J. T. S.
Vol. 12 (2018), pp.85-92
https://doi.org/10.56424/jts.v12i01.10589

Study on Einstein-Sasakian Decomposable Recurrent Space of First Order

K. S. Rawat and Sandeep Chauhan

Department of Mathematics
H. N. B. Garhwal University Campus,
Badshahi Thaul, Tehri (Garhwal)-249199 (Uttarakhand), India
e-mail: drksrawathnbgu@gmail.com, sandeepschauhan687@gmail.com
(Received: February 13, 2018, Accepted: June 26, 2018)

Abstract

Takano [2] have studied decomposition of curvature tensor in a recurrent space. Sinha and Singh [3] have been studied and defined decomposition of recurrent curvature tensor field in a Finsler space. Singh and Negi studied decomposition of recurrent curvature tensor field in a Käehlerian space. Negi and Rawat [6] have studied decomposition of recurrent curvature tensor field in Käehlerian space. Rawat and Silswal [11] studied and defined decomposition of recurrent curvature tensor fields in a Tachibana space. Rawat and Kunwar Singh [12] studied the decomposition of curvature tensor field in Käehlerian recurrent space of first order. Further, Rawat and Chauhan [23] studied the decomposition of curvature tensor field in Einstein- Käehlerian recurrent space of first order.

In the present paper, we have studied the decomposition of curvature tensor fields R^h_{ijk} in terms of two non-zero vectors and a tensor field in Einstein-Sasakian recurrent space of first order and several theorem have been established and proved.

Key Words: Sasakian space, Einstein space, Einstein-Sasakian space, recurrent space, Curvature tensor, Projective curvature tensor.
2000 AMS Subject Classification :

1. Introduction

An n-dimensional Sasakian space S_n (or, normal Contact metric spaces) is a Riemannian space which admits a unit Killing Vector field η_i satisfying

Okumura [4]

$$\nabla_i \nabla_j n_k = \eta_j g_{ik} - \eta_k g_{ij} \tag{1.1}$$

It is well known that the Sasakian space is orientable and odd dimensional. Also, we know that an ndimensional Käehlerian space K_n is a Riemannian space, which admits a structure tensor field F_i^h satisfying

$$F_j^h F_h^i = -\delta_j^i, \tag{1.2}$$

$$F_{ij} = -F_{ji}, \qquad (F_{ij} = F_i^a g_{aj}) \tag{1.3}$$

and

$$F_{i,j}^h = 0 \tag{1.4}$$

where the comma (,) followed by an index denotes the operator of Covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian Curvature tensor field denoted by R_{ijk}^h, is given by

$$R_{ijk}^h = \partial_i \begin{Bmatrix} h \\ j\ k \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ i\ k \end{Bmatrix} + \begin{Bmatrix} h \\ i\ \alpha \end{Bmatrix} \begin{Bmatrix} \alpha \\ j\ k \end{Bmatrix} - \begin{Bmatrix} h \\ j\ \alpha \end{Bmatrix} \begin{Bmatrix} \alpha \\ i\ k \end{Bmatrix}, \tag{1.5}$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

The Ricci-tensor and the Scalar curvature in S_n are respectively given by

$$R_{ij} = R_{ij\alpha}^\alpha \qquad \text{and} \qquad R = R_{ij} g^{ij}. \tag{1.6}$$

If we define a tensor S_{ij} by

$$S_{ij} = F_i^\alpha R_{\alpha j}, \tag{1.7}$$

Then, we have

$$S_{ij} = -S_{ji}, \tag{1.8}$$

$$F_i^\alpha S_{\alpha j} = -S_{i\alpha} F_j^\alpha, \tag{1.9}$$

and

$$F_i^\alpha S_{jk,\alpha} = R_{ji,k} - R_{ki,j}. \tag{1.10}$$

It has been verified in Yano [10] that the metric tensor g_{ij} and the Ricci-tensor denoted by R_{ij} are hybrid in i and j. Therefore, we get

$$g_{ij} = g_{rs} F_i^r F_j^s, \tag{1.11}$$

and

$$R_{ij} = R_{rs} F_i^r F_j^s, \tag{1.12}$$

The holomorphically projective curvature tensor P_{ijk}^h is defined by

$$P_{ijk}^h = R_{ijk}^h + \frac{1}{(n+2)} (R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}S_i^h + 2S_{ij}F_k^h) \tag{1.13}$$

where $S_{ij} = F_i^a R_{aj}.$

Let us suppose that a Sasakian space is Einstein one, and then the Ricci tensor satisfies

$$R_{ij} = \frac{R}{n} g_{ij}, \qquad R_{,a} = 0$$

from which, we obtain

$$R_{ij,a} = 0, \qquad S_{ij,a} = 0$$

and

$$S_{ij} = \frac{R}{n} F_{ij}.$$

The Bianchi identity for Einstein-Sasakian space are given by

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0 \tag{1.14}$$

and

$$R^h_{ijk,a} + R^h_{ika,j} + R^h_{iaj,k} \tag{1.15}$$

The commutative formulae for the curvature tensor fields are given as follows:

$$T^i_{,jk} - T^i_{,kj} = T^a \, R^i_{ajk}, \tag{1.16}$$

$$T^h_{i,ml} - T^h_{i,lm} = T^a_i R^h_{aml} - T^h_a R^a_{iml}. \tag{1.17}$$

A Einstein-Sasakian space is said to be Einstein-Sasakian recurrent space of first order, if its curvature tensor field satisfy the condition

$$R^h_{ijk,a} = \lambda_a R^h_{ijk} \tag{1.18}$$

where λ_a is a non-zero vector and is known as recurrence vector field.

The following relations follow immediately from equation (1.14),

$$R_{ij,a} = \lambda_a R_{ij} \tag{1.19}$$

and

$$R_{,a} = \lambda_a R. \tag{1.20}$$

2. Decomposition of Recurrent Curvature Tensor R^h_{ijk}

Let us consider the decomposition of recurrent curvature tensor field R^h_{ijk} in the following way

$$R^h_{ijk} = \phi^h_\alpha A^\alpha \psi_{ijk} \tag{2.1}$$

where A^α, is a non-zero vector field and ϕ^h_α, ψ_{ijk} are two non-zero tensor fields, such that

$$\lambda_h \phi^h_\alpha = Q_\alpha \tag{2.2}$$

and

$$\lambda_h A^h = 1. \tag{2.3}$$

Here, Q_α is called non-zero decomposed vector field.

Now, we shall proof the following:

Theorem 2.1. Using the decomposition (2.1), the Bianchi identities for R^h_{ijk} is expressed as in the following way

$$\psi_{ijk} + \psi_{jki} + \psi_{kij} = 0 \quad (\psi_{ijk} = -\psi_{ikj}) \tag{2.4}$$

and

$$\lambda_a \psi_{ijk} + \lambda_j \psi_{ika} + \lambda_k \psi_{iaj} = 0. \tag{2.5}$$

Proof. By the equations (1.14), (1.15) and (2.1), we have

$$\phi^h_\alpha A^\alpha (\psi_{ijk} + \psi_{jki} + \psi_{kij}) = 0. \tag{2.6}$$

and

$$\phi^h_\alpha A^\alpha (\lambda_a \psi_{ijk} + \lambda_j \psi_{ika} + \lambda_k \psi_{iaj}) = 0. \tag{2.7}$$

The identities (2.4) and (2.5) follow immediately from these equation and the fact that $\phi^h_\alpha A^\alpha \neq 0$.

Theorem 2.2. The vector field A^α and tensor field R^h_{ijk}, R_{ij}, ψ_{ijk} satisfies the following relation through the decomposition (2.1):

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = Q_\alpha A^\alpha \psi_{ijk}. \tag{2.8}$$

Proof. By the help of Ricci identity, (1.18) and (1.19), we have

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik}, \tag{2.9}$$

multiplying equation (2.1) by λ_h and from the equation (2.2), we obtain

$$\lambda_a R^a_{ijk} = Q_\alpha A^\alpha \psi_{ijk}. \tag{2.10}$$

From equations (2.9) and (2.10), we get the required result (2.8).

Theorem 2.3. The quantities λ_a and field ϕ^h_α, act as recurrent vector and recurrent tensor fields respectively and their recurrent relations can be written as in the following way through the decomposition (2.1):

$$\lambda_{a,m} = \mu_m \lambda_a \tag{2.11}$$

and

$$\phi_(\alpha, m)^h = \mu_m \phi^h_\alpha. \tag{2.12}$$

Proof. Firstly differentiating equation (2.8) covariantly with w.r. to x^m, then using (2.1) and (2.8), we obtain the relation

$$\lambda_{a,m}\phi_\alpha^h A^\alpha \psi_{ijk} = \lambda_{i,m}R_{jk} - \lambda_{j,m}R_{ik}. \tag{2.13}$$

Transvecting equation (2.13) with λ_a and using the equation (2.9), we get

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik}) = \lambda_a(\lambda_{i,m}R_{jk} - \lambda_(j,m)R_{ik}). \tag{2.14}$$

Again, transvecting the equation (2.13) with λ_h, we get

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik})\lambda_h = \lambda_a\lambda_h(\lambda_{i,m}R_{jk} - \lambda_{j,m}R_{ik}). \tag{2.15}$$

Since, the right hand side of the equation (2.15) symmetric in a and h, then we have

$$\lambda_{a,m}\lambda_h = \lambda_{h,m}\lambda_a, \tag{2.16}$$

provided that

$$\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0.$$

The vector field $\lambda_a \neq 0$, then there exists a proportional vector μ_m, such that

$$\lambda_{a,m} = \mu_m\lambda_a.$$

Now, differentiating (2.2) w.r. to x^m and from the condition (2.11), we find

$$\lambda_h\phi_{\alpha,m}^h = (Q_{\alpha,m} - \mu_m Q_\alpha) \tag{2.17}$$

from the above equation, it is clear that

$$\lambda_h\phi_{\alpha,m}^h = \lambda_a\phi_(\alpha,m)^a. \tag{2.18}$$

Since, $\lambda_a \neq 0$ is recurrence vector field, then we can get a proportional vector field μ_m such that

$$\phi_{\alpha,m}^h = \mu_m\phi_\alpha^h$$

which complete the proof.

Theorem 2.4. The decomposition vector field Q_α and the tensor field ϕ_α^h, act as recurrent vector field and recurrent tensor fields respectively and their recurrent form can be written as in the following way through the decomposition (2.1):

$$Q_{\alpha,m} = 2\mu_m Q_\alpha \tag{2.19}$$

and

$$(\lambda_m - 2\mu_m)\,\psi_{ijk} = \psi_{i,k,m}. \tag{2.20}$$

Proof. Differentiating (2.2) covariantly w.r. to x^m, and using the equations (2.2), (2.11) and (2.12), we obtain the required result (2.19). Further, differentiating (2.1) covariantly w.r.to x^m, and using (1.18), (2.2), (2.11) and (2.12), we obtain the recurrent form (2.20).

Theorem 2.5. The curvature tensor R^h_{ijk} is equal to holomorphically projective curvature tensor field under the decomposition (2.1) if

$$\delta^h_j \psi_{ik} - \delta^h_i \psi_{jk} + \psi_{ak}(F^h_j F^a_i - F^h_i F^a_j) + 2F^h_k F^a_i \psi_{aj} = 0. \tag{2.21}$$

Proof. Contract the indices h and k in the relation (2.1), we have

$$R_{ij} = \phi^k_\alpha A^\alpha \psi_{ijk}. \tag{2.22}$$

In view of equation (2.22), we have

$$S_{ij} = F^a_i R_{aj} = F^a_i \phi^r_\alpha A^\alpha \psi_{ajr}. \tag{2.23}$$

Using the relations equations (2.22) and (2.23) in (2.23), we find

$$\Delta^h_{ijk} = \frac{\phi^r_\alpha A^\alpha}{(n+2)} \{\psi_{ikr}\delta^h_j - \psi_{jkr}\delta^h_i + \psi_{akr}(F^h_j F^a_i - F^h_i F^a_j) + 2F^h_k F^a_i \psi_{ajr}\}. \tag{2.24}$$

If in (2.22) $\Delta^h_{ijk} = 0$, then $P^h_{ijk} = R^h_{ijk}$, so equate the equation (2.24) to zero, we have

$$\psi_{ikr}\delta^h_j - \psi_{jkr}\delta^h_i + \psi_{akr}(F^h_j F^a_i - F^h_i F^a_j) + 2F^h_k F^a_i \psi_{ajr} = 0. \tag{2.25}$$

Multiplying the above equation by A^α and using the relation $\psi_{ijk}A^k = \psi_{ij}$, we get the required relation.

Theorem 2.6. Using the decomposition (2.1), the scalar curvature R satisfies the following relation:

$$\lambda_k R = g^{ij} Q_\alpha A^\alpha \psi_{ijk} \tag{2.26}$$

or

$$R_{,k} = g^{ij} Q_\alpha A^\alpha \psi_{ijk}.$$

Proof. Multiplying (2.22) by g^{ij} on both sides, we obtain

$$g^{ij} R_{ij} = g^{ij} \phi^k_\alpha A^\alpha \psi_{ijk} \tag{2.27}$$

$$R = g^{ij} \phi^k_\alpha A^\alpha \psi_{ijk}.$$

Now, transvecting (2.27), with λ_k and using (2.2), we get

$$\lambda_k R = g^{ij} Q_\alpha A^\alpha \psi_{ijk},$$

which completes the proof of the theorem.

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