Ricci Solitons and the Spacetime of General Relativity

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Abstract

The vector fields associated with Ricci flow and Ricci solitons in Riemann manifold has been studied and the correspondence between these vector fields and symmetries of spacetime manifold of general relativity has been established. The relationships between the symmetries of Petrov type D and N pure radiation fields and Ricci solitons have been explored. The solitons corresponding to Schwarzschild solution and Reissner-Nordstrom spacetime have been obtained.

1. Introduction

The Ricci flow (named after Gregorio Ricci-Curbastro) was introduced by Richard S. Hamilton in 1982 [1] to study compact three dimensional manifolds with positive Ricci curvature and he calls equation defining Ricci flow as evolution equation. Hamilton proved many important and remarkable results for Ricci flow, and laid the foundation for the programme to approach the Poincare's and Thurston's conjectures [Poincare's conjecture (Henry Poincare in 1904) is a theorem about the characterization of 3-sphere, which is the hypersurface that bounds the unit ball in 4-dimensional space. The conjecture states that: "Every simply connected closed 3-manifold is homeomorphic to the 3-sphere" (that is, the conjecture says that the 3-sphere is the only type of bounded three dimensional space possible that contains no holes). While Thurston's conjecture states that every closed 3-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structures (that is, a complete characterization of geometric structure on 3-dimensional manifold). Thus Thurston's conjecture extends Poincare's conjecture]. Hamilton's idea was to

define a kind of non-linear diffusion equation which would tend to smooth out singularities in the metric. The Ricci flow has been extensively studied since 1982. Some recent work has focussed on the question of precisely how higher dimensional Riemannian manifolds evolve under Ricci flow, and in particular, what types of parametric singularities may form.

The concept of Ricci solitons was introduced by Hamilton [2]. They are natural generalizations of Einstein metrics, which have been a subject of intense study in differential geometry and geometric analysis. Ricci solitons also correspond to special solutions of Hamilton's Ricci flow. They can be viewed as fixed points of the Ricci flow and that is why it is very important to understand the geometry of Ricci solitons and to classify them both topologically and geometrically. The motivation for the study of Ricci solitons comes from different problems. The ultimate aim of the different geometric evolution equations is to produce (or deduce the existence of) manifolds with an optimal behavior with respect to the given invariants: the Ricci flow makes it possible to construct Einstein metrics under certain conditions, whereas the mean curvature flow makes it possible to deform certain surfaces in other ones whose mean curvature is constant. However, there are conditions under which the initial structure does not evolve under the flow but remains as a fixed point of it. Ricci solitons are the geometric fixed points (modulo homotheties and diffeomorphisms) of the Ricci flow. Moreover, since they appear as singular models for the flow, analyzing their geometry is an important step towards an understanding of the Ricci flow itself. During the last two decades, a lot of work has been done on Ricci solitons (for a detailed survey of the work done on Ricci solitons, Cao [3]).

Motivated by the roles of Ricci flow and Ricci solitons in differential geoemetry, in this paper, we have made a detailed study of Ricci solitons on the spacetime of general relativity. In Section 2, we have discussed the concepts of Ricci flow and Ricci solitons along with some immediate consequences of the definition of Ricci solitons. The importance of Ricci solitons in the study of symmetries of the spacetime has been dealt in Section 3, and a number of results has been obtained for the spacetime of general relativity. The role of Ricci solitons in the study of Petrov type D and N pure radiation fields has been investigated in Section 4. In Sections 5 and 6, the geometry of Schwarzchild and Reissner-Nordström solitons, respectively has been discussed and it is seen that the solitons are responsible for the deformation in the metric and hence in the geometry of the spacetime under consideration.

2.   Ricci flow and Ricci solitons

The Ricci flow is a way of evolving a Riemannian metric over time t. If we consider the metric tensor (and the associated Ricci tensor) to be a function of a variable t, then the Ricci flow is defined by the partial differential equation (geometric evolution equation)

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \tag{1}$$

This equation may be considered as the "heat equation" for Riemanninan space.

An immediate consequence of Ricci flow is that:
"If the space is Ricci-flat (i.e., $R_{ij} = 0$) then $\dfrac{\partial g_{ij}}{\partial t} = 0$ and the Ricci flow leaves the metric unchanged. Conversely, any metric which is unchanged by Ricci flow is Ricci-flat."

In differential geometry, the Ricci flow is an intrinsic geometric flow (a process which deforms the metric of a Riemannian manifold). It is a kind of diffusion equation. To see how the evolution equation defining the Ricci flow is analogous to diffusion equation, consider a 2-dimensional metric (in exponential isothermal co-ordinate chart - as these co-ordinates provide an example of a conformal co-ordinate chart, because angles, but not the distances, are represented correctly) given by

$$ds^2 = e^{2p(x,y)}[dx^2 + dy^2] \tag{2}$$

Here

$$g_{11} = g_{22} = e^{2p}, g^{11} = g^{22} = e^{-2p}, g = e^{4p}, g_{ij} = g^{ij} = 0, i \neq j$$

and

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = p_x; \Gamma^1_{12} = \Gamma^1_{21} = -\Gamma^2_{11} = \Gamma^2_{22} = p_y$$

where $p_x = \dfrac{\partial p}{\partial x}$ and $p_y = \dfrac{\partial p}{\partial y}$. Using the definition of Riemann curvature tensor

$$R^i_{jkl} = -\frac{\partial \Gamma^i_{jk}}{\partial x^l} + \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \Gamma^a_{jk}\Gamma^i_{al} + \Gamma^b_{jl}\Gamma^i_{bk}$$

we have

$$R^1_{212} = -(p_{xx} + p_{yy}) = -\nabla^2 p$$

While the definition of Ricci tensor

$$R_{jk} = R^i_{jki} = -\frac{\partial \Gamma^i_{jk}}{\partial x^i} + \frac{\partial \Gamma^i_{ji}}{\partial x^k} - \Gamma^a_{jk}\Gamma^i_{ai} + \Gamma^b_{ji}\Gamma^i_{bk}$$

leads to

$$R_{11} = R_{22} = -\nabla^2 p$$

which can be expressed as

$$R_{xx} = R_{yy} = -\nabla^2 p$$

Thus

$$\frac{\partial p}{\partial t} = -\nabla^2 p = -R_{ij} \tag{3}$$

which is analogous to the heat equation (the best known of all diffusion equations)

$$\frac{\partial u}{\partial t} = \nabla^2 u \tag{4}$$

Soliton. A solution of an evolution equation that evolves along symmetries of the equation is called a *soliton* (or, self-similar solution).

If we imagine the Ricci flow as the flow of some vector field on the spaces of metrics, then the solitons may act as attractors to this flow. In case of 2-dimensional sphere, the flow is known to converge to a soliton.

To illustrate what we mean by "evolving along symmetries", consider one dimensional heat equation

$$f_t = f_{xx}$$

It can easily be seen that the following vector fields are infinitesimal symmetries of the equation - that is, each one generates a one-parameter group of transformations that transform solutions to solutions:

$$X_1 = \frac{\partial}{\partial x} \quad - \quad \text{translation in space}$$

$$X_2 = \frac{\partial}{\partial t} \quad - \quad \text{translation in time}$$

$$X_3 = f\frac{\partial}{\partial f} \quad - \quad \text{scaling in} f$$

$$X_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} \quad - \quad \text{scaling in space and time}$$

For the fundamental solution

$$f(x,t) = \frac{1}{\sqrt{t}}e^{-x^2/4t}$$

we see that

$$f(\lambda x, \lambda^2 t) = \lambda^{-2} f(x,t)$$

for any $\lambda > 0$ so this solution is a soliton that evolves along the vector field $X = X_4 - 2X_3$ [In fact, X is tangent to the graph of $f(x,t)$].

Einstein metric. A Riemannian metric is an *Einstein metric* if

$$R_{ij} = \lambda g_{ij} \tag{5}$$

for some constant λ. If $\lambda = \dfrac{R}{n}$, where $R = g^{ij} R_{ij}$ is the scalar curvature then the Riemannian manifold is an Einstein manifold.

Ricci soliton. A Ricci sliton is a natural generalization of Einstein metric and is defined on a (pseudo-) Riemannian manifold (M, g, ξ^i) by

$$R_{ij} - \frac{1}{2}\mathcal{L}_\xi \, g_{ij} = \lambda g_{ij} \tag{6}$$

where ξ^i is a smooth vector field on M, λ is a real number and $\mathcal{L}_\xi$ denotes the Lie derivative along the vector field ξ^i.

The presence of constant λ in equation (6) means that the metric is not fixed by the flow (up to a diffeomorphism) but is expanded/contracted by a scalar λ. This is similar to the fact that Einstein metrics (5) are not fixed points of the flow, they are fixed upto a scalar. They become fixed points for the normalized Ricci flow. Accordingly, a Ricci soliton is said to be *shrinking*, *steady* (or *translating*) and *expanding* if $\lambda > 0, \lambda = 0$ and $\lambda < 0$, respectively. If ξ is a gradient field, i.e., $\xi = \nabla f$, then equation (6) can be expressed as

$$R_{ij} - \nabla_i \nabla_j f = \lambda g_{ij} \tag{7}$$

and in this case, the Ricci soliton is said to be *gradient*. The smooth function f on M is called a *potential function* of the Ricci soliton. If ξ^i in equation (6) is zero then the soliton is trivial.

Some of the aspects, among many others (cf., [3]), that why we are interested in the study of Ricci solitons on a Riemanninan manifold are as follows:
(i) for a given vector field ξ^i find the nature of the Riemannian manifold.
(ii) find the nature of the vector field ξ^i if the properties of the Ricci tensor are given.

If M is the spacetime manifold of dimension four then some of the immediate consequences of equation (6) are as follows:

(a) If ξ^i is a Killing vector field [cf., equation (8)], then equation (6) leads to $R_{ij} = \lambda g_{ij}$ and the spcetime M is an Einstein manifold provided that $\lambda = \dfrac{R}{4}$.

(b) For a steady Ricci soliton (i.e., $\lambda = 0$), equation (6) yields

$$R_{ij} = \frac{1}{2}\mathcal{L}_\xi\ g_{ij}$$

and thus if ξ^i is a Killing vector field then the spacetime is Ricci-flat and conversely.

Remarks.

(i) It may be noted that the solution to Ricci-flat metric is the Schwarzchild exterior solution which describes the spacetime due to an isolated, static and spherically symmetric gravitating mass and the metric is

$$ds^2 = -(1 - \frac{2m}{r})^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + (1 - \frac{2m}{r})dt^2$$

If we consider $\lambda = \Lambda$ in equation (5) as the cosmological constant, then the corresponding Schwarzchild solution is

$$ds^2 = -(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3})^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + (1 - \frac{2m}{r} - \frac{\Lambda r^2}{3})dt^2$$

so that when $m = 0$ and $\Lambda = 0$, this solution reduces to flat spacetime whose metric is

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2$$

(ii) It is known that (cf., [4]) if a space has maximum number of Killing vectors then it is said to be *maximally symmetric space,* and is a space of constant curvature; and consequently an Einstein space. The significance of spaces of constant curvature is very well known in cosmology. To obtain a model of the universe, certain simplifying assumptions have to be made and one such assumption is that the universe is isotropic and homogeneous. This is known as *cosmological principle.* By *isotropy* we mean that all spatial directions are equivalent, while *homogeneity* means that it is impossible to distinguish one place in the universe from the other. This cosmological principle, when translated into the language of Riemannian geometry, asserts that the three dimensional position space is a space of maximal symmetry; and consequently a space of constant curvature whose curvature can depend upon time. In the spaces of constant curvature no points and no directions are preferred, that is, the spaces of constant curvature are isotropic and homogeneous and thus these spaces find their applications in

cosmology. The cosmological solutions of Einstein equations which contain a three dimensional space-like surface of a constant curvature are the Friedmann-Lemaitre-Robertson-Walker (FLRW) metrics, while a four dimensional space of constant curvature is the deSitter model of the universe. de Sitter universe possess a three dimensional space of constant curvature and thus belongs to Friedmann-Lemaitre-Robertson-Walker metrics.

3.   Symmetries of the spacetime and Ricci solitions

The construction of gravitational potential satisfying Einstein field equations is the principal aim of all investigations in the gravitational physics and this has often be achieved by imosing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. The geometrical symmetries of the spacetime are expressible through the vanishing of the Lie derivative of certain tensor with respect to a vector. The role of symmetries in general theory of relativity has been introduced by Katzin, Levine, Davis and co-workers in a series of papers ([5]-[8]). These symmetries, also known as collineations, were further studied by Ahsan ([9]-[13]), Ahsan and Ali [14], [15] and Ahsan and Husain [16]. So far more than twentysix different types of collineations have been studied and the literature on such collineation is very large and still expanding with results of elegance (cf., [13], [14] and [15]). Here we shall consider only those symmetries which are required for our investigation and we have

(i) Motion. A spacetime is said to admit motion if there exists a vector field ξ^i such that

$$\mathcal{L}_\xi \, g_{ij} = \xi_{i;j} + \xi_{j;i} = 0 \tag{8}$$

Equation (8) is known as Killing equation and the vector field ξ^i is called a Killing vector field.

(ii) Conformal motion (Conf M). If

$$\mathcal{L}_\xi \, g_{ij} = \sigma g_{ij} \tag{9}$$

where σ is a scalar, then the spacetime is said to admit conformal motion.

(iii) Special conformal motion (SCM). A spacetime is said to admit SCM if

$$\mathcal{L}_\xi \, g_{ij} = \sigma g_{ij}, \; \sigma_{;ij} = 0 \tag{10}$$

(iv) Homothetic motion (HM). A spacetime is said to admit homothetic motion if

$$\mathcal{L}_\xi \, g_{ij} = \sigma g_{ij} \tag{11}$$

where σ is a constant.

(v) Weyl conformal collineation (WCC). A spacetime is said to admit WCC if

$$\mathcal{L}_\xi \, C^i_{jkl} = 0 \tag{12}$$

where C^i_{jkl} is the Weyl conformal curvature tensor.

(vi) Curvature collineation (CC). If

$$\mathcal{L}_\xi \, R^i_{jkl} = 0 \tag{13}$$

then the spacetime is said to admit CC, where R^i_{jkl} is the Riemann curvature tensor.

(vii) Ricci collineation (RC). A spacetime is said to admit RC if

$$\mathcal{L}_\xi \, R_{ij} = 0 \tag{14}$$

where R_{ij} is the Ricci tensor.

(viii) Affine collineation (AC). If

$$\mathcal{L}_\xi \, \Gamma^i_{jk} = \xi^i_{;jk} + R^i_{jmk}\xi^m = 0 \tag{15}$$

then the spacetime is said to admit an AC.

(ix) Conformal collineation (Conf C). A spacetime is said to admit Conf C if there is vector field ξ^i such that

$$\mathcal{L}_\xi \, \Gamma^i_{jk} = \delta^i_j \sigma_{;k} + \delta^i_k \sigma_{;j} - g_{jk} g^{il} \sigma_{;l}, \ \sigma_{;jk} = 0 \tag{16}$$

(x) Special conformal collineation (S Conf C). If

$$\mathcal{L}_\xi \, \Gamma^i_{jk} = \delta^i_j \sigma_{;k} + \delta^i_k \sigma_{;j} - g_{jk} g^{il} \sigma_{;l}, \ \sigma_{;jk} = 0; \ \sigma_{;jk} = 0 \tag{17}$$

then the spacetime admits S Conf C along the vector field ξ^i.

We shall now investigate the role of Ricci solitons in the study of symmetries of the spacetime.

If ξ^i is a Killing vector field, equation (6) then reduces to

$$R_{ij} = \lambda g_{ij} \tag{18}$$

Taking the covariant derivative of both sides of this equation, we get

$$\nabla_k R_{ij} = 0 \tag{19}$$

We thus have

Theorem 1. If the vector field ξ^i associated with the Ricci soliton (M, g, ξ^i) is Killing then the manifold M is Ricci parallel.

Now taking the Lie derivative of equation (18), we get

$$\mathcal{L}_\xi \ R_{ij} = \lambda \ \mathcal{L}_\xi \ g_{ij}$$

We thus have

Theorem 2. If the vector field ξ^i associated with the Ricci soliton is Killing then the spacetime admits Ricci collineation.

From equations (6) and (8), we have

$$2R_{ij} = \xi_{i;j} + \xi_{j;i} + 2\lambda g_{ij} \tag{20}$$

Contracting this equation with g^{ij}, we get

$$R = \xi^i_{;i} + \lambda n = \theta + \lambda n \tag{21}$$

where $\theta = \xi^i_{;i}$ is the expansion scalar. We thus have

Corollary 1. If the Ricci soliton is steady then the scalar curvature is expanding.

Equation (21) can also be expressed as

$$\text{div } \xi^i = \xi^i_{;i} = R - \lambda n \tag{22}$$

where $R = g^{ij} R_{ij}$ is the Ricci scalar.. From equations (20) and (22), we have

$$n^{-1} R g_{ij} - R_{ij} = -\frac{1}{2} \mathcal{L}_\xi \ g_{ij} + n^{-1} \xi^i_{;i} g_{ij} \tag{23}$$

Now choosing λ as $(n^{-1}R)$ in equation (5), equations (23) and (9) lead to

Lemma 1 ([17]). The vector field associated with Ricci soliton (M, g, ξ^i) is conformally Killing if and only if M is an Einstein manifold.

It is known that [18]

$$\mathcal{L}_\xi \ \Gamma^i_{jk} = \frac{1}{2} g^{il} (\nabla_j \mathcal{L}_\xi \ g_{kl} + \nabla_k \mathcal{L}_\xi \ g_{lj} - \nabla_l \mathcal{L}_\xi \ g_{jk}) \tag{24}$$

If ξ^i is conformally Killing vector field, then equations (9) and (24) lead to

$$\mathcal{L}_\xi \, \Gamma^i_{jk} = \delta^i_j \sigma_{;k} + \delta^i_k \sigma_{;j} - g^{il} g_{jk} \sigma_{;l} \tag{25}$$

Thus we have

Lemma 2. If ξ^i is a conformal Killing vector field then it is also a conformal collineation vector field.

From Lemmas 1 and 2, we thus have

Theorem 3. A vector field ξ^i associated with a Ricci soliton is conformal collineation vector field if and only if M is an Einstein manifold.

While using equation (10), we have

Corollary 2. If $\sigma_{;jk} = 0$, the vector field associated with a Ricci soliton is special conformal collineation vector field if and only if the manifold is an Einstein manifold.

From equation (23), we have

$$R_{ij} - \frac{R}{n} g_{ij} = \frac{1}{2} \mathcal{L}_\xi \, g_{ij} - \frac{1}{n} \theta g_{ij}$$

where $\theta = \xi^i_{;i}$. Also, an Einstein space is characterized by $R_{ij} = \dfrac{R}{n} g_{ij}$ and thus we have

Theorem 4. For Einstein spaces, the vector field ξ^i associated with Ricci soliton is Killing if and only if it is expansion-free.

Moreover, if ξ^i defines a homothetic motion then by taking the covariant derivative of equation (11), we get

$$\nabla_k \mathcal{L}_\xi \, g_{ij} = 0$$

and equation (6) now yields

$$\nabla_k R_{ij} = 0$$

Thus we have

Theorem 5. If the vector field ξ^i associated with Ricci soliton defines a homothetic motion then the manifold is Ricci parallel.

It is known that [4] for the spacetime of general relativity, the Weyl conformal tensor C^i_{jkl} is given by

$$C^i_{jkl} = R^i_{jkl} + \frac{1}{2}(\delta^i_k R_{jl} - \delta^i_l R_{jk} + g_{jl}R^i_k - g_{jk}R^i_l) + \frac{R}{6}(\delta^i_l g_{jk} - \delta^i_k g_{jl}) \qquad (26)$$

where R^i_{jkl} is the Riemann curvature tensor. Also, we have ([18])

$$\mathcal{L}_\xi \ R^i_{jkl} = \nabla_j \mathcal{L}_\xi \Gamma^i_{kl} - \nabla_k \mathcal{L}_\xi \Gamma^i_{jl} \qquad (27)$$

which on using equation (25) leads to

$$\mathcal{L}_\xi \ R^i_{jkl} = -2\delta^i_{[j}\nabla_{k]}\sigma_{;l} - 2(\nabla_{[j}\sigma^{;i})g_{k]}l \qquad (28)$$

Contraction of this equation yields

$$\mathcal{L}_\xi R_{kl} = -(n-2)\nabla_k \sigma_{;l} - g_{kl}\nabla_m \sigma^{;m} \qquad (29)$$

Multiplying this equation by g^{kl}, we get

$$\mathcal{L}_\xi R = -2\sigma^l_{;l}R - 2(n-1)\nabla_m \sigma^{;m}$$

which for 4-dimensional spacetime reduces to

$$\mathcal{L}_\xi R = -2\phi R - 6\nabla_m \sigma^{;m} \qquad (30)$$

where $\phi = \text{div } \sigma$.

Now taking the Lie derivative of equation (26) and using equations (28) - (30), we get

$$\mathcal{L}_\xi C^i_{jkl} = 0 \qquad (31)$$

Thus, from equation (12), we have

Lemma 3. For the spacetime of general relativity every conformal Killing vector field is Weyl conformal collineation vector field.

From Lemmas 1 and 3, we thus have

Theorem 6. A vector field ξ associated with Ricci soliton is Weyl conformal collineation vector field if and only if the spacetime is an Einstein space.

Remark. For further details, see [19].

4. Petrov type D and N pure radiation fields and Ricci solitons

The study of Petrov type D gravitational field is an important activity in general relativity as most of the physically significant metrics belong to this type of field. The most familiar members of this class are Schwarzchild exterior

solution, Reissner-Nordström metric, Kerr solution, Gödel solution and Vaidya's metric of a radiating star. Recently Ahsan and Ali [14] have studied the symmetries of Petrov type D pure radiation fields and have established a number of relations between different types of collineations. From [14], we have

Lemma 4. In pure radiation type D fields, conformal motion, special conformal motion and homothetic motion all degenerate to motion.

Thus, from Lemmas 1 and 4, we have

Theorem 7. Type D pure radiation fields do admit motion along a vector field ξ^i associated to Ricci soliton (M, g, ξ^i) if and only if M is an Einstein space.

Moreover, for a Killing vector field, equation (6) yields

$$R_{ij} = \lambda g_{ij}$$

Now taking the Lie derivative of this equation with respect to ξ^i, we have

Theorem 8. Type D pure radiation fields admit Ricci collineation along a vector field ξ^i associated to Ricci soliton (M, g, ξ^i) if and only if M is an Einstein space.

Remark. A number of similar results can easily be obtained as motion implies almost all symmetries of the spacetime (cf., [5]).

While on the other hand, Petrov type N solutions of Einstein vacuum equations are amongst the most interesting, rather difficult and little explored of all empty spacetime metrics (cf., [20], [21]). From the physical point of view, they represent spacetimes filled up entirely with gravitational radiation while mathematically they form a class of solutions of Einstein equations which should be possible to determine explicitly. The behaviour of gravitation radiation from a bounded source is an important physical problem. Even reasonably far from the source, however, twisting type N solutions of the vacuum field equations are required for an exact description of that radiation. Such solutions would provide small laboratories in which to understand better the complete nature of singularities of type N solutions and could also be used to check numerical solutions that include the gravitational radiation (cf., [22]).

Different types of symmetries of Petrov type N gravitational fields have been the subject of interest since last few decades (cf., [23]). Recently, Ahsan and Ali [15] have focussed their attention on the interaction of pure electromagnetic

radiation field and pure gravitational radiation field, and have made a detailed investigation of different types of symmetries (collineations) for such radiation fields. These fields are termed as pure radiation fields.

From [15], we have

Lemma 5. For type N pure radiation fields, conformal motion, special conformal motion and homothetic motion all degenerate to motion.

Thus from Lemmas 1 and 5, we have

Theorem 9. Type N pure radiation fields admit motion along the vector field ξ^i associated to Ricci soliton (M, g, ξ^i) if and only if M is an Einstein space.

From equation (6) we also have

Theorem 10. A Killing vector field ξ^i associated to Ricci soliton (M, g, ξ^i) is a Ricci collineation vector field for type N pure radiation field if and only if M is an Einstein space.

From the definition of affine collineation and equation (24), we have

Theorem 11. Pure radiation type N fields admit affine collineation along a Killing vector ξ^i associated to Ricci soliton (M, g, ξ^i) if and only M is an Einstein space.

Moreover, from the definition of Lie derivative (cf., [4]), we have

$$\mathcal{L}_\xi R^i_{jkl} = \xi^h R^i_{jkl;h} - R^h_{jkl}\xi^i_{;h} + R^i_{hkl}\xi^i_{;j} + R^i_{jhl}\xi^h_{;k} + R^i_{jkh}\xi^i_{;l} \tag{32}$$

Now using the definition of Christoffel symbol, Killing vector field ξ^i and equation (13), we have

Theorem 12. A Killing vector field associated to Ricci soliton (M, g, ξ^i) is a curvature collineation vector field for type N pure radiation fields if M is an Einstein space.

Remark. Similar type of other results can easily be obtained for pure radiation type N fields (cf., [15]).

5. Schwarzschild soliton

Recently, Ali and Ahsan (cf., [24], [25]) have made a detailed geometric study of soliton corresponding to Schwarzschild exterior solution. The Schwarzchild

soliton is given by

$$ds^{*2} = -\Big(1 - \frac{2m}{r}\Big)^{\sqrt{2}} dt^2 + dr^2 + (r^2 - 2mr)(d\theta^2 + \sin^2\theta d\phi^2) \qquad (33)$$

Using the six dimensional formalism, they have obtained the eigen values ω_i $(i = 1, 2, .., 6)$ of the characteristic equation $\mid R_{AB} - \omega g_{AB} \mid = 0$. It is seen that the curvatures of the two and three dimensional surface of the Schwarzchild soliton are related to the eigen values which are expressed in terms of the physical parameters m and r. Moreover, for Schwarzchild soliton it is observed that not only the Gaussian curvature differ with that of Schwarzchild metric but also the dependence of curvature on eigen values is not similar. Thus, the deformation in metric of a spacetime is cause for change in geometry or gravitational field.

6.   Reissner-Nordström soliton

When the electromagnetic field is taken into consideration in Schwarzchild exterior solution we get Reissner-Nordström solution and the metric is

$$ds^2 = -\Big(\frac{r^2 + e^2 - 2mr}{r^2}\Big)dt^2 + \Big(\frac{r^2}{r^2 + e^2 - 2mr}\Big)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \ (34)$$

It has been shown by Ali and Ahsan [19] that the Reissner-Nordström soliton is given by

$$ds^{*2} = -\Big(\frac{r^2 + e^2 - 2mr}{r^2}\Big)^{\sqrt{2}} dt^2 + dr^2 + (r^2 - 2mr - e^2)(d\theta^2 + \sin^2\theta d\phi^2) \ (35)$$

We have

Case (i). When $e = 0$ (i.e., in the absence of charge), equation (34) reduces to Schwarzchild soliton given by equation (33).

Case (ii). When $m = 0$ and $e = 0$, equation (34) leads to

$$ds^{*2} = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2)$$

which is the soliton for the flat spacetime.

Remark. From the above discussions it may be noted that solitons are responsible for the deformation in the metric and hence in the geometry as well as gravitational field.

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