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Symmetry of Finsler-Kropina Space with Very Special Relativity

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Abstract

The spacetime is defined by means of isometric group of transformations and Killing vectors of spacetime are the generators of isometric group[18]. In this paper, we derived the Killing vectors of Finsler-Kropina spacetime and we studied the symmetry of very special relativity.

Key Words: Finsler-Berwald space, Very Special relativity, Killing vectors, Lorentz violation, Isometric group.
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1. Introduction

Finsler geometry is an interesting branch of differential geometry and has active recently due to the excellent works of many geometers[1, 2, 3]. Notably, the model of gravity and cosmology based on Finsler geometry and many applications of Finsler geometry in the field of modern Physics and Biology be the very important motivation to study the geometry. Recently, in [5], D. Bao, C. Robles and Z. Shen used the Randers metrics in Finsler geometry to study the Zermelo Navigation on Riemannian manifolds. According to the E. Cartan the theory of Riemannian symmetric space plays a very important role in Riemannian geometry. Finsler geometry is a natural generalization of Riemannian geometry could provide new sight on modern physics.

Specially, the study of Lorentz invariance violation is the fundamental standard model of particle physics and is the study of possible spacetime symmetry. In the recent years, while investigating Lorentz invariance there are two sub branches developed in special relativity, they are doubly special relativity (DSR)([1] [2] [3] [17] [18]) and very special relativity (VSR) [6]. In DSR, planck-scale effects have been taken into consideration with two invariant parameters, they are planckian parameter k and speed of light c. In VSR, Coleman and Glashow developed the perturbative framework to investigate the effects of Lorentz invariant in quantum field theory.

Very interestingly, physicist showed that these sub divisions of special relativity have relation with Finsler geometry[12]. The isometric group of VSR is the subgroup of poincare group, which includes the spacetime translations and proper subgroups of Lorentz transformation. Along with that one more theory developed which is de-Sitter (dS)/Anti de-Sitter(AdS) invariant special relativity(dSSR) [10, 9], which states that the principle of relativity should be generalized to the spacetime with constant curvature having radius R in Riemannian manifold. In [8], Girelli, Liberati and Sinduni showed that the Modified Dispersion Relation(MDR) in DSR can be studied through the framework of Finsler geometry. Very recently, the authors Xin Li, Zhe Chang and Xiaohuon Mo [27], have studied isometric group of (α, β) type Finsler space and studied symmetry of Very Special Relativity. They also found that the Killing equation of various Finsler metrics such as Randers metric, Berwald metric, Funk metric, etc.

Recently, the authors Xin Li, Zhe Chang and Xiaohuon Mo[25], have studied Isometric group of (α, β) type Finsler space and the symmetry of very special relativity. They also found that the Killing vectors of Finsler-Funk space and proved that the 4 dimensional Finsler-Funk space with constant curvature has just 6 independent Killing vectors.

Thus, the symmetry of Finslerian space time is important for further study. Therefore, the way of describing spacetime symmetry in a covariant language i.e., the symmetry should not depend on any particular choice of coordinate system, involves the concept of isometric transformation. In fact, the symmetry of space time is described by the so called isometric group. The generators of isometric group is directly connected with the Killing vectors [16]. In this paper, we use solutions of the Killing equation to find the symmetry of a class of Finslerian spacetime. In particular, we derived the Killing vectors of generalized Finsler-Kropina space. We studied the symmetry of Finsler-Kropina space and symmetry of VSR metric.

2. Preliminaries

Let (M, F) be an n-dimensional Finsler space. We denote the tangent space at $x \in M$ by T_xM and the tangent bundle of M by TM. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. The formal definition of Finsler space as follows;

Definition 2.1. A Finsler space is a triple $F^n = (M, D, F)$, where M is an n-dimensional manifold, D is an open subset of a tangent bundle TM and F is a Finsler metric defined as a function $F : TM \to [0, \infty)$ with the following properties:
i. Regular: F is C^∞ on the entire tangent bundle $TM\backslash\{0\}$.
ii. Positive homogeneous: $F(x, \lambda y) = \lambda F(x, y)$.
iii. Strong convexity: The $n \times n$ Hessian matrix

$$g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$$

is positive definite at every point on $TM\backslash\{0\}$, where $TM\backslash\{0\}$ denotes the tangent vector y is non zero in the tangent bundle TM.

An important class of Finsler metrics is (α, β)-metrics [5]. The first Finsler space with (α, β)-metric were introduced by the Physicist G.Randers in 1940 are called Randers space.

Definition 2.2. A Finsler space $F^n = (M, F(x, y))$ is called with (α, β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to R$ is given by,

$$F^2(x, y) = L(\alpha(x, y), \beta(x, y)), \tag{2.1}$$

where $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric on M, and $\beta(x, y) = b_i(x)y^i$ is a 1-form on M.

Randers metric $L = \alpha + \beta$, Kropina metric $L = \frac{\alpha^2}{\beta}$, Matsumoto metric $L = \alpha + \beta + \frac{\beta}{\alpha}$, etc are the examples of Finsler (α, β)-metrics. If we neglect the function L to be homogeneous of order two with respect to the (α, β) variables, then we have a Lagrange space with (α, β)-metric. In this paper we consider the generalized Kropina metric $L = \frac{\alpha^(m+1)}{\beta^m}$.

Since, Finsler geometry is a natural generalization of Riemannian geometry, first we discuss about notion of the Killing vectors in Riemannian space [24].

Consider the coordinate transformation $x \to \bar{x}$, the Riemannian metric $g_{ij}(x)$ is defined as

$$\bar{g}_{ij}(\bar{x}) = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j} g_{kl}(x). \tag{2.2}$$

A transformation $x \to \bar{x}$ is called isometry if and only if the transformation of the metric $g_{ij}(x)$ satisfies $\bar{g}_{ij}(\bar{x}) = g_{ij}(x)$. Thus we have to verify the isometric transformations do form a group. Now it is sufficient to identify the isometric transformation under the infinitesimal coordinate transformation

$$\bar{x}^i = x^i + \epsilon V^i, \tag{2.3}$$

where $|\epsilon| \ll 1$. To first order in $|\epsilon|$, the equation (2.2) can be written as,

$$V^m \frac{\partial g_{ij}}{\partial x^m} + g_{mi}\frac{\partial V^m}{\partial x^j} + g_{mj}\frac{\partial V^m}{\partial x^i} = 0. \tag{2.4}$$

Then by making use of the notions of covariant derivatives with respect to Riemannian connection, we get,

$$V_{i|j} + V_{j|i} = 0, \tag{2.5}$$

where " | " denotes the covariant derivative. A vector field V_i satisfies equation (2.5) is called Killing vector. The group of isometric transformations directly connected to the Killing vectors. Thus, the problem of finding isometric group it is sufficient to find the dimension of the linear space of Killing vectors.

In Riemannian geometry, with the help of covariant derivatives Ricci identities given as

$$V_{k|i|j} - V_{k|j|i} = -V_l R^l_{kji}, \tag{2.6}$$

where R^l_{kji} is the Riemannian curvature tensor and the first Bianchi identity for the Riemannian curvature tensor given by

$$R^l_{kji} + R^l_{jik} + R^l_{ikj} = 0. \tag{2.7}$$

From equations (2.6) and (2.7), we obtain

$$V_{k|i|j} = V_l R^l_{jik}. \tag{2.8}$$

Clearly, all the derivatives of V_i can be expressed by the linear combinations of V_i and $V_{i|j}$. If the Killing vector V_i and $V_{i|j}$ at an arbitrary point on Riemannian space is given, then V_i and $V_{i|j}$ at any other point is determined by integration of the system of ordinary differential equations. Thus, the Killing vectors form a space of dimension $n(n+1)/2$ on n dimensional Riemannian space. If a metric admits that the maximum number $n(n+1)/2$ of Killing vectors, then its

Riemannian space must be homogeneous and isotropic and is called maximally symmetry space.

Example 1. Minkowskian space: The Killing equation (2.5) of a given Minkowskian metric $\eta_{ij}(x)$ becomes

$$\frac{\partial V_i}{\partial x^j} + \frac{\partial V_j}{\partial x^i} = 0. \tag{2.9}$$

The solution of (2.9) is

$$V^i = Q^i_j x^j + C^i, \tag{2.10}$$

where $Q_{ij} = \eta_{ki} Q^k_j$ is an arbitrary constant skew symmetric matrix and C^i is an constant vector.

Example 2. Spherical and hyperbolic: Spherical and hyperbolic metric is given as

$$g_{ij} = \left(\frac{\eta_{ij}}{1 + k(x.x)} - k \frac{x_i x_j}{(1 + k(x.x))^2} \right), \tag{2.11}$$

where $x_i \equiv \eta_{ij} x^j$. The Killing equation (4.5), now reads as

$$\frac{\partial V_i}{\partial x^j} + \frac{\partial V_j}{\partial x^i} + \frac{2k}{1 + k(x.x)} (x_i V_j + x_j V_i) = 0. \tag{2.12}$$

The solution of the above equation is

$$V^i = g^{ij} V_j = Q^i_j x^j + C^i + k(x.c) x^i, \tag{2.13}$$

where the index of Q and C are raised and lowered by Minkowskian metric μ^{ij} and its inverse matrix μ_{ij}.

3. Killing Vectors in Finsler-Kropina Space

In this section, we find the Killing vectors in the Finsler space, for this we should construct the isometric transformation of Finsler manifold. Let us consider the coordinate transformation

$$\bar{x}^i = x^i + \epsilon V^i, \tag{3.1}$$

and together with the corresponding transformation for y.

$$\bar{y}^i = y^i + \epsilon \frac{\partial V^i}{\partial x^j} y^j, \tag{3.2}$$

Under the coordinate transformation (3.1) and (3.2), to first order in $| \epsilon |$, we have the Finsler structure,

$$\bar{F}(\bar{x}, \bar{y}) = \bar{F}(x, y) + \epsilon V^i \frac{\partial F}{\partial x^i} + \epsilon y^j \frac{\partial V^i}{\partial x^j} \frac{\partial F}{\partial y^i}. \tag{3.3}$$

A Finsler structure is called isometry if and only if

$$F(x,y) = \bar{F}(x,y), \tag{3.4}$$

under the transformation (3.1) and (3.2). Then, deducing from (3.3), we obtain Killing equation $K_V(F)$ in Finsler space

$$K_V(F) = V^i \frac{\partial F}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j} \frac{\partial F}{\partial y^i} = 0. \tag{3.5}$$

Searching the Killing vectors for general Finsler manifold is a difficult task. Here, we give the Killing vectors for a class of Finsler space (α, β) space with metric defining as in [20]

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \tag{3.6}$$

where $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a differential one form, and $\phi(s)$ is a smooth function. In particular, consider Finsler-Kropina metric, given by

$$F = \frac{\alpha^2}{\beta}$$

. This can be written in the standard form $F = \alpha\phi(s)$, where $\phi(s) = 1/s$. Then, the Killing equation (3.5) in Kropina (α, β) space is given as follows

$$\begin{aligned} 0 &= K_V(\alpha)\phi(s) + \alpha K_V(\phi(s)), \\ &= \left(\phi(s) - s\frac{\partial \phi(s)}{\partial s}\right) K_V(\alpha) + \frac{\partial \phi(s)}{\partial s} K_V(\beta) \\ &= \frac{2\alpha}{\beta} K_V(\alpha) - \frac{\alpha^2}{\beta^2} K_V(\beta). \end{aligned} \tag{3.7}$$

By making use of the Killing equation (3.5), we obtain

$$K_V(\alpha) = \frac{1}{2\alpha}(V_{i|j} + V_{j|i}) y^i y^j, \tag{3.8}$$

$$K_V(\beta) = (V^i \frac{\partial b_j}{\partial x^i} + b_i \frac{\partial V_i}{\partial x^j}) y^j, \tag{3.9}$$

here " | " denotes the covariant derivative with respect to the Riemannian metric α.

Clearly, the coefficients in equation (3.7) are non zero. Therefore, $K_V(\alpha) = 0$ and $K_V(\beta) = 0$. Then we have solutions of the Killing equation are,

$$V_{i|j} + V_{j|i} = 0, \tag{3.10}$$

$$V^i \frac{\partial b_j}{\partial x^j} + b_i \frac{\partial V_i}{\partial x^j} = 0. \tag{3.11}$$

The first equation (3.10) is clearly the Riemannian Killing equation(4.5). The second equation (3.11) can be regarded as the constraint for the Killing vectors that satisfy the Killing equation (3.10). Therefore, in general, the dimension of the linear space formed by Killing vectors of Kropina (α, β) metric is lower than the Riemannian one.

Now, we investigated Killing vectors of Finsler space with generalized Kropina metric.

Theorem 3.1. Let $F = \frac{\alpha^{n+1}}{\beta^n}$ be the Finsler-Kropina metric, then Killing vectors of the Finsler-Kropina space are becomes

$$V^i = Q^i_j x^j + C^i, \tag{3.12}$$

with the constraint $b_i Q^i_j = 0$.

Proof. Now, consider the special generalized Finsler-Kropina metric given by [21];

$$F = \frac{\alpha^{n+1}}{\beta^n}, \tag{3.13}$$

where $\alpha = \sqrt{g_{ij} y^i y^j}$ and g_{ij} is the Riemannian metric tensor. This metric can be written as,

$$F = (g_{ij} y^i y^j)^{\frac{n+1}{2}} (\beta^{-n}), \tag{3.14}$$

Then by using the equation (3.7) we get the Killing equation as follows,

$$K_V(F) = V^i \frac{\partial F}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j} \frac{\partial F}{\partial y^i} = 0, \tag{3.15}$$

$$y^{nu} \frac{\partial V^\mu}{\partial x^\nu} [\frac{(1-n) y_\mu (b_\nu y^\nu)^n + n(\eta_{\mu\nu} y^\mu y^\nu)^{1/2} b_\mu (b_\nu y^\nu)^{n-1}}{(\eta_{\mu\nu} y^\mu y^\nu)^{(1+n)/2}}] = 0, \tag{3.16}$$

Then the solution of the above differential equation is gives the Killing vectors,

$$V^i = Q^i_j x^j + C^i. \tag{3.17}$$

And the second Killing equation (3.11) gives the constraint for Killing vector V^i,

$$b_i Q^i_j = 0. \tag{3.18}$$

4. Symmetry of Finsler Space with very Special Relativity

One important physical example of (α, β) space is VSR. When we take $\phi(s) = s^m$, where m is an arbitrary constant, the Finsler structure takes the form proposed by Gibbons et al[9].

$$\begin{aligned} F &= \alpha^{1-m}\beta^{m}, \\ &= (\eta_{ij}y^iy^j)^{(1-m)/2}(b_ky^k)^m, \end{aligned} \tag{4.1}$$

where η_{ij} is Minkowskian metric and b_k is a constant vector, the metric (4.1) is called VSR metric. One immediately obtain from the first Killing equation (3.10) of (α, β) space as,

$$V^i = Q^i_j x^j + C^i. \tag{4.2}$$

And the second Killing equation (3.11) gives the constraint for Killing vector V^i,

$$b_i Q^i_j = 0. \tag{4.3}$$

We use the following result proved by [15]:

Lemma 4.1. The VSR metric is invariant under the group of two-dimensional Euclidean motion ($E(2)$).

Here, we consider Finsler Kropina metric and proved the metric is invariant or symmetric with VSR metric as follws;

Theorem 4.2. The Finsler-Kropina metric is invariant under the group of two-dimensional Euclidean motion ($E(2)$).

In the previous section we obtained the Killing equations (3.10) and (3.11) for Kropina space are with respect to the arbitrary direction of y^i. It shows that there is no preferred direction exists in spacetime. Suppose the spacetime taken with a special direction, then the Killing equation (3.7) will have a special solution. Now consider the VSR metric, which is first suggested by Bogoslovsky[4] and suppose taking the null direction to be preferred direction, then we deduce from Killing equation (3.7) that

$$\begin{aligned} 0 &= s^m\left(\frac{1-n}{2\alpha}\left(\frac{\partial V_i}{\partial x^j}+\frac{\partial V_j}{\partial x^i}\right)y^iy^j + ms^{-1}b_k\frac{\partial V^k}{\partial x^r}y^r\right), \\ &= s^m\frac{1}{\alpha\beta}\left(\frac{1-n}{2}\left(\frac{\partial V_i}{\partial x^j}+\frac{\partial V_j}{\partial x^i}\right)b_r + m\eta_{ij}b^k\frac{\partial V_k}{\partial x^r}\right)y^iy^jy^r. \end{aligned} \tag{4.4}$$

This equation has a special solution

$$V_+ = (Q_{+-} + m\eta_{+-})x^- + C_+, \tag{4.5}$$

where Q_{+-} is an antisymmetrical matrix and it satisfies

$$b_- = -b^+Q_{+-}. \tag{4.6}$$

It implies that the Lorentz transformation for b^+ is

$$(\delta^+_+ + \epsilon(m\delta^+_+ + Q^+_+))b^+ = (1 + \epsilon(n+1))b^+, \tag{4.7}$$

this shows that the null direction b^+(or b_-) is invariable under the Lorentz transformation. Therefore, if the null direction of spacetime taken as a preferred direction, then the symmetry corresponding to Q_{+-} is restored. In this case, under the transformations of the group DISIM$_b(2)$ proposed by Gibbons et al[9] the VSR metric is invariant. Then, in Kropina space the Killing equation (3.7) reduces to

$$\frac{2\alpha}{\beta}K_V(\alpha) - \frac{\alpha^2}{\beta^2}K_V(\beta) = 0. \tag{4.8}$$

Since the $K_V(\alpha)$ contains irrational term of y^i and $K_V(\beta)$ only contains rational term of y^i, the equation (4.8) satisfies if and only if $K_V(\alpha) = 0$ and $K_V(\beta) = 0$. If Kropina space is flat, its Killing vectors satisfies the same Killing equation with VSR metric.

5. Conclusion

In the recent years Physicists developed two theories of relativity, such as Very Special Relativity(VSR) and Doubly Special Relativity(DSR), then many physicists and geometers found relation with the Finsler geometry. Girelli, Liberati and Sindoni [8, 9] showed that the modified dispersion relation (MDR) in DSR can be explained in terms of the framework of Finsler geometry. The symmetry of the MDR was studied under the Hamiltonian formalism. Gibbons, Gomis and Pope [9], showed that the Finslerian line element is invariant under the isometric transformation groups.

Thus, the symmetry of Finslerian spacetime is an interesting application of relativity. The spacetime symmetries are described under the group of isometric transformations independent of the choice of coordinate system. But, the generators of isometric group are related to the collection of Killing vector on the space[16]. In this paper we developed the isometric group of Finsler spacetime by determining the Killing vectors on the space with generalized Kropina metric. Then in the last section we deduced that the Killing vectors of Finsler-Kropina space are coincides with the Killing vectors of VSR metric and we showed that the Killing vectors are invariant under Lorentz transformation. Thus the Finsler-Kropina space is symmetric with Very Special Relativity.

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