^2+3|\beta|^2)}{2\alpha^6}\right], \tag{3.18}$$

$$\rho_{-1} = \left[\frac{F}{\alpha}+\frac{2|\beta|^2}{\alpha^2}\right]. \tag{3.19}$$

Now, the fundamental metric tensors of $\mathbb{R}$-complex Finsler space with special $(\alpha,\beta)$ metric are given by[18]:

$$g_{i\bar{j}}=\rho_0 a_{i\bar{j}}+\frac{\rho'_{-2}}{\rho_0}l_i l_{\bar{j}}+\frac{\mu''_0}{\rho_0}b_i b_{\bar{j}}+\frac{\mu''_{-2}}{\rho_0}\eta_i\eta_{\bar{j}} \tag{3.20}$$

By using the equations (3.16) to (3.19) in (3.20) we have

$$\begin{aligned} g_{ij}=&\frac{F}{\alpha^3}(\alpha^2-|\beta|^2)a_{ij}+\frac{\alpha^3(8F\alpha^5|\beta|^4-8\alpha^4|\beta|^4+8\alpha^2|\beta|^2+F\alpha^3-\alpha^4}{F(\alpha^2-|\beta|^2)8\alpha^6|\beta|^2}\\ &\frac{+2\alpha^2|\beta|^2-F\alpha|\beta|^2-|\beta|^4)}{}l_il_j+\frac{\alpha^3(8F\alpha|\beta|-F\alpha+\alpha^2-|\beta|^2)}{F(\alpha^2-|\beta|^2)(4\alpha^2|\beta|)}b_ib_j\\ &+\frac{\alpha^3(-F\alpha+\alpha^2-|\beta|^2)}{2F^3(\alpha^2-|\beta|^2)^2}\eta_i\eta_j. \end{aligned} \tag{3.21}$$

$$\begin{aligned} g_{i\bar{j}}=&\frac{F}{\alpha^3}(\alpha^2-|\beta|^2)a_{i\bar{j}}+\frac{\alpha^3(8F\alpha^5|\beta|^4-8\alpha^4|\beta|^4+8\alpha^2|\beta|^2+F\alpha^3-\alpha^4}{F(\alpha^2-|\beta|^2)8\alpha^6|\beta|^2}\\ &\frac{+2\alpha^2|\beta|^2-F\alpha|\beta|^2-|\beta|^4)}{}l_il_{\bar{j}}+\frac{\alpha^3(8F\alpha|\beta|-F\alpha+\alpha^2-|\beta|^2)}{F(\alpha^2-|\beta|^2)(4\alpha^2|\beta|)}b_ib_{\bar{j}}\\ &+\frac{\alpha^3(-F\alpha+\alpha^2-|\beta|^2)}{2F^3(\alpha^2-|\beta|^2)^2}\eta_i\eta_{\bar{j}} \end{aligned} \tag{3.22}$$

The metric tensor $g_{ij}$ and $g_{i\bar{j}}$ are equivalent to

$$g_{ij}=a_{ij}+Pl_il_j+Qb_ib_j+R\eta_i\eta_j, \tag{3.23}$$

$$g_{i\bar{j}} = a_{i\bar{j}} + Pl_i l_{\bar{j}} + Qb_i b_{\bar{j}} + R\eta_i \eta_{\bar{j}}, \tag{3.24}$$

where

$$P = \frac{\alpha^3(8F\alpha^5|\beta|^4 - 8\alpha^4|\beta|^4 + 8\alpha^2|\beta|^2 + F\alpha^3 - \alpha^4}{F(\alpha^2 - |\beta|^2)8\alpha^6|\beta|^2} \frac{+2\alpha^2|\beta|^2 - F\alpha|\beta|^2 - |\beta|^4)}{}, \tag{3.25}$$

$$Q = \frac{\alpha^3(8F\alpha|\beta| - F\alpha + \alpha^2 - |\beta|^2)}{F(\alpha^2 - |\beta|^2)(4\alpha^2|\beta|)}, \tag{3.26}$$

$$R = \frac{\alpha^3(-F\alpha + \alpha^2 - |\beta|^2)}{2F^3(\alpha^2 - |\beta|^2)^2}. \tag{3.27}$$

we use the following proposition (2.1) for further calculations. The solution of the non-Hermitian metric $Q_{i\bar{j}}$ as follows.

**Theorem 3.1.** For a non-Hermitian $\mathbb{R}$-Complex Finsler space with special $(\alpha, \beta)$-metric $F = \alpha + \frac{|\beta|^2}{\alpha}$, then they have the following:
i) The contravarient tensor $g^{i\bar{j}}$ of the fundamental tensor $g_{i\bar{j}}$ is:

$$\begin{aligned} g^{\bar{j}i} = &\frac{F\alpha^3}{\alpha^2 - |\beta|^2}\Bigg[a^{ji} + \frac{P}{1+P\gamma} + \frac{P^2 Q\epsilon^2}{\tau}(1+P\gamma)^2\eta^i\eta^{\bar{j}} + \frac{Q}{\tau}b^i b^{\bar{j}} \\ &+ \frac{PQ\epsilon}{\tau(1+P\gamma)}(b^i\eta^{\bar{j}} + \eta^i b^{\bar{j}}) + \frac{U^2\eta^i\eta^{\bar{j}} + UV(\eta^i b^{\bar{j}} + b^i\eta^{\bar{j}}) + V^2 b^i b^{\bar{j}}}{1 + (U\gamma + V\epsilon)\sqrt{R}}\Bigg], \end{aligned} \tag{3.28}$$

where,

$$\begin{aligned} U =& \left[1 + \left(\frac{P}{1+P\gamma} + \frac{P^2 Q\epsilon^2}{\tau(1+P\tau)^2}\right)\right]\gamma + \frac{PQ\epsilon}{\tau(1+P\tau)^2}, \\ \text{and} \quad V =& \frac{Q}{\tau} + \frac{PQ\epsilon\tau}{\tau(1+P\gamma)}. \end{aligned} \tag{3.29}$$

ii) $det(a_{i\bar{j}} + Pl_i l_{\bar{j}} + Qb_i b_{\bar{j}} + R\eta_i\eta_{\bar{j}}) = \Big[1 + (U\gamma - V\epsilon)\sqrt{R}\Big] \Big[1 + \omega + \frac{P\epsilon^2}{1+P\mu}\Big](1 + P\gamma)det(Q_{i\bar{j}})$
where, $R = \frac{\alpha^3(-F\alpha+\alpha^2-|\beta|^2)}{2F^3(\alpha^2-|\beta|^2)^2}$.

**Proof.** We prove this theorem by following three steps:
Step 1:- We write $g_{i\bar{j}}$ from (3.22)in the form as;

$$g_{i\bar{j}} = \left[a_{i\bar{j}} + Pl_i l_{\bar{j}} + Qb_i b_{\bar{j}} + R\eta_i\eta_{\bar{j}}\right]. \tag{3.30}$$

We take $Q_{i\bar{j}} = a_{i\bar{j}}$ and $C_i = \sqrt{P}l_i$. By applying the proposition 2.1 we obtain $Q^{i\bar{j}} = a^{ji}$, $C^2 = C_iC^i = \sqrt{P}l_i \times Q^{\bar{j}i} \times C_j = \sqrt{P}l_i \times a^{\bar{j}i} \times \sqrt{P}l_j = P \times l_i a^{i\bar{j}} l_{\bar{j}} = P\gamma$, and $1 + C^2 = (1 + P\gamma)$.

So, the matrix $H_{i\bar{j}} = a_{i\bar{j}} + Pl_il_{\bar{j}}$, is invertible with

$$H^{i\bar{j}} = a^{\bar{j}i} + \tfrac{1}{1+P\gamma}\eta^i\eta^{\bar{j}},$$

$$det(a_{i\bar{j}} + Pl_il_{\bar{j}}) = (1 + P\gamma) = det(a_{i\bar{j}}).$$

Step 2: Now, we consider

$$Q_{i\bar{j}} = a_{i\bar{j}} + Pl_il_{\bar{j}}, \text{ and } C_i = \sqrt{Q}b_i,$$

again applying the proposition 2.1, then wehave

$$Q^{\bar{j}i} = a^{\bar{j}i} + \frac{P\eta^i\eta^{\bar{j}}}{1 + P\gamma},$$

$$C^2 = C_iC^i = Q^{\bar{j}i} \times C_{\bar{j}} = \sqrt{Q}b_i \left[a^{i\bar{j}} + \frac{P\eta^i\eta^{\bar{j}}}{1 + P\gamma}\sqrt{Q}b^{\bar{j}}\right],$$

$$c^2 = Q\left[\omega + \frac{P\epsilon^2}{1 + P\gamma}\right].$$

Therefore,

$$1 + C^2 = 1 + C\left[\omega + \frac{P\epsilon^2}{1 + P\gamma}\right] \neq 0,$$

where $\quad \epsilon = b_j\eta^j, \quad \omega = b_jb^j.$

It results that the inverse of $H_{ij} = a_{i\bar{j}} + Pl_il_{\bar{j}} + Qb_ib_{\bar{j}}$ exists and it is

$$H^{\bar{j}i} = Q^{\bar{j}i} + \frac{1}{1 + C^2}C^iC^{\bar{j}},$$

$$H^{\bar{j}i} = a^{\bar{j}i} + \frac{P\eta^i\eta^{\bar{j}}}{1 + P\gamma} + \frac{P\left[b^i + \frac{P\epsilon\eta^i}{1+P\gamma}\right]\left[b^{\bar{j}} + \frac{P\epsilon\eta^{\bar{j}}}{1+P\gamma}\right]}{\tau}, \tag{3.31}$$

$$H^{\bar{j}i} = a^{\bar{j}i} + \left(\frac{P}{1 + P\gamma} + \frac{P^2Q\epsilon^2}{\tau(1 + PQ\gamma)^2}\right)\eta^i\eta^{\bar{j}} + \frac{PQ\epsilon}{1 + B\gamma}(b^i\eta^{\bar{j}} + b^{\bar{j}}\eta^i) + \frac{Q}{\tau}b^ib^{\bar{j}}, \tag{3.32}$$

where,

$$\tau = 1 + Q\left[\omega + \tfrac{P\epsilon^2}{1+P\gamma}\right].$$

and

$$det\left[a_{i\bar{j}} + Pl_i l_{\bar{j}} + Qb_i b_{\bar{j}}\right] = \left[1 + Q\left(\omega + \frac{P\epsilon^2}{1+P\gamma}\right)\right](1+P\gamma)det(a_{i\bar{j}}). \quad (3.33)$$

Step 3: We put

$$Q_{\bar{j}i} = a_{i\bar{j}} + Pl_i l_{\bar{j}} + Qb_i b_{\bar{j}}, \quad (3.34)$$

and

$$C_i = \sqrt{R}\eta_i,$$

clearly observe that and obtain

$$Q^{\bar{j}i} = a^{\bar{j}i} + \left(\frac{P}{1+P\gamma} + \frac{P^2 Q\epsilon^2}{\tau(1+P\gamma)}\right)\eta^i\eta^{\bar{j}} + \frac{PQ\epsilon}{1+P\gamma}(b^i\eta^{\bar{j}} + b^{\bar{j}}\eta^i) + \frac{Q}{1+P\gamma}b^i b^{\bar{j}}, \quad (3.35)$$

and $C_i = U\eta^i + Vb^{\bar{j}}$,
where

$$U = \left[1 + \left(\frac{P}{1+P\gamma} + \frac{P^2 Q\epsilon^2}{\tau(1+P\mu)^2}\right)\right]\gamma + \frac{PQ\epsilon}{\gamma(1+P\gamma)^3}, \quad (3.36)$$

$$V = \frac{Q}{\tau} + \frac{PQ\epsilon\gamma}{\gamma(1+P\mu)}. \quad (3.37)$$

and

$$C^2 = (U\gamma + V\epsilon)\sqrt{R},$$

$$1 + C^2 = \left[a^{\bar{j}i} + \left(\frac{P}{1+P\gamma} + \frac{QP^2\epsilon^2}{\gamma(1+P\gamma)} + \frac{PQ\epsilon}{1+P\gamma} + \frac{Q}{1+P\gamma}\right)\right]\sqrt{R} \neq 0.$$

Clearly, the matrix $H_{i\bar{j}}$ is invertible.

$$C^i = a^{\bar{j}i} + \left\{\frac{P\eta^i\eta^{\bar{j}}}{1+P\gamma} + \frac{Q\left[b^i + \frac{P\epsilon\eta^i}{1+P\gamma}\right]\left[b^{\bar{j}} + \frac{P\epsilon\eta^{\bar{j}}}{1+P\gamma}\right]}{\gamma}\right\}\eta_{\bar{j}},$$

and

$$C^{\bar{j}} = a^{\bar{j}i} + \left\{\frac{P\eta^i\eta^{\bar{j}}}{1+P\gamma} + \frac{Q\left[b^i + \frac{P\epsilon\eta^i}{1+P\gamma}\right]\left[b^{\bar{j}} + \frac{P\epsilon\eta^{\bar{j}}}{1+P\gamma}\right]}{\gamma}\right\}\eta_i,$$

where

$$C^iC^{\bar{j}} = U^2\eta^i\eta^{\bar{j}} + UV(\eta^i b^{\bar{j}} + \eta^{\bar{j}} b^i) + V^2 b^i b^{\bar{j}}.$$

Therefore, we obtain the inverse of $H_{i\bar{j}}$ as:

$$H^{\bar{j}i} = \left[a^{ji} + \left(\frac{P}{1+P\gamma} + \frac{P^2Q\epsilon^2}{\tau(1+P\gamma)^2}\right)\right]\eta^i\eta^j + \frac{Q}{\tau}b^ib^j + \frac{PQ\epsilon}{\tau(1+P\gamma)}(b^i\eta^j + b^j\eta^i) + \frac{U^2\eta^i\eta^j + UV(\eta^i b^j + \eta^j b^i) + V^2 b^i b^j}{1+(U\gamma+V\epsilon)\sqrt{R}} \tag{3.38}$$

$$det(a_{i\bar{j}} + Pl_il_{\bar{j}} + Qb_ib_{\bar{j}} + R\eta_i\eta_{\bar{j}}) = \left[1+(U\gamma+V\epsilon)\sqrt{R}\right]\left[1+\omega + \frac{P\epsilon^2}{1+P\gamma}\right](1+P\gamma)det(a_{i\bar{j}}). \tag{3.39}$$

But $g_{i\bar{j}} = H_{i\bar{j}}$, with $H_{i\bar{j}}$ from last step. Thus

$$g^{ji} = \frac{1}{\rho_0}H^{i\bar{j}}. \tag{3.40}$$

Then we get,

$$g^{\bar{j}i} = \frac{F\alpha^3}{\alpha^2 - |\beta|^2}\left[a^{ji} + \frac{P}{1+P\gamma} + \frac{P^2Q\epsilon^2}{\tau}(1+P\gamma)^2\eta^i\eta^{\bar{j}} + \frac{Q}{\tau}b^ib^{\bar{j}} + \frac{PQ\epsilon}{\tau(1+P\gamma)}(b^i\eta^{\bar{j}} + \eta^i b^{\bar{j}}) + \frac{U^2\eta^i\eta^{\bar{j}} + UV(\eta^i b^{\bar{j}} + b^i\eta^{\bar{j}}) + V^2 b^i b^{\bar{j}}}{1+(U\gamma+V\epsilon)\sqrt{R}}\right], \tag{3.41}$$

Therefore, from equation (3.36) in (3.24) and the equation (3.32), then we obtained claims (i) and (ii) are desired.

## 4. Holomorphic curvature of Complex Finsler space with special $(\alpha, \beta)-$metric

The holomprphic curvature is the correspondent of the holomorphic sectional curvature in Complex Finsler geometry. Our goal is to find a notation of Complex Finsler spaces with special $(\alpha, \beta)$-metric. By anology with the naming from the real case [2], we shall call it the holomorphic flag curvature and we shall introduce it with respect to Chern-Finsler connection (c.n.c).

The holomorphic curvature $K_F(z, \eta)$ depends on the position $z \in M$ alone. In view of definition (2.1) we obtain the holomorphic curvature of Complex Finsler space with metric equation3.1 if $R_{i\bar{j}} = -g_{l\bar{j}}\delta_{\bar{h}}(N_k^l)\bar{\eta}^h$, where, $N_k^l$ is the Chern-Finsler connection coefficients.

To find Riemannian curvature $R_{\bar{j}k}$, we need the Chern Finsler connection(c.n.c) coefficients.

Now, by direct computations, we get the Chern-Finsler (c.n.c) connection coefficients;

$$\begin{aligned} N^l_K = & \frac{\alpha^4}{(\alpha+|\beta|)^2}\Bigg[a^{\bar{j}i} + \left(\frac{\alpha FB}{\alpha^3+F\alpha B\mu} + \frac{C(\alpha FB\epsilon)^2}{\delta(\alpha^3+\alpha FB\mu)^2}\right)\eta^i\eta^{\bar{j}} + \frac{C}{\delta}b^i b^{\bar{j}} \\ & + \frac{FBC\epsilon}{\delta(1+B\mu)}(b^i\eta^{\bar{j}} + b^{\bar{j}}\eta^i) + \frac{M^2\eta^i\eta^{\bar{j}} + MN(\eta^i b^{\bar{j}} + b^i\eta^{\bar{j}} + N^2 b^i b^{\bar{j}})}{L}\times \\ & \Bigg\{\frac{\alpha^3 A_1 - A(A(\frac{3}{2}\alpha\eta^i\eta^{\bar{j}}))}{\alpha^3}a_{i\bar{j}}F(A_2)\,l_i l_{\bar{j}} + 2\left(\frac{\alpha^4-\alpha^3 F}{\alpha}\right)A_3 b_i b_{\bar{j}} \\ & + \frac{(\alpha^4 - 2\alpha^3|\beta|)A_4' - a_4(A_5)}{(A_4)^2}\eta_i\eta_{\bar{j}}\Bigg\}\Bigg], \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} A &= 2\alpha^4 + 2|\beta|^4 + 2\alpha^2|\beta|^2 + 8\alpha^3|\beta| + 8\alpha|\beta|^3 - \alpha^5 + 3\alpha|\beta|^4 \\ & \quad -2\alpha^3|\beta|^2 - 3\alpha^4|\beta| - 4\alpha^2|\beta| + \alpha^5 \\ A_1 &= \Big(4\alpha^2 + 2|\beta|^2 + 12\alpha|\beta| + \frac{4|\beta|^3}{\alpha} - \frac{5\alpha^3}{2} + \frac{3}{2}\frac{|\beta|^4}{\alpha} - 3\alpha|\beta|^2 \\ & \quad +2\alpha^2|\beta| - |\beta|\Big)\eta^i\eta^{\bar{j}} + \left(8|\beta|^3 + 4\alpha^2|\beta| + 8\alpha^3 + 24\alpha^2|\beta|^2 + 2\alpha|\beta|^3 \right. \\ & \quad \left. -4\alpha^3|\beta| - 3\alpha^4 - 4\alpha^2 + 5|\beta|^4\right)\eta^i \\ A_2 &= (12\alpha^2 - 4\alpha + 3|\beta|^2 - \alpha|\beta| - 6\alpha^2|\beta| - 9|\beta| + \alpha + 2|\beta|)\eta^i\eta^{\bar{j}} \\ & \quad +(12 - 2\alpha^2 - 4\alpha^3 - 4|\beta|)\eta^i \\ A_3 &= (3\alpha^3 + \alpha|\beta|^2 + 3|\beta|)\eta^i\eta^{\bar{j}} + 12(|\beta|^3 + 3\alpha^2|\beta| + 3\alpha|\beta|)\eta^i \\ A_4 &= 4\alpha^7 - 2\alpha^4 + 4\alpha^5|\beta|^2 + 12\alpha^3|\beta|^3 + 8\alpha^6|\beta| + 16\alpha^4|\beta|^2 + 2\alpha^2|\beta|^4 \\ & \quad +4\alpha^5|\beta| - 2\alpha^4|\beta| + 8\alpha|\beta|^4 \\ A_4' &= (24\alpha^6 + 14\alpha^5 - 4\alpha^2 + 2|\beta|^4 + 4\frac{|\beta|^4}{\alpha} + |\beta| + 10\alpha^3|\beta|^2 \\ & \quad +18\alpha|\beta|^3 2\alpha^2|\beta|^2 + 20\alpha^3|\beta| + 4\alpha^2|\beta|)\eta^i\eta^{\bar{j}} \\ & \quad +(8\alpha^6 + 12\alpha^5 + 4\alpha + 2\alpha^4 + 32|\beta|^3 + 10|\beta|^4 + 36\alpha^3|\beta|^2 +)\eta^i \\ A_5 &= 2\alpha^3\eta^i + 3\alpha|\beta|\eta^i\eta^{\bar{j}}. \end{aligned}$$

Now using equation (4.1) and $(g^{\bar{j}i})$ on $R_{\bar{j}k}$(see definition(**??**) we get the Riemann curvature tensor $R_{\bar{j}k}$ as,

$$
\begin{aligned}
R_{\bar{j}k} &= \{A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11} + A_{12} \\
&+A_{13} + A_{14} + A_{15} + A_{16}\}\,\eta^i\bar{\eta}^j + \{A_{17} + A_{18} + A_{19} + A_{20} + A_{21} \\
&+A_{22} + A_{23}\}l_i l_{\bar{j}} + \{A_{24} + A_{25} + A_{26} + A_{27} + A_{28} + A_{29} + A_{30} \\
&+A_{31} + A_{32} + A_{33} + A_{34} + A_{35} + A_{36}\}b^i b^{\bar{j}} + \{A_{37} + A_{38} + A_{39} \\
&+A_{40} + A_{41} + A_{42} + A_{43} + A_{44} + A_{45} + A_{46} + A_{47} + A_{48} + A_{49}\}b^i\bar{\eta}^j \\
&+\{A_{50} + A_{51} + A_{52} + A_{53} + A_{54} + A_{55} + A_{56} + A_{57}\}\eta^i\bar{b}^j + \{A_{58} \\
&+A_{59} + A_{60} + A_{61} + A_{62}\}\eta_{\bar{j}} + \{A_{63} + A_{64} + A_{65} + A_{66} + A_{67}\}l_i\bar{b}^j \\
&+A_{68}l_i\bar{b}^j p_0|k + \{A_{69} + A_{70}\}p_0|k + A_{71}l_i|k + A_{72}\eta_{\bar{j}|k} + A_{73}p_2|k. \qquad (4.2)
\end{aligned}
$$

where,

$$
\begin{aligned}
A_1 &= \frac{\sigma_1}{1+\sigma_1\gamma}p_0 a_{\bar{j}|k}, & A_2 &= \frac{\sigma_1^2\sigma_2\epsilon^2}{\tau(1+\sigma_1\gamma)^2}p_0 a_{i\bar{j}|k}, \\
A_3 &= \frac{U^2 p_0}{[1+U\gamma+V\epsilon]\sqrt{\sigma_3}}a_{i\bar{j}|k}, & A_4 &= \frac{\sigma_1}{1+\sigma_1\gamma}l_i l_{\bar{j}} p_i|k, \\
A_5 &= \frac{\sigma_1^2\sigma - 2\epsilon^2}{\tau(1+\sigma_1\gamma)^2}l_i l_{\bar{j}} p_1|k, & A_6 &= \frac{U^2}{[1+(U\gamma+V\epsilon)]\sqrt{\sigma_3}}l_i l_{\bar{j}} p_1|k, \\
A_7 &= \frac{\sigma_1 p_1}{1+\sigma_1\gamma}l_i l_{\bar{j}|k}, & A_8 &= \frac{\sigma_1^2\sigma_2\epsilon^2 p_1}{\tau(1+\sigma_1\gamma)}l_{\bar{j}} l_{i|k}, \\
A_9 &= \frac{\sigma_2 p_1}{\tau}l_{\bar{j}} l_{i|k}, & A_{10} &= \frac{U^2 p_1}{[1+(U\gamma+V\epsilon)]\sqrt{\sigma_3}}l_{\bar{j}} l_{i|k}, \\
A_{11} &= \frac{\sigma_1 p_1}{1+\sigma_1\gamma}l_i l_{j|k}, & A_{12} &= \frac{\sigma_1^2\sigma_2\epsilon^2 p_1}{\tau(1+\sigma_1\gamma)^2}l_i l_{j|k}, \\
A_{13} &= \frac{U^2 p_1}{[1+(U\gamma+V\epsilon)]\sqrt{\sigma_3}}l_i l_{j|k}, & A_{14} &= \frac{\sigma_1 p_2}{(1+\sigma_1\gamma)}\eta_{\bar{j}}\eta_{i|k}, \\
A_{15} &= \frac{\sigma_1^2\sigma_2\epsilon^2 p_2}{\tau(1+\sigma_1\gamma)}\eta_{\bar{j}}\eta_{i|k}, & A_{16} &= \frac{U^2 p_2}{[1+(U\gamma+V\epsilon)]\sqrt{\sigma_3}}\eta_j\eta_{i|k}, \\
A_{17} &= a^{\bar{j}ip_{1|k}}, & A_{18} &= \frac{\sigma_2}{\tau}b^i\bar{b}^j p_{1|k}, \\
A_{19} &= \frac{\sigma_1\sigma_2\epsilon}{\tau(1+\sigma_1\gamma)}b^i\bar{\eta}^j p_{i|k}, & A_{20} &= \frac{\sigma_1\sigma_2\epsilon}{\tau(1+\sigma_1\gamma)}\bar{b}^j\eta^i p_{1|k}, \\
A_{21} &= \frac{UV}{[1+(U\gamma+V\epsilon)]\sqrt{\sigma_3}}\eta^i\bar{b}^j p_{1|k}, & A_{22} &= \frac{UV}{[1+(U\gamma+V\epsilon)]\sqrt{\sigma_3}}\bar{b}^j\eta^i p_{1|k}, \\
A_{23} &= \frac{V^2}{[1+(U\gamma+V\epsilon)]\sqrt{\sigma_3}}l_{\bar{j}} l_{i|k}, & A_{24} &= \frac{\sigma_2}{\tau}a_{i\bar{j}}, \\
A_{25} &= \frac{V^2}{[1+(U\gamma+V\epsilon)]\sqrt{\sigma_3}}p_{0|k}a_{i\bar{j}}, & A_{26} &= \frac{\sigma_2 p_0}{\tau}a_{i\bar{j}|k},
\end{aligned}
$$

$$A_{27} = \frac{V^2 p_0}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} a_{i\bar{j}|k}, \qquad A_{28} = \frac{V^2 p_1}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} l_{\bar{j}} l_{i|k},$$

$$A_{29} = \frac{\sigma_2 p_1}{\tau} l_i l_{\bar{j}|k}, \qquad A_{30} = \frac{V^2}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} l_i l_{\bar{j}|k},$$

$$A_{31} = \frac{\sigma_2 p_2}{\tau} \eta_{\bar{j}} \eta_{\bar{j}} \eta_{i|k}, \qquad A_{32} = \frac{V^2}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} \eta_{\bar{j}} \eta_{i|k},$$

$$A_{33} = \frac{p_2 \sigma_3}{\tau} \eta_i \eta_{j|k}, \qquad A34 = \frac{V^2 p_2}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} \eta^i \eta_{\bar{j}|k},$$

$$A_{35} = \frac{\sigma_2}{\tau} \eta_i \eta_{\bar{j}} p_{|k2}, \qquad A_{36} = \frac{V^2}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} \eta_i \eta_{\bar{j}} p_{2|k},$$

$$A_{37} = \frac{\sigma_1 \sigma_2 \epsilon}{\tau(1 + \sigma_1 \gamma)} p_{0|k}, \qquad A_{38} = \frac{UV}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} p_{0|k} l_i b^i,$$

$$A_{39} = \frac{\sigma_1 \sigma_2 \epsilon}{\tau(1 + \sigma_1 \gamma)} l_i l_{\bar{j}|k}, \qquad A_{40} = \frac{\sigma_1 \sigma_2 \epsilon p_0}{\tau(1 + \sigma_1 \gamma)} a_{i\bar{j}|k},$$

$$A_{41} = \frac{p_1 \sigma_1 \sigma_2 \epsilon}{\tau(1 + \sigma_1 \gamma)}, \qquad A_{42} = \frac{UVp_1}{[1 + (U\gamma + V\epsilon)]} l_{\bar{j}} l_{i|k},$$

$$A_{43} = \frac{UVp_1}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} l_i l_{\bar{j}|k}, \qquad A_{44} = \frac{\sigma_1 \sigma_2 p_2}{\tau(1 + \sigma_1 \gamma)} \eta_{\bar{j}} \eta_{i|k},$$

$$A_{45} = \frac{\sigma_1 \sigma_2 \epsilon p_2}{\tau(1 + \sigma_1 \gamma)} \eta_{\bar{j}} \eta_{i|k}, \qquad A_{46} = \frac{UVp_2}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} \eta_{\bar{j}} \eta_{i|k},$$

$$A_{47} = \frac{UVp_2}{[1 + (U\gamma + V\epsilon)]\sigma_3} \eta_i \eta_{j|k}, \qquad A_{48} = \frac{\sigma_1 \sigma_2 \epsilon}{\tau(1 + \sigma_1 \gamma)} \eta_i \eta_{\bar{j}} p_{2|k},$$

$$A_{49} = \frac{UV}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} \eta_i \eta_{\bar{j}} p_{2|k}, \qquad A_{50} = \frac{\sigma_1 \sigma_2 \epsilon p_0}{\tau(1 + \sigma_1 \gamma)} a_{i\bar{j}|k},$$

$$A_{51} = \frac{UVp_0}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} a_{i\bar{j}|k}, \qquad A_{52} = \frac{\sigma_1 \sigma_2 \epsilon p_1}{l}{}_{\bar{j}} l_{i|k},$$

$$A_{53} = \frac{UVp_1}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} l_{\bar{j}} l_{i|k}, \qquad A_{54} = \frac{\sigma_1 \sigma_2 \epsilon p_1}{\tau(1 + \sigma_1 \gamma)} l_i l_{\bar{j}|k},$$

$$A_{55} = \frac{UVp_1}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} l_i l_{\bar{j}|k}, \qquad A_{56} = \frac{UVp_2}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} \bar{\eta}^j \eta_{i|k},$$

$$A_{57} = \frac{\sigma_1 \sigma_2 \epsilon p_2}{\tau(1 + \sigma_1 \gamma)} \eta_i \eta_{\bar{j}|k}, \qquad A_{58} = \frac{\sigma_1 p_2}{1 + \sigma_1 \gamma} \eta_{\bar{j}|k},$$

$$A_{59} = \frac{\sigma_1^2 \sigma_2 \epsilon^2 p_2}{\tau(1 + \sigma_1 \gamma)^2} \eta_{\bar{j}|k}, \qquad A_{60} = \frac{U^2 p_2}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} \eta_{\bar{j}|k},$$

$$A_{61} = \frac{\sigma_1}{1 + \sigma_1 \gamma} \eta_{\bar{j}} p_{2|k}, \qquad A_{62} = \frac{\sigma_1^2 \sigma_2 \epsilon^2}{\tau(1 + \sigma_1 \gamma)^2} \eta_{\bar{j}} p_{2|k},$$

$$A_{63} = \frac{UVp_2}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} \eta_{\bar{j}|k}, \qquad A_{64} = \frac{\sigma_1 \sigma_2 \epsilon}{\tau(1 + \sigma_1 \gamma)} \eta_{\bar{j}} p_2||k,$$

$$A_{65} = \frac{UV}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} \eta_{\bar{j}} p_{2|k}, \qquad A_{66} = \frac{\sigma_1 \sigma_2 \epsilon p_2}{\tau(1 + \sigma_1 \gamma)} \eta_{\bar{j}|k},$$

$$A_{67} = \frac{UV}{[1 + (U\gamma + V\epsilon)]\sqrt{\sigma_3}} l_{\bar{j}} p_{0|k}, \qquad A_{68} = \frac{\sigma_1 \sigma_2 \epsilon}{\tau(1 + \sigma_1 \gamma)} l_i \bar{b}^j p_{0|k},$$

$$A_{69} = \frac{\sigma_1 \alpha^2}{1+\sigma_1 \gamma} p_{0|k}, \quad A_{70} = \frac{U^2 \alpha^2}{[1+(U\gamma + V\epsilon)]\sqrt{\sigma_3}} p_{0|k},$$
$$A_{71} = p_1 a^{\bar{j}} l_{\bar{j}} l_{i|k}, \quad A_{72} = p_2 l^{\bar{j}} \eta_{\bar{j}|k},$$
$$A_{73} = l^{\bar{j}} \eta_j p_{2|k}.$$

Therefore, all of above we can be stated as:

**Theorem 4.1.** The Ricci curvature of Complex FInsler space with special $(\alpha, \beta)$-metric $F = \alpha + \frac{|\beta|^2}{\alpha}$ is given by,

$$\begin{aligned} R_{\bar{j}k} = \; & \Im_0 \eta^i \bar{\eta}^j + \Im_1 l_i l_{\bar{j}} + \Im_2 b^i b^{\bar{j}} + \Im_3 b^i \bar{\eta}^j + \Im_5 \eta_{\bar{j}} + \Im_6 l_i \bar{b}^j + \Im_7 l_i \bar{b}^j p_{o|k} \\ & + \Im_8 p_{0|k} + \Im_9 l_{i|k} + \Im_{10} \eta_{\bar{j}|k}, \end{aligned} \tag{4.3}$$

where, $\Im_i$, $\;i = 0, 1, 2, 3......10$ are the terms of $A-$tensors.

**Remark 4.2.** Since from theorem (4.1) we have curvature $R_{\bar{j}k}$, this together with eqution (2.4), then obtain the holomorphic curvature of (3.1).

## 5. Conclusion

The concepts of holomorphic curvature is main role in Complex Finsler geometry. In this paper, we investigated the holomorphic curvature of complex Finsler-Square metric(i.e $F = \frac{(\alpha+|\beta|)^2}{\alpha}$) through the coefficients of Chern-Finsler connection. Finally, we proved the Complex Finsler space with special $(\alpha, \beta)-$metric is not a Käler and Weakly Käler.

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