J. T. S.
Vol. 13 (2019), pp.31-43
https://doi.org/10.56424/jts.v13i01.10602

On Lanczos Potential for Spherically Symmetric Spacetimes

Ravi Panchal[+], A. C. Patel[*] and A. H. Hasmani[**]

[+] Government Polytechnic, Valsad-396001, India.
[*]Department of Applied Sciences and Humanities, Hasmukh Goswami College of Engineering, Vahelal-382330, India
[**]Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar-388120, India
e-mail: [+]ravipanchal1712@gmail.com, [*]acck007@gmail.com, [**]ah_hasmani@spuvvn.edu
(Received: December 22, 2019, Accepted: April 23, 2020)

Abstract

By taking particular choice of unit timelike velocity vector, Lanczos potential for non-vacuum and non-static general spherically symmetric spacetime has been obtained. The results thus obtained are applied to some particular spherical symmetric spacetimes and a linear relationship between Lanczos scalars and spin coefficients has been obtained for these spacetimes.

Keywords: Lanczos Potential; Newman-Penrose Formalism; Spherically Symmetric Spacetimes.
AMS Classification: 83C20; 83C50; 83C60.

1. Introduction

Lanczos potential tensor L_{ijk} is a rank three tensor and we can produce rank four gravitational field tensor C_{hijk} by taking combinations of first order covariant derivative of L_{ijk}. This phenomena is analogous to the fact in electromagnetism that rank two electromagnetic field tensor F_{ij} can be obtained using rank one potential tensor A_i. So, study of Lanczos potential becomes important in relativity theory. Relationship between Weyl tensor and Lanczos potential tensor is known as Weyl-Lanczos relations. In order to obtain Lanczos potential for a given geometry, we require to solve Weyl-Lanczos relations

---

This paper is dedicated to Late Prof. Subhash Bhatt who has been a source for inspiration for research in Mathematics to all of us.

[21, 1]. Weyl-Lanczos relations are combinations of Lanczos potential tensor, its first order covariant derivatives and Weyl curvature tensor; which makes it difficult to solve them for an arbitrary metric.

For arbitrary perfect fluid spacetimes satisfying certain conditions, Novello and Velloso [21] have developed an algorithm to obtained Lanczos potential. In adopting this method for finding Lanczos potential, one of the challenges is to identify field of observer appropriate to given spacetime which satisfy those conditions. Dolan and Muratori [9] have unified the treatment of the Lanczos potential and the Ernst potential for vacuum stationary axially symmetric and cylindrically symmetric spacetimes by taking particular choice of tetrad and coordinate system. Edgar and Höglund [10] have proved that Lanczos potential for the Weyl candidate tensor does not generally exist for higher than four dimensions. Zund [28] have translated Weyl Lanczos relations into spinor version. By comparing Newman-Penrose (NP) form of Weyl-Lanczos relations and Newman-Penrose field equations, Parga, Chavoya and Bonilla [6] have obtained Lanczos potential for arbitrary Petrov type N, III or O spacetimes. Also, they have conjectured that, by choosing appropriate tetrad, it is possible to establish relationship between Lanczos scalars and spin coefficients. Ahsan [1] has presented a detailed study of the Lanczos potential in his monograph. Also many solutions of Weyl-Lanczos relations are obtained [2, 3, 4, 7, 8, 15, 17, 18, 25, 27]. Since, there is no standard method for finding Lanczos potential, a large class of solutions is still uncovered, particularly for non-vacuum spacetimes. We hope that it will be fruitful to find Lanczos potentials for unknown situations, which may lead us to Lanczos potential for arbitrary spacetimes. Due to this reason, Hasmani, Panchal and Patel [12, 13, 14] have obtained Lanczos potential for different non-vacuum spacetimes using different techniques. For more details related to Lanczos potential theory, refer Mora and Sánchez [19], Ahsan [1], O'Donnell [22], O'Donnell and Pye [24].

In this paper, using general observer method, we have obtained Lanczos potential for spherically symmetric spacetime of general nature which is Petrov type D, non-vacuum and non-static. Then, the obtained results are applied to Schwarzschild, Schwarzschild-de Sitter and Kantowski-Sachs spacetimes. For brevity, we have followed the terminology as used in [12, 13, 14].

2. Lanczos Potential using General Observer Quantities

We consider a unit timelike velocity vector u^i. Its covariant derivative can be decomposed as [11, 21]

$$u_{i;j} = a_i u_j + \frac{\theta}{3} h_{ij} + \sigma_{ij} + \omega_{ij}, \tag{2.1}$$

in which $a_i,\ \theta,\ h_{ij},\ \sigma_{ij}$ and ω_{ij} are known as acceleration, expansion, projection, shear and vorticity, respectively (the NP version of these kinematical quantities and the equations satisfied by them has been given in [1]).

These quantities satisfy following constraint equations

$$\frac{2}{3}\theta_{;i}\ h^i{}_j - (\sigma^p{}_q + \omega^p{}_q)_{;p}\ h^q{}_j - a^p\ (\sigma_{jp} + \omega_{jp}) = R_{ip}\ u^i\ h^p{}_j, \tag{2.2}$$

$$\omega^i{}_{;i} + 2\omega^i\ a_i = 0, \tag{2.3}$$

$$\frac{1}{2} h^\epsilon{}_{(i} h^p{}_{j)}\ \eta_\epsilon{}^{q\gamma\nu}\ u_\nu (\omega_{pq} + \sigma_{pq})_{;\gamma} - a_{(i}\omega_{j)} = -H_{ij} \tag{2.4}$$

along with following three dynamical equations

$$\dot{\theta} + \frac{\theta^2}{3} + \sigma^2 - \omega^2 - a^i{}_{;i} = R_{ij}\ u^i\ u^j, \tag{2.5}$$

$$\begin{aligned} & h_i{}^p\ h_j{}^q\ \dot{\sigma}_{pq} + \frac{1}{3} h_{ij} \left[\frac{1}{2}\omega^2 - \sigma^2 + a^p{}_{;p}\right] + a_i\ a_j - \frac{1}{2} h_i{}^p\ h_j{}^q\ a_{(p;q)} \\ & + \frac{2}{3}\theta\ \sigma_{ij} + \sigma_{ip}\ \sigma^p{}_j - \omega_i\ \omega_j = R_{ipjq}\ u^p u^q - \frac{1}{3} R_{pq}\ u^p u^q\ h_{ij}, \end{aligned} \tag{2.6}$$

$$h_i{}^p\ h_j{}^q\ \dot{\omega}_{pq} - \frac{1}{2} h_i{}^p\ h_j{}^q\ a_{[p;q]} + \frac{2}{3}\theta\ \omega_{ij} + \sigma_{ip}\ \omega^p{}_j - \sigma_{jp}\ \omega^p{}_i = 0, \tag{2.7}$$

where an overhead dot represents derivative projected in the u^i direction, H_{ij} is magnetic part of the Weyl tensor, $\sigma^2 = \sigma_{ij}\sigma^{ij}$ and $\omega^2 = \omega_{ij}\omega^{ij}$.

The Weyl tensor C_{hijk} can be generated from Lanczos potential tensor L_{ijk} using following set of equations, known as Weyl-Lanczos equations [1]

$$\begin{aligned} C_{hijk} \ = \ & L_{hij;k} - L_{hik;j} + L_{jkh;i} - L_{jki;h} + L_{(hk)} g_{ij} + L_{(ij)} g_{hk} - L_{(hj)} g_{ik} \\ & - L_{(ik)} g_{hj} + \frac{2}{3} L^{pq}{}_{p;q} (g_{hj} g_{ik} - g_{hk} g_{ij}), \end{aligned} \tag{2.8}$$

in this expression

$$L_{ij} = L_i{}^k{}_{j;k} - L_i{}^k{}_{k;j}. \tag{2.9}$$

Lanczos potential tensor L_{ijk} has following properties

$$\begin{aligned} L_{ijk} &= -L_{jik}, \\ L_{i\ t}^{\ t} &= 0, \quad (\text{or}, g^{kl}L_{kil} = 0) \\ L_{ijk} + L_{jki} + L_{kij} &= 0, \\ L_{ij\ ;k}^{\ \ k} &= 0. \end{aligned} \tag{2.10}$$

By projecting null tetrad on Lanczos potential tensor, we get following scalars which are known as Lanczos Potential scalars [1].

$$\begin{aligned} L_0 &= L_{ijk}l^i m^j l^k, & L_4 &= L_{ijk}l^i m^j m^k, \\ L_1 &= L_{ijk}l^i m^j \bar{m}^k, & L_5 &= L_{ijk}l^i m^j n^k, \\ L_2 &= L_{ijk}\bar{m}^i n^j l^k, & L_6 &= L_{ijk}\bar{m}^i n^j m^k, \\ L_3 &= L_{ijk}\bar{m}^i n^j \bar{m}^k, & L_7 &= L_{ijk}\bar{m}^i n^j n^k. \end{aligned} \tag{2.11}$$

Using equations (2.1)-(2.7) for Weyl-Lanczos relations (2.8), Novello and Velloso [21] have proved the following theorem.

Theorem 2.1. [21] If in a given spacetime there is a field of observers u^j that is shear-free and irrotational, then the Lanczos potential is given by

$$L_{ijk} = a_i u_j u_k - a_j u_i u_k \tag{2.12}$$

up to a gauge.

It may be noted that Newman-Penrose formalism [20, 1] and related formalisms [22, 28] have also played a considerable amount of role in the development of Lanczos potential theory.

3. Lanczos Potential for Spherically Symmetric Spacetimes

The line element for general spherically symmetric spacetime is

$$ds^2 = -e^{2\lambda}dr^2 - R^2 d\theta^2 - R^2 \sin\theta^2 d\phi^2 + e^{2\nu}dt^2, \tag{3.1}$$

where λ, ν and R are real functions of r and t. This spacetime is non-static and non-vacuum.

Chosen null tetrad corresponding to the spacetimes is

$$\begin{aligned} l^j =&\frac{1}{\sqrt{2}}e^{-\lambda}\delta^j_1+\frac{1}{\sqrt{2}}e^{-\nu}\delta^j_4, & n^j =&-\frac{1}{\sqrt{2}}e^{-\lambda}\delta^j_1+\frac{1}{\sqrt{2}}e^{-\nu}\delta^j_4,\\ m^j =&\frac{1}{\sqrt{2}R}\delta^j_2+i\frac{\csc\theta}{\sqrt{2}R}\delta^j_3, & \bar{m}^j =&\frac{1}{\sqrt{2}R}\delta^j_2-i\frac{\csc\theta}{\sqrt{2}R}\delta^j_3. \end{aligned} \quad (3.2)$$

We choose unit timelike vector as $u^j = \frac{(m^j-\bar{m}^j)}{\sqrt{2}}$ and due to this, non-vanishing components of acceleration vector are

$$a_1 = -\frac{R'}{R}, \quad a_2 = -\cot\theta, \quad a_4 = -\frac{\dot{R}}{R}. \quad (3.3)$$

Through out in this section, prime ($'$) denotes partial derivative with respect to r and overhead dot denotes partial derivative with respect to t.

The non-vanishing spin coefficients are given by

$$\begin{aligned} \rho =&-\frac{e^{-\nu}\dot{R}+e^{-\lambda}R'}{\sqrt{2}R}, & \mu =&\frac{e^{-\nu}\dot{R}-e^{-\lambda}R'}{\sqrt{2}R},\\ \epsilon =&\frac{e^{-\nu}\dot{\lambda}+e^{-\lambda}\nu'}{2\sqrt{2}}, & \gamma =&\frac{-e^{-\nu}\dot{\lambda}+e^{-\lambda}\nu'}{2\sqrt{2}},\\ \beta =&-\alpha=\frac{\cot\theta}{2\sqrt{2}R}. \end{aligned} \quad (3.4)$$

and the only non-vanishing NP Weyl scalars is

$$\begin{aligned} \Psi_2 =& -\frac{1}{6R^2}e^{-2(\lambda+\nu)}[e^{2\lambda}\{\dot{R}^2+R\dot{R}(-\dot{\lambda}+\dot{\nu})+R(-\ddot{R}+R(\dot{\lambda}^2-\dot{\lambda}\dot{\nu}+\ddot{\lambda}))\}\\ &+e^{2\nu}\{e^{2\lambda}-R'^2+RR'(-\lambda'+\nu')+R(R''+R((\lambda'-\nu')\nu'-\nu''))\}]. \end{aligned} \quad (3.5)$$

which shows that, the metric (3.1) is of Petrov type D.

The following are non-vanishing complex tetrad components of Ricci tensor,

$$\begin{aligned} \Phi_{00} =&\frac{1}{2R}e^{-2(\lambda+\nu)}[e^{2\lambda}\{-\dot{R}(\dot{\lambda}+\dot{\nu})+\ddot{R}\}-2e^{\lambda+\nu}(\dot{\lambda}R'+\dot{R}\nu'-\dot{R}')\\ &+e^{2\nu}\{-R'(\lambda'+\nu')+R''\}],\\ \Phi_{11} =&\frac{1}{4R^2}e^{-2(\lambda+\nu)}[e^{2\lambda}\{-\dot{R}^2+R^2(\dot{\lambda}^2-\dot{\lambda}\dot{\nu}+\ddot{\lambda})\}+e^{2\nu}\{e^{2\lambda}-R'^2\\ &+R^2(-\lambda'\nu'+\nu'^2+\nu'')\}], \quad (3.6)\\ \Phi_{22} =&\frac{1}{2R}e^{-2(\lambda+\nu)}[e^{2\lambda}\{-\dot{R}(\dot{\lambda}+\dot{\nu})+\ddot{R}\}+2e^{\lambda+\nu}(\dot{\lambda}R'+\dot{R}\nu'-\dot{R}')\\ &+e^{2\nu}\{-R'(\lambda'+\nu')+R''\}], \end{aligned}$$

$$\Lambda = \frac{1}{12R^2} e^{-2(\lambda+\nu)}[-e^{2\lambda}\{\dot{R}^2 + 2R\dot{R}(\dot{\lambda} - \dot{\nu}) + R(2\ddot{R} + R(\dot{\lambda}^2 - \dot{\lambda}\dot{\nu} + \ddot{\lambda}))\}$$
$$+ e^{2\nu}\{-e^{2\lambda} + R'^2 + 2RR'(-\lambda' + \nu') + 2RR'' + R^2(-\lambda'\nu' + \nu'^2 + \nu'')\}].$$

For the chosen unit timelike vector, the field of observer for metric (3.1) is shear free and irrotational. We use theorem (2.1) and to exhibit L_{ijk} in the Lanczos gauge, we consider

$$L_{ijk} = (a_i u_j u_k - a_j u_i u_k) - \frac{1}{3}(a_i g_{jk} - a_j g_{ik}). \quad (3.7)$$

Thus, we get non-zero independent components of the Lanczos potential tensor as follows

$$\begin{aligned} L_{121} &= \frac{1}{3} e^{2\lambda} \cot\theta, & L_{122} &= -\frac{1}{3} RR', \\ L_{133} &= \frac{2}{3} RR' \sin^2\theta, & L_{141} &= \frac{e^{2\lambda}\dot{R}}{3R}, \\ L_{144} &= \frac{e^{2\nu}R'}{3R}, & L_{233} &= \frac{1}{3} R^2 \sin 2\theta, \\ L_{242} &= \frac{1}{3} R\dot{R}, & L_{244} &= \frac{1}{3} e^{2\nu \cot\theta}, \\ L_{343} &= -\frac{2}{3} R\dot{R} \sin^2\theta. \end{aligned} \quad (3.8)$$

The Lanczos potential for the spacetime depends on r, θ and t coordinates, only. Now, using (2.11), we convert Lanczos potential tensor in Lanczos potential scalars and following are non-zero Lanczos potential scalars

$$\begin{aligned} L_1 &= \frac{e^{-\nu}\dot{R} + e^{-\lambda}R'}{6\sqrt{2}R}, & L_6 &= -\frac{e^{-\nu}\dot{R} - e^{-\lambda}R'}{6\sqrt{2}R}, \\ L_2 &= \frac{\cot\theta}{3\sqrt{2}R}, & L_5 &= -\frac{\cot\theta}{3\sqrt{2}R}, \\ L_3 &= \frac{e^{-\nu}\dot{R} - e^{-\lambda}R'}{2\sqrt{2}R}, & L_4 &= -\frac{e^{-\nu}\dot{R} + e^{-\lambda}R'}{2\sqrt{2}R}. \end{aligned} \quad (3.9)$$

The linear relationship between Lanczos scalars and spin coefficients is as follows

$$\begin{aligned} L_1 &= -\frac{1}{3}L_4 = -\frac{\rho}{6}, \\ L_6 &= -\frac{1}{3}L_3 = -\frac{\mu}{6}, \\ L_2 &= -L_5 = \frac{2}{3}\beta. \end{aligned} \quad (3.10)$$

Lanczos potential scalars obtained here are proportional to only three spin coefficients namely, ρ, μ and β. The following are some examples which supports these results.

3.1. **Schwarzschild Spacetime.** The Schwarzschild spacetime is given by

$$ds^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin\theta^2 d\phi^2 + \left(1-\frac{2m}{r}\right) dt^2. \tag{3.11}$$

For Schwarzschild metric, we chose null tetrad as

$$\begin{aligned} l^j =& \frac{1}{\sqrt{2}}\left(1-\frac{2m}{r}\right)^{-\frac{1}{2}} \delta^j_1 + \frac{1}{\sqrt{2}}\left(1-\frac{2m}{r}\right)^{\frac{1}{2}} \delta^j_4, \\ n^j =& -\frac{1}{\sqrt{2}}\left(1-\frac{2m}{r}\right)^{-\frac{1}{2}} \delta^j_1 + \frac{1}{\sqrt{2}}\left(1-\frac{2m}{r}\right)^{\frac{1}{2}} \delta^j_4, \\ m^j =& \frac{1}{\sqrt{2}R}\delta^j_2 + i\frac{\csc\theta}{\sqrt{2}R}\delta^j_3, \\ \bar{m}^j =& \frac{1}{\sqrt{2}R}\delta^j_2 - i\frac{\csc\theta}{\sqrt{2}R}\delta^j_3. \end{aligned} \tag{3.12}$$

For a choice of velocity vector

$$u^j = \frac{(m^j - \bar{m}^j)}{\sqrt{2}} = \left\{0, 0, \frac{i\csc\theta}{r}, 0\right\}, \tag{3.13}$$

the field of observer becomes expansion-free, shear-free and rotation-free; the acceleration vector is

$$a_j = \left\{-\frac{1}{r}, -\cot\theta, 0, 0\right\}. \tag{3.14}$$

The following are non-vanishing spin coefficients

$$\rho = \mu = -\frac{1}{\sqrt{2}r}\left(1-\frac{2m}{r}\right)^{\frac{1}{2}}, \epsilon = \gamma = \frac{m}{2\sqrt{2}r^2}\left(1-\frac{2m}{r}\right)^{-\frac{1}{2}}, \beta = -\alpha = \frac{\cot\theta}{2\sqrt{2}R}. \tag{3.15}$$

The only non-vanishing NP Weyl scalars is

$$\Psi_2 = -\frac{m}{r^3}. \tag{3.16}$$

Thus, the metric (3.11) is of Petrov type D. All components of complex tetrad components of Ricci tensor vanish. Using theorem (2.1) and equations (2.11), non-vanishing Lanczos potential scalars for Schwarzschild metric are given by

$$L_1 = -\frac{1}{3}L_3 = -\frac{1}{3}L_4 = L_6 = -\frac{\rho}{6},$$
$$L_2 = -L_5 = \frac{2}{3}\beta \tag{3.17}$$

or

$$L_1 = -\frac{1}{3}L_3 = -\frac{1}{3}L_4 = L_6 = \frac{1}{6\sqrt{2}r}\left(1-\frac{2m}{r}\right)^{\frac{1}{2}},$$
$$L_2 = -L_5 = \frac{\cot\theta}{3\sqrt{2}r}. \tag{3.18}$$

That is, the Lanczos potential scalars for Schwarzschild metric depend on θ and radial coordinate r.

3.2. **Schwarzschild-de Sitter Spacetime.** The Schwarzschild-de Sitter spacetime is given by

$$ds^2 = -\left(1-\frac{2m}{r}-\Lambda\frac{r^2}{3}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin\theta^2 d\phi^2 + \left(1-\frac{2m}{r}-\Lambda\frac{r^2}{3}\right)dt^2, \tag{3.19}$$

For Schwarzschild-de Sitter spacetime, we chose null tetrad as

$$\begin{aligned}
l^j &= \frac{1}{\sqrt{2}}\left(1-\frac{2m}{r}-\Lambda\frac{r^2}{3}\right)^{-\frac{1}{2}}\delta_1^j + \frac{1}{\sqrt{2}}\left(1-\frac{2m}{r}-\Lambda\frac{r^2}{3}\right)^{\frac{1}{2}}\delta_4^j,\\
n^j &= -\frac{1}{\sqrt{2}}\left(1-\frac{2m}{r}-\Lambda\frac{r^2}{3}\right)^{-\frac{1}{2}}\delta_1^j + \frac{1}{\sqrt{2}}\left(1-\frac{2m}{r}-\Lambda\frac{r^2}{3}\right)^{\frac{1}{2}}\delta_4^j,\\
m^j &= \frac{1}{\sqrt{2}R}\delta_2^j + i\frac{\csc\theta}{\sqrt{2}R}\delta_3^j,\\
\bar{m}^j &= \frac{1}{\sqrt{2}R}\delta_2^j - i\frac{\csc\theta}{\sqrt{2}R}\delta_3^j.
\end{aligned} \tag{3.20}$$

For a choice of velocity vector

$$u^j = \frac{(m^j - \bar{m}^j)}{\sqrt{2}} = \left\{0, 0, \frac{i\csc\theta}{r}, 0\right\}, \tag{3.21}$$

the field of observer becomes expansion-free, shear-free and rotation-free; non-vanishing components of acceleration vector are

$$a_1 = -\frac{1}{r}, \quad a_2 = -\cot\theta. \tag{3.22}$$

The following are non-vanishing spin coefficients

$$\rho=\mu=-\frac{1}{\sqrt{2}r}\left(1-\frac{2m}{r}-\Lambda\frac{r^2}{3}\right)^{\frac{1}{2}},\quad \epsilon=\gamma=\frac{3m-\Lambda r^2}{6\sqrt{2}r^2}\left(1-\frac{2m}{r}-\Lambda\frac{r^2}{3}\right)^{-\frac{1}{2}},$$
$$\beta=-\alpha=\frac{\cot\theta}{2\sqrt{2}R}. \tag{3.23}$$

The only non-vanishing NP Weyl scalars is

$$\Psi_2=-\frac{m}{r^3}. \tag{3.24}$$

Thus, for the metric (3.19) the Weyl tensor is of Petrov type D. Also, it is non-vacuum. Using (3.9) non-zero Lanczos potential scalars for Schwarzschild-de Sitter metric are as follows

$$L_1=-\frac{1}{3}L_3=-\frac{1}{3}L_4=L_6=-\frac{\rho}{6},$$
$$L_2=-L_5=\frac{2}{3}\beta. \tag{3.25}$$

That is,

$$L_1=-\frac{1}{3}L_3=-\frac{1}{3}L_4=L_6=\frac{1}{6\sqrt{2}r}\left(1-\frac{2m}{r}-\Lambda\frac{r^2}{3}\right)^{\frac{1}{2}},$$
$$L_2=-L_5=\frac{\cot\theta}{3\sqrt{2}r}. \tag{3.26}$$

This means, the Lanczos potential scalars for this metric depend on only two spin coefficients ρ and β.

3.3. **Kantowski-Sachs Spacetimes.** The Kantowski-Sachs metric is given by

$$ds^2=-X^2d\chi^2-Y^2(d\theta^2+\sin\theta^2)d\phi^2+dt^2, \tag{3.27}$$

where X and Y are function of t. For Kantowski-Sachs metric, we chose null tetrad as

$$l^j=\frac{1}{\sqrt{2}X}\delta_1^j+\frac{1}{\sqrt{2}}\delta_4^j,\qquad n^j=-\frac{1}{\sqrt{2}X}\delta_1^j+\frac{1}{\sqrt{2}}\delta_4^j,$$
$$m^j=\frac{1}{\sqrt{2}Y}\delta_2^j+i\frac{\csc\theta}{\sqrt{2}Y}\delta_3^j,\qquad \bar{m}^j=\frac{1}{\sqrt{2}Y}\delta_2^j-i\frac{\csc\theta}{\sqrt{2}Y}\delta_3^j. \tag{3.28}$$

For a choice of velocity vector

$$u^j=\frac{(m^k-\bar{m}^k)}{\sqrt{2}}=\left\{0,0,\frac{i\csc\theta}{Y},0\right\}, \tag{3.29}$$

the field of observer becomes expansion-free, shear-free and rotation-free; non-vanishing components of acceleration vector are

$$a_2 = -\cot\theta, \quad a_4 = -\frac{\dot{Y}}{Y}. \tag{3.30}$$

The following are non-vanishing spin coefficients

$$\rho = -\mu = -\frac{1}{\sqrt{2}}\frac{\dot{Y}}{Y}, \qquad \epsilon = -\gamma = \frac{1}{2\sqrt{2}}\frac{\dot{Y}}{Y}, \qquad \beta = -\alpha = \frac{1}{2\sqrt{2}}\frac{\cot\theta}{Y}. \tag{3.31}$$

The only non-vanishing component of Weyl scalar is

$$\Psi_2 = \frac{1}{6}\left(\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} - \frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} - \frac{1}{Y^2} - \frac{\dot{Y}^2}{Y}\right). \tag{3.32}$$

The non-zero components of complex tetrad components of Ricci tensor are

$$\begin{aligned}
\Phi_{00} = \Phi_{22} =& \frac{1}{2}\left(\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} - \frac{\ddot{Y}}{Y}\right),\\
\Phi_{11} =& \frac{1}{4}\left(\frac{1}{Y^2} + \frac{\dot{Y}^2}{Y^2} - \frac{\ddot{X}}{X}\right),\\
\Lambda =& \frac{1}{12}\left(2\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} + \frac{\ddot{X}}{X} + 2\frac{\ddot{Y}}{Y} + \frac{1}{Y^2} + \frac{\dot{Y}^2}{Y}\right).
\end{aligned} \tag{3.33}$$

Using (3.9) Lanczos potential for Kantowski-Sachs metric is given by

$$\begin{aligned}
L_1 = -\frac{1}{3}L_3 = -\frac{1}{3}L_4 = L_6 = -\frac{\rho}{6},\\
L_2 = -L_5 = \frac{2}{3}\beta.
\end{aligned} \tag{3.34}$$

That is,

$$\begin{aligned}
L_1 = -\frac{1}{3}L_3 = -\frac{1}{3}L_4 = L_6 = \frac{1}{6\sqrt{2}}\frac{\dot{Y}}{Y},\\
L_2 = -L_5 = \frac{1}{2\sqrt{2}}\frac{\cot\theta}{Y}.
\end{aligned} \tag{3.35}$$

For this spacetime, the Lanczos potential scalars depend only on ρ and β. Also, the potential depend on θ and t.

4. Conclusion

Lanczos potential for non-vacuum Petrov type D spherically symmetric spacetimes have been obtained. In all examples considered here, Lanczos potential depends on at most three of spin coefficients namely, ρ, μ and β. Nonlinearity involved in the Weyl-Lanczos relations, makes it is difficult to solve them and due to this reason there is no standard method for finding Lanczos potential for arbitrary metric. As a consequence, large class of solutions are still uncovered, especially of non-vacuum nature. It is hoped that, by exploring Lanczos potential for various spacetimes, we may reach to Lanczos potential for general situation. The results obtained here supports the conjuncture that there is linear relationship between Lanczos scalars and spin coefficients.

Acknowledgments

The authors are thankful to the learned anonymous referee for his valuable comments which enhanced the content of the manuscript.

References

[1] Ahsan, Zafar:The Potential of Fields in Einstein's Theory of Gravitation, Springer Nature Pvt. Ltd. Singapore 2019, ISBN:978-981-13-8975-7, doi:http://doi.org/10.1007/978-981-13-8976-4.

[2] Ahsan, Zafar and Bilal, Mohd.: *A Solution of Weyl-Lanczos Equations for Arbitrary Petrov Type D Vacuum Spacetimes*, International Journal of Theoretical Physics, 49 No. 11 (2010), 2713–2722, doi:10.1007/s10773-010-0464-5, http://dx.doi.org/10.1007/s10773-010-0464-5.

[3] Ahsan, Zafar and Bilal, Mohd.: *Lanczos Potential and Perfect Fluid Spacetimes*, International Journal of Theoretical Physics, 50 No. 6 (2011), 1752–1768, doi:10.1007/s10773-011-0684-3, http://dx.doi.org/10.1007/s10773-011-0684-3.

[4] Andersson, F. and Edgar, S. B.: *Spin Coefficients as Lanczos Scalars: Underlying Spinor Relations*, Journal of Mathematical Physics, 41 No. 5 (2000), 2990-3001, doi:http://dx.doi.org/10.1063/1.533285.

[5] Chandrasekhar, Subrahmanyan: title=The Mathematical Theory of Black Holes, Oxford University Press, New York, 1998.

[6] Ares de Parga, Gonzálo and Chavoya A., Oscar and López Bonilla, José L.: *Lanczos Potential*, Journal of Mathematical Physics, 30 No. 6 (1989), 1294-1295, doi:http://dx.doi.org/10.1063/1.528306, http://scitation.aip.org/content/aip/ journal/jmp/30/6/10.1063/1.528306.

[7] Dolan, P. and Kim, C. W.: *Some Solutions of the Lanczos Vacuum Wave Equation*, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 447 No. 1931 (1994), 577-585, doi:10.1098/rspa.1994.0156, http://rspa.royalsocietypublishing.org/content/447/1931/577, eprint:http://rspa.royal societypublishing.org/content/447/1931/577.full.pdf.

[8] Dolan, P. and Kim, C. W.: *The Wave Equation for the Lanczos Potential. I*, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 447 No. 1931 (1994), 557-575, doi:10.1098/rspa.1994.0155, http://rspa.royalsocietypublishing.org/content/447/1931/557, eprint:http://rspa.royal societypublishing.org/content/447/1931/557.full.pdf.

[9] Dolan, P. and Muratori, B. D.: *The Lanczos Potential for Vacuum Space-Times with an Ernst Potential*, Journal of Mathematical Physics, 39 No. 10 (1998), 5406-5420, doi:http://dx.doi.org/10.1063/1.532580, http://scitation.aip.org/ content/aip/journal/jmp/39/10/10.1063/1.532580.

[10] Edgar, S. Brian and Höglund, A.: *The Lanczos Potential for Weyl-Candidate Tensors Exists Only in Four Dimensions*, General Relativity and Gravitation, 32 No. 12 (2000), 2307–2318, doi:10.1023/A:1001951609641, http://dx.doi.org/10.1023/A:1001951609641.

[11] Ellis, George F. R.: *Republication of: Relativistic Cosmology*, General Relativity and Gravitation, 41 No. 3 (2009), 581–660, doi:10.1007/s10714-009-0760-7, http://dx.doi.org/10.1007/s10714-009-0760-7.

[12] Hasmani, A. H. and Panchal, Ravi: *Lanczos Potential for Some Non-Vacuum Spacetimes*, The European Physical Journal Plus, 131 No. 9 (2016), 1–6, doi:10.1140/epjp/i2016-16336-7, http://dx.doi.org/10.1140/epjp/i2016-16336-7.

[13] Hasmani, A. H., Patel, A. C. and Panchal, Ravi: *Lanczos potential for Weyl metric*, PRAJNA-E-Journal of Pure & Applied Sciences, Accepted, 24 (2017), 11–14.

[14] Panchal, Ravi: *FLRW Metric and Lanczos Potential*, GIT-Journal of Engineering and Technology, 11 (2018), 26.

[15] Holgersson, David: title=*Lanczos potentials for perfect fluid cosmologies*, Linköping University Thesis, 2004.

[16] Lanczos, C.: *The Splitting of the Riemann Tensor*, Rev. Mod. Phys., 34 (3) (1962), 379–389, doi:10.1103/RevModPhys.34.379", http://link.aps.org/doi/10.1103/RevModPhys.34.379.

[17] Maher, W. F. and Zund, J. D.: *A Spinor Approach to the Lanczos Spin Tensor*, Il Nuovo Cimento A (1965-1970), 57 No. 4 (1968), 638–648, doi:10.1007/BF02751371, http://dx.doi.org/10.1007/BF02751371.

[18] Mora, César and Sánchez, Rubén: *Lanczos Potential for the Van Stockung Space-Time*, International Journal of Theoretical Physics, 48 No. 5 (2008), 1357–1368, doi:10.1007/s10773-008-9907-7, http://dx.doi.org/10.1007/s10773-008-9907-7.

[19] Sánchez, César Moraand Rubén: *An heuristic review of Lanczos potential*, Lat. Am. J. Phys. Educ., 1 No. 1 (2007), 78.

[20] Newman, Ezra and Penrose, Roger: *An Approach to Gravitational Radiation by a Method of Spin Coefficients*, Journal of Mathematical Physics, 3 No. 3 (1962), 566-578, doi:http://dx.doi.org/10.1063/1.1724257, http://scitation.aip.org/content/aip/journal/jmp/3/3/10.1063/1.1724257.

[21] Novello, M. and Velloso, A. L.: *The Connection Between General Observers and Lanczos Potential*, General Relativity and Gravitation, 19 No. 12 (1987), 1251–1265, doi:10.1007/BF00759104, http://dx.doi.org/10.1007/BF00759104.

[22] O'Donnell, Peter: Introduction to 2-Spinors in General Relativity, World Scientific, 2003.

[23] O'Donnell, Peter: *A Method for Finding Lanczos Coefficients with the Aid of Symmetries*, Czechoslovak Journal of Physics, 54 No. 9 (2004), 889–896, doi:10.1023/B:CJOP.0000042642.14014.80, http://dx.doi.org/10.1023/B:CJOP.0000042642.14014.80.

[24] O'donnell, P. and Pye, H.: *A Brief Historical Review of the Important Developments in Lanczos Potential Theory*, Electronic Journal of Theoretical Physics, 7 No. 24 (2010), 327–350.

[25] Roberts, M. D.: *The Physical Interpretation of the Lanczos Tensor*, Il Nuovo Cimento B (1971-1996), 110 No. 10 (1995), 1165–1176, doi:10.1007/BF02724607, http://dx.doi.org/10.1007/BF02724607.

[26] Stephani, Hans and Kramer, Dietrich and MacCallum, Malcolm and Hoenselaers, Cornelius and Herlt, Eduard: Exact Solutions of Einstein's Field Equations, Cambridge University Press, 2009.

[27] Takeno, Hyöitiró: *On the Spintensor of Lanczos*, Tensor, 15 (1964), 103–119.

[28] Zund, J. D.: *The Theory of the Lanczos Spinor*, Annali di Matematica Pura ed Applicata, 104 No. 1 (1975), 239–268. doi:10.1007/BF02417018, http://dx.doi.org/10.1007/BF02417018.