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On Cyclic Ricci-recurrent Spaces

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(Dedicated to the memory of the late Professor Witold Roter)

Abstract

The object of the present paper is to study cyclic Ricci-recurrent spaces. Some basic geometric properties of such a space are obtained. Among others we study conformally symmetric cyclic Ricci-recurrent spaces. Also we study decomposibility and conformal deformation of cyclic Ricci-recurrent spaces. Finally, the existence of such space is ensured by an interesting example.

Keywords: Ricci symmetric, Ricci-recurrent, cyclic Ricci parallel, cyclic Ricci-recurrent, conformally symmetric, conformally flat space, scalar curvature, Codazzi tensor.
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1. Introduction

Let M^n be a Riemannian space of dimension n with Riemannian metric g. Then M^n is said to be Ricci symmetric or Ricci parallel if its Ricci tensor S_{ij} of type (0, 2) satisfies the condition $S_{ij,k} = 0$, where ',' denotes the covariant differentiation with respect to the metric tensor g. The class of Ricci parallel spaces is very natural generalization of the class of spaces of constant scalar curvature. Again generalizing the notion of Ricci parallel space, Patterson [5] introduced the notion of Ricci-recurrent space and later studied by Roter [6] and various authors. A Riemannian space is said to be Ricci-recurrent if S_{ij} satisfies the condition $S_{ij,k} = A_k S_{ij}$, where A_k is a nowhere vanishing 1-form. Again by the decomposition of the covariant derivative $S_{ij,k}$ of S_{ij}, Gray [4] introduced two important classes $\mathcal{A}$, $\mathcal{B}$, which lie between the class of Ricci-parallel spaces

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and the spaces of constant scalar curvature, namely (i) the class $\mathcal{A}$ is the class of spaces whose Ricci tensor is cyclic parallel and (ii) the class $\mathcal{B}$ is the class of spaces whose Ricci tensor is of Codazzi type. The spaces of class $\mathcal{A}$ are said to be cyclic Ricci parallel spaces. Generalizing the notion of cyclic Ricci parallel space, in the present paper we introduce the notion of *cyclic Ricci-recurrent space*. A Riemannian space M^n,$(n > 2)$ is said to be *cyclic Ricci-recurrent* if S_{ij} satisfies the condition

$$S_{ij,k} + S_{jk,i} + S_{ki,j} = A_k S_{ij} (\text{ or } = A_i S_{jk} \text{ or } = A_j S_{ki}), \tag{1.1}$$

where A_k is a nowhere vanishing 1-form associated to the vector field ρ such that $A_k = \rho^m g_{km}$.

The present paper is organized as follows. Section 2 deals with preliminaries. Section 3 is concerned with some basic properties of cyclic Ricci-recurrent spaces. It is proved that if the Ricci tensor of a cyclic Ricci-recurrent space is a Codazzi one then the scalar curvature of the space vanishes. Also it is shown that a cyclic Ricci-recurrent space is Ricci-recurrent if and only if its Ricci tensor is a Codazzi one.

In section 4, we investigate conformally symmetric cyclic Ricci-recurrent spaces and proved that a conformally symmetric cyclic Ricci-recurrent space is Ricci-recurrent if and only if its scalar curvature vanishes. Section 5 deals with decomposibility of cyclic Ricci-recurrent space and it is shown that a decomposable Riemannian space is cyclic Ricci-recurrent if and only if one of the decomposition spaces is cyclic Ricci-recurrent and the other is Ricci flat. Section 6 is devoted to the study of conformal deformation of cyclic Ricci-recurrent space. Finally, the last section provides the existence of a cyclic Ricci-recurrent space with vanishing scalar curvature.

2. Preliminaries

In the sequel each Latin index runs over $1, 2, \cdots, n$ and each Greek index runs over $2, 3, \cdots, n-1$ and we shall need the following results:

Transvecting (1.1) with g^{ij} and using the well known relation $S^r_{j,r} = \frac{1}{2} K_{,j}$ we get

$$2K_{,j} = K A_j, \quad A_r S^r_j = 2K_{,j}, \tag{2.1}$$

where K is the scalar curvature of the space.

Moreover, using the condition $R^r_{ijk,r} = S_{ij,k} - S_{ik,j}$ (which is a consequence of the second Bianchi identity) and (1.1) we obtain

$$S_{ij,l} + R^r_{ijl,r} + R^r_{jil,r} = 4S_{ij,l} - A_l S_{ij} \tag{2.2}$$

and

$$S_{il,j} + S_{lj,i} = A_l S_{ij} - S_{ij,l}. \tag{2.3}$$

From (1.1) we have

$$A_k S_{ij} = A_i S_{jk} \tag{2.4}$$

hence

$$S_{ij} = A_i \Theta_j \tag{2.5}$$

where $\Theta_j = \tau^k S_{kj}$ and τ is so choosen that $\tau^r A_r = 1$.

Since the Ricci tensor is symmetric, we get

$$S_{ij} = pA_i A_j, \tag{2.6}$$

where $p = \tau^r \Theta_r$. Hence we have

$$\text{rank } S_{ij} \leq 1. \tag{2.7}$$

We now suppose that the Ricci tensor does not vanish. Then putting

$$B_j = \sqrt{\epsilon p} A_j, \tag{2.8}$$

where $\epsilon = 1$ or -1, we obtain from (2.6) that

$$S_{ij} = \epsilon B_i B_j. \tag{2.9}$$

Lemma 2.1. Let us assume that B_i and T_{jk} are numbers satisfying

$$B_i T_{jk} + B_j T_{ik} = 0. \tag{2.10}$$

If not all B_i are zero then $T_{jk} = 0$ for $j, k = 1, 2, \cdots, n$.

Proof. Let $B_\alpha \neq 0$. Putting $i = j = \alpha$ in (2.10) we get $B_\alpha T_{\alpha k} = 0$, whence $T_{\alpha k} = 0$ for all k.

Now let $i = \alpha$ then we have $B_\alpha T_{jk} = 0$ and hence $T_{jk} = 0$ for all j and k.

3. Some basic properties of cyclic Ricci-recurrent Spaces

This section deals with various basic properties of cyclic Ricci-recurrent spaces.

Theorem 3.1. If the Ricci tensor of a cyclic Ricci-recurrent space M is a Codazzi one then the scalar curvature K of the space vanishes.

Proof. Obviously if at a point of M the Ricci tensor vanishes then at that point $K = 0$. Therefore suppose that $S_{ij} \neq 0$.

Since the Ricci tensor is of Codazzi type [3] then we have

$$S_{ij,k} = S_{ik,j} = S_{kj,i}, \tag{3.1}$$

which yields

$$K_{,j} = 0 \text{ for all } j. \tag{3.2}$$

Using (3.1) in (1.1) we obtain

$$S_{ij,k} = \frac{1}{3} A_k S_{ij}. \tag{3.3}$$

Differentiating (2.9) covariantly and using (3.3) we get

$$B_i B_{j,k} + B_j B_{i,k} = \frac{1}{3} A_k S_{ij}, \tag{3.4}$$

which can be written as

$$B_i (B_{j,k} - \frac{1}{6} A_k B_j) + B_j (B_{i,k} - \frac{1}{6} A_k B_i) = 0. \tag{3.5}$$

In view of Lemma 2.1 it follows from (3.5) that

$$B_{j,k} = \frac{1}{6} A_k B_j. \tag{3.6}$$

Again from (2.9) we get

$$K = \epsilon B^r B_r. \tag{3.7}$$

Differentiating (3.7) covariantly and using (3.2) we obtain

$$2\epsilon B^r B_{r,k} = 0. \tag{3.8}$$

Using (3.6) in (3.8) we get

$$\frac{1}{3} \epsilon A_k B^r B_r = 0. \tag{3.9}$$

Since $A_k \neq 0$ therefore from (3.7) and (3.9) we have $K = 0$ and hence the proof is complete.

We now assume that a cyclic Ricci-recurrent space is Ricci-recurrent and the Ricci tensor does not vanish at every point of a subset U of M. Then in view of (2.9) and

$$S_{ij,k} = \Phi_k S_{ij}, \tag{3.10}$$

we obtain

$$B_i B_{j,k} + B_j B_{i,k} = \Phi_k B_i B_j, \tag{3.11}$$

which yields by Lemma 2.1

$$B_i B_{j,k} = \frac{1}{2}\Phi_k B_j. \tag{3.12}$$

In view of (2.9) and (3.10) it follows from (1.1) that

$$B_i B_j \Phi_k + B_j B_k \Phi_i + B_i B_k \Phi_j = q B_i B_j B_k, \tag{3.13}$$

where $q = \frac{1}{\sqrt{\epsilon p}}$. But (3.13) can be written as

$$B_i B_j a_k + B_j B_k a_i + B_i B_k a_j = 0, \tag{3.14}$$

where $a_j = \Phi_j - \frac{1}{3} q B_j$.

Suppose now $a_\alpha \neq 0$ then (3.14) implies that $3B_\alpha B_\alpha a_\alpha = 0$ and hence $B_\alpha = 0$.

Moreover putting $k = \alpha$ in (3.14) and using the last result we obtain $a_\alpha B_i B_j = 0$, which yields $B_j = 0$, a contradiction. Thus a_α must be equal to zero and hence

$$\Phi_j = \frac{1}{3} q B_j. \tag{3.15}$$

Using (3.15) in (3.12) we get

$$B_{j,k} = \frac{1}{6} q B_j B_k \tag{3.16}$$

From (2.9) and (3.16) we have

$$S_{ij,k} = S_{ik,j}, \tag{3.17}$$

which implies that the Ricci tensor of a cyclic Ricci-recurrent space is at U a Codazzi one.

Suppose now that the Ricci tensor vanishes at some point x of M. Then from (3.10) we obtain

$$S_{ij,k} = 0 = S_{ik,j}, \tag{3.18}$$

that is, the Ricci tensor of a cyclic Ricci-recurrent space is at x a Codazzi one. Thus if a cyclic Ricci-recurrent space is Ricci-recurrent then the Ricci tensor of the space is Codazzi one. Also from (3.3) it follows that if the Ricci tensor of a cyclic Ricci-recurrent space is Codazzi one then the space is Ricci-recurrent. Hence we can state the following:

Theorem 3.2. A cyclic Ricci-recurrent space is Ricci-recurrent if and only if its Ricci tensor is a Codazzi one.

Moreover we now consider that A_j is locally a gradient, that is, $A_{i,j} = A_{j,i}$. Then there exists a function, say A, such that $A_{,j} = A_j$.

Define now ψ as follows:

$$\psi_j = e^{-\frac{1}{6}A} B_j. \tag{3.19}$$

Then in view of (3.6) we have

$$\psi_{j,k} = 0. \tag{3.20}$$

Also from (3.19) we get

$$\psi^r \psi_r = e^{-\frac{1}{3}A} B^r B_r = 0, \tag{3.21}$$

which is an immediate consequence of (2.9) and hence $K = 0$. Thus we can state the following:

Theorem 3.3. If the covector A of a cyclic Ricci-recurrent space is locally a gradient and its Ricci tensor does not vanish and it is a Codazzi one then the manifold is Ricci-recurrent and it admits locally a null parallel vector field.

4. Conformally Symmetric cyclic Ricci-recurrent Spaces

This section deals with conformally symmetric and conformally flat cyclic Ricci-recurrent spaces.

Theorem 4.1. Let (M, g) be conformally symmetric cyclic Ricci-recurrent space. Then M is Ricci-recurrent if and only if the scalar curvature of M vanishes.

Proof. The Weyl conformal curvature tensor C_{hijk} of type (0, 4) is given by

$$\begin{aligned} C_{hijk} &= R_{hijk} - \frac{1}{n-2}[S_{ij}g_{hk} - S_{hj}g_{ik} + S_{hk}g_{ij} - S_{ik}g_{hj}] \\ &+ \frac{K}{(n-1)(n-2)}[g_{ij}g_{hk} - g_{hj}g_{ik}] \end{aligned} \tag{4.1}$$

and hence

$$\begin{aligned} C_{hijk,l} &= R_{hijk,l} - \frac{1}{n-2}[S_{ij,l}g_{hk} - S_{hj,l}g_{ik} + S_{hk,l}g_{ij} - S_{ik,l}g_{hj}] \\ &+ \frac{K_{,l}}{(n-1)(n-2)}[g_{ij}g_{hk} - g_{hj}g_{ik}]. \end{aligned} \tag{4.2}$$

Since the space is conformally symmetric, we have [2]

$$C_{hijk,l} = 0 \tag{4.3}$$

and hence

$$C_{hijk,l} + C_{kijl,h} + C_{lijh,k} = 0. \tag{4.4}$$

In view of (4.2) and (1.1), (4.4) yields

$$R_{hijk,l} + R_{kijl,h} + R_{lijh,k} - \frac{1}{n-2}[A_l g_{ij} S_{hk} - g_{ik} S_{hj,l} + g_{hk} S_{ij,l} - g_{hj} S_{ik,l} - g_{il} S_{kj,h} - g_{kj} S_{il,h} - g_{ih} S_{lj,k} + g_{lh} S_{ij,k} - g_{lj} S_{ih,k})] + \frac{1}{(n-1)(n-2)} [K_{,l}(g_{ij} g_{hk} - g_{hj} g_{ik}) + K_{,h}(g_{ij} g_{kl} - g_{kj} g_{il}) + K_{,k}(g_{ij} g_{lh} - g_{ih} g_{lj})] = 0. \tag{4.5}$$

Transvecting (4.5) with g^{hk} and making use of Lemma 2.1, we get

$$4S_{ij,l} - A_l S_{ij} - \frac{1}{n-2}[2K_{,l} g_{ij} + nS_{ij,l} - \frac{1}{2}K_{,j} g_{il} - \frac{1}{2}K_{,i} g_{jl} + S_{ij,l} - A_l S_{ij}] + \frac{1}{(n-1)(n-2)}[(n+1)K_{,l} g_{ij} - K_{,i} g_{lj} - K_{,j} g_{il}] = 0, \tag{4.6}$$

whence, by a quite elementary computation, we find

$$(n-3)[3(n-1)S_{ij,l} - (n-1)A_l S_{ij} - K_{,l} g_{ij} + \frac{1}{2}K_{,j} g_{il} + \frac{1}{2}K_{,i} g_{jl}] = 0. \tag{4.7}$$

Consequently we have finally

$$3(n-1)S_{ij,l} - (n-1)A_l S_{ij} - K_{,l} g_{ij} + \frac{1}{2}K_{,j} g_{il} + \frac{1}{2}K_{,i} g_{jl} = 0. \tag{4.8}$$

If $K_{,j} = 0$ or by (2.1), $K = 0$ then from (4.8) we get

$$S_{ij,l} = \frac{1}{3}A_l S_{ij}, \tag{4.9}$$

that is the space is Ricci-recurrent. Conversely, if the space is Ricci-recurrent then (4.9) holds and using (4.9) in (4.8) we obtain $K_{,j} = 0$ and hence from (2.1) we have $K = 0$. This completes the proof.

Corollary 4.1. Let M be conformally flat cyclic Ricci-recurrent space. Then M is Ricci-recurrent if and only if the scalar curvature of M vanishes.

Proof. Evidently every conformally flat space is conformally symmetric. Hence the relation (4.8) holds for conformally flat cyclic Ricci-recurrent spaces. Thus every conformally flat cyclic Ricci-recurrent space with vanishing scalar curvature is Ricci-recurrent. Conversely, if a conformally flat and cyclic Ricci-recurrent space is Ricci-recurrent then, which follows from (4.8), the space must have a vanishing scalar curvature.

Theorem 4.2. *(i)* In a conformally symmetric cyclic Ricci-recurrent space with vanishing scalar curvature, coordinates can be locally choosen so that the metric takes the form

$$ds^2 = \Omega(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1 dx^n, \quad (4.10)$$

$$\Omega = (Gk_{\lambda\mu} + a_{\lambda\mu})x^\lambda x^\mu, \quad (4.11)$$

where $[k_{\lambda\mu}]$ is a symmetric and non-singular matrix of constants, $a_{\lambda\mu}$ is a symmetric matrix of constants satisfying $k^{\lambda\omega}a_{\lambda\omega} = 0$ with $[k^{\lambda\omega}] = [k_{\lambda\omega}]^{-1}$ and G is a non-zero and non-constant function of x^1 only.

(ii) Let $\mathbb{R}^n$ be endowed with the metric satisfying *(4.10)* and *(4.11)*, where $[k_{\lambda\mu}]$ and $[a_{\lambda\mu}]$ are as above and G is a function of x^1 only such that $0 \neq G \neq$ constant. Then $\mathbb{R}^n$ is conformally symmetric and cyclic Ricci-recurrent. Moreover its scalar curvature vanishes.

Proof. Assume that the scalar curvature of a conformally symmetric cyclic Ricci-recurrent space M vanishes. Then by Theorem 4.1, the space M is Ricci-recurrent with recurrence vector field $\tau_j = \frac{1}{3}A_j$.

Here and in the sequel we assume that the Ricci tensor does not vanish.

Adati and Miyazawa [1] proved that in a conformally symmetric Ricci-recurrent space τ_j is locally a gradient. Hence there exists a function, say τ such that $\tau_{,j} = \tau_j$.

Define now ψ as follows:

$$\psi_j = e^{-\frac{1}{2}\tau}B_j, \quad (4.12)$$

where B_j satisfies (3.12). But by Theorem 3.3 it follows that ψ_j is parallel and null. Therefore M admits locally a null parallel vector field.

From (2.6) and (2.9) we have

$$\epsilon B_i B_j = pA_i A_j,$$

whence $A_j = \sigma B_j$. Hence by (4.12) and $\tau_j = \frac{1}{3}A_j$ we get $\tau_j = \alpha\psi_j$. Therefore the recurrence vector field τ_j is codirectional with a parallel null vector field ψ_j.

From [6] it follows that the curvature tensor of a conformally symmetric Ricci-recurrent space has locally the form

$$R_{jhkm} = \tau_h\tau_k S_{mj} - \tau_h\tau_j S_{mk} + \tau_m\tau_j S_{hk} - \tau_m\tau_k S_{hj}, \quad (4.13)$$

where

$$S_{ij} = S_{ji} = a^r a^s R_{rijs} \text{ and } a^r\tau_r = 1.$$

Again Walker ([7],[8]) proved that if a pseudo-Riemannian space with the curvature tensor of the form (4.13) admits a null parallel vector field ψ^i satisfying $\tau_j = \alpha\psi_j$ then one can choose coordinates so that the metric can be written as

$$ds^2 = \theta(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1dx^n, \tag{4.14}$$

where $k_{\lambda\mu}$ are constants, $\det[k_{\lambda\mu}] \neq 0$ and θ is independent of x^n.

In this coordinate system the null parallel vector field is of the form $\psi^i = \delta^i_n$, whence in view of $\tau_j = \alpha\psi_j$, we have

$$\tau_j = \alpha g_{ij}\psi^i = \alpha g_{jn} = \alpha\delta^1_j.$$

The recurrence vector τ_j is therefore a gradient of some function $\tau(x^1)$ and so α is a function of x^1 only.

In the metric (4.14) the only components of R and S not identically zero are these related to

$$R_{1\lambda\mu1} = \frac{1}{2}\theta_{.\lambda\mu}, \quad S_{11} = \frac{1}{2}k^{\beta\omega}\theta_{.\beta\omega}, \tag{4.15}$$

where the dot denotes partial differentiation with respect to coordinates.

Moreover one can easily show that

$$C_{1\lambda\mu1} = \frac{1}{2}\theta_{.\lambda\mu} - \frac{1}{2(n-2)}k_{\lambda\mu}k^{\beta\omega}\theta_{.\beta\omega},\ R_{1\lambda\mu1,j} = \frac{1}{2}\theta_{.\lambda\mu j} \tag{4.16}$$

and

$$C_{1\lambda\mu1,j} = \frac{1}{2}\theta_{.\lambda\mu j} - \frac{1}{2(n-2)}k_{\lambda\mu}k^{\beta\omega}\theta_{.\beta\omega j},\ S_{11,j} = \frac{1}{2}k^{\beta\omega}\theta_{.\beta\omega j}. \tag{4.17}$$

All other components of C and the covariant derivative of S, R and C are identically zero.

Since the space is, by assumption, conformally symmetric and Ricci-recurrent we obtain

$$\theta_{.\lambda\mu j} = \frac{1}{n-2}k_{\lambda\mu}(k^{\beta\omega}\theta_{.\beta\omega})_{.j}, \tag{4.18}$$

$$(k^{\beta\omega}\theta_{.\beta\omega})_{.j} = \alpha\delta^1_j k^{\beta\omega}\theta_{.\beta\omega}. \tag{4.19}$$

From (4.18) and (4.19) we have

$$\theta_{.\lambda\mu} = 2Gk_{\lambda\mu} + 2a_{\lambda\mu}, \tag{4.20}$$

where $a_{\lambda\mu}$ are constants such that $k^{\beta\omega}a_{\beta\omega} = 0$ and G is a function of x^1 only. Hence

$$\theta = Gk_{\lambda\mu}x^\lambda x^\mu + a_{\lambda\mu}x^\lambda x^\mu + k_\lambda x^\lambda + \chi, \tag{4.21}$$

k and χ being functions of x^1 only.

Consider now a transformation [7] of the form

$$x'^{\lambda} = x^{\lambda} - k^{\lambda\mu}\sigma_{\mu}, \quad x'^{n} = x^{n} + \rho_{\lambda}x^{\lambda} + \eta \tag{4.22}$$

from $x^2, x^3, \cdots, x^n$, $x'^2, x'^3, \cdots, x'^n$, where ρ_λ, σ_λ and η are functions of x^1 satisfying

$$\rho_{\lambda} = \frac{1}{2}\int k_{\lambda}dx^1, \quad \sigma_{\lambda} = \int \rho_{\lambda}dx^1, \quad \eta = \frac{1}{2}\int (\chi + k^{\beta\omega}\rho_{\beta}\rho_{\omega})dx^1. \tag{4.23}$$

Transforming (4.14) and (4.21) and omitting the primes, we obtain (4.10) and (4.11) for the metric of a conformally symmetric cyclic Ricci-recurrent space.

(ii) From (4.20) it follows that

$$G = \frac{1}{2(n-2)}k^{\beta\omega}\theta_{.\beta\omega}.$$

But (4.17) implies

$$S_{11,1} = \frac{1}{G}G_{.1}S_{11}.$$

The last condition shows that the space is Ricci-recurrent and the Ricci tensor of this space is a Codazzi one. Hence $\mathbb{R}^n$ is cyclic Ricci-recurrent. Moreover from (4.17) and (4.20) it follows that $\mathbb{R}^n$ is also conformally symmetric and because of $g^{11} = 0$ and $K = g^{ij}S_{ij} = g^{11}S_{11} = 0$, its scalar curvature vanishes. This completes the proof.

5. Decomposable Cyclic Ricci-recurrent Spaces

A Riemannian space M^n is decomposable [9] if it can be expressed as a product $M_1^p \times M_2^{n-p}$ for some $p(2 \leq p \leq n-2)$, i.e., if coordinates can be found so that its metric takes the form

$$ds^2 = \sum_{a,b=1}^{p} g_{ab}dx^a dx^b + \sum_{\alpha,\beta=p+1}^{n} g_{\alpha\beta}dx^{\alpha}dx^{\beta}, \tag{5.1}$$

where the g_{ab} are functions of $x^1, x^2, \cdots, x^p$ only and the $g_{\alpha\beta}$ are functions of x^{p+1}, x^{p+2}, $\cdots$, x^n only. Latin letters $a, b, c, d, \cdots$ range over the indices $1, 2, \cdots, p$ and Greek letters $\alpha, \beta, \gamma, \delta, \cdots$ range over the indices $p+1, p+2, \cdots, n$. The two parts of (5.1) are the metrics of M_1^p and M_2^{n-p}, called the decomposition spaces of M^n. From the above form of the metric it is easily seen that the Christoffel symbols and the components of the curvature tensor and its covariant derivatives in M^n are zero unless all suffixes belong to the same range $1, 2, \cdots, p$ or $p+1, p+2, \cdots, n$. When all the suffixes belong to the same range , say

$1, 2, \cdots, p$, then the symbols and the tensor components are the same for M_1^p as for M^n and covariant differentiation in M_1^p is the same as in M^n with respect to $x^1, x^2, \cdots, x^p$. When one of the decomposition spaces, say, M_2^{n-p} is flat then M^n described as a flat extension of M_1^p.

Let us consider a decomposable Riemannian space $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$ which is cyclic Ricci-recurrent. Then we have from (1.1) that

$$S_{ab,c} + S_{bc,a} + S_{ca,b} = A_c S_{ab}, \tag{5.2}$$

$$S_{\alpha\beta,\gamma} + S_{\beta\gamma,\alpha} + S_{\gamma\alpha,\beta} = A_\gamma S_{\alpha\beta}. \tag{5.3}$$

Taking $c = \gamma$ in (5.2) we get

$$A_\gamma S_{ab} = 0, \tag{5.4}$$

which implies that $S_{ab} = 0$, since $A_\gamma \neq 0$ and hence the decomposition M_1^p is Ricci flat and the decomposition M_2^{n-p} is cyclic Ricci-recurrent. Again taking $\gamma = c$ in (5.3) we obtain M_2^{n-p} is Ricci flat and M_1^p is cyclic Ricci-recurrent. Conversely, if M_1^p is Ricci flat and M_2^{n-p} is cyclic Ricci-recurrent then $M^n = M_1^p \times M_2^{n-p}$ is cyclic Ricci-recurrent. This leads to the following:

Theorem 5.1. Let $M^n = M_1^p \times M_2^{n-p}$ be a decomposable Riemannian space. Then M is cyclic Ricci-recurrent if and only if one of the decomposition spaces is cyclic Ricci-recurrent and the other is Ricci flat.

6. Conformal Mapping of cyclic Ricci-recurrent Spaces

Let M be an n-dimensional smooth space with metric tensors g and $\bar{g}$ relative to a neighbourhood U with local coordinates x^i, we have

$$\bar{g}_{ij} = e^{2\sigma} g_{ij}, \tag{6.1}$$

where σ is a smooth function of the coordinates x^i. Clearly the angle between any two directions at point of U is independent of the choice of metric g or $\bar{g}$. We say that these two spaces (M, g) and $(M, \bar{g})$ are conformally related.

From (6.1) it follows that

$$\bar{g}^{ij} = e^{-2\sigma} g^{ij}. \tag{6.2}$$

A straightforward calculation shows that the Christoffel symbols are related by

$$\overline{[ij,k]} = e^{2\sigma}([ij,k] + g_{ik}\sigma_{,j} + g_{jk}\sigma_{,i} - g_{ij}\sigma_{,k}), \tag{6.3}$$

$$\bar{\Gamma}_{ij}^l = \Gamma_{ij}^l + \delta_i^l \sigma_{,j} + \delta_j^l \sigma_{,i} - g_{ij} g^{lm} \sigma_{,m}, \tag{6.4}$$

where $\sigma_{,i} = \frac{\partial \sigma}{\partial x^i}$ and $[ij,k] = g_{hk}\Gamma_{ij}^h$.

In Eisenhart's notation, the covariant form of the curvature tensor has components

$$R_{hijk} = \frac{\partial}{\partial x^j}[ik,h] - \frac{\partial}{\partial x^k}[ij,h] + \Gamma^l_{ij}[hk,l] - \Gamma^l_{ik}[hj,l]. \qquad (6.5)$$

If we substitute for the analogous expression derived from $\bar{g}$ we find

$$\begin{aligned} e^{-2\sigma}\bar{R}_{hijk} &= R_{hijk} + g_{hk}\sigma_{ij} + g_{ij}\sigma_{hk} - g_{hj}\sigma_{ik} \\ &- g_{ik}\sigma_{hj} + \Delta_1\sigma(g_{hk}g_{ij} - g_{hj}g_{ik}), \end{aligned} \qquad (6.6)$$

where

$$\sigma_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,j} \qquad (6.7)$$

and $\Delta_1\sigma$ is the first Beltrami operator defined by

$$\Delta_1\sigma = g^{ij}\sigma_{,i}\sigma_{,j}. \qquad (6.8)$$

Contracting (6.6) over the indices h and k and using (6.2) we obtain

$$\bar{S}_{ij} = S_{ij} + (n-2)\sigma_{ij} + [\Delta_2\sigma + (n-2)\Delta_1\sigma]g_{ij}, \qquad (6.9)$$

where $\Delta_2\sigma$ is the second Beltrami operator defined by

$$\Delta_2\sigma = g^{ij}\sigma_{,ij}. \qquad (6.10)$$

Again taking contraction of (6.9) we obtain

$$\bar{K} = e^{-2\sigma}[K + 2(n-1)\Delta_2\sigma + (n-1)(n-2)\Delta_1\sigma]. \qquad (6.11)$$

We now suppose that both (M,g) and $(M,\bar{g})$ are cyclic Ricci-recurrent spaces. Then we have the relation (1.1) and

$$\bar{S}_{ij,k} + \bar{S}_{jk,i} + \bar{S}_{ki,j} = \bar{A}_k\bar{S}_{ij}(\text{ or } = \bar{A}_i\bar{S}_{jk} \text{ or } = \bar{A}_j\bar{S}_{ki}), \qquad (6.12)$$

where $\bar{A}_k$ is a nowhere vanishing 1-form such that $\bar{A}_k = \rho^m\bar{g}_{km}$.

From (6.9) we have

$$\bar{S}_{ij} = S_{ij} + 2(n-1)\sigma_{ij} + n(n-1)\sigma_{,i}\sigma_{,j} \qquad (6.13)$$

and hence

$$\bar{S}_{ij,k} = S_{ij,k} + 2(n-1)\sigma_{ij,k} + n(n-1)\{\sigma_{,ik}\sigma_{,j} + \sigma_{,i}\sigma_{,jk}\}. \qquad (6.14)$$

By virtue of (6.14) we obtain from (6.12) that

$$\begin{aligned} (\bar{A}_k - A_k)Sij &= (n-1)[2(\sigma_{ij,k} + \sigma_{jk,i} + \sigma_{ki,j}) \\ &+ n(\sigma_{,ik}\sigma_{,j} + \sigma_{,i}\sigma_{,jk} + \sigma_{,ji}\sigma_{,k} \\ &+ \sigma_{,j}\sigma_{,ki} + \sigma_{,kj}\sigma_{,i} + \sigma_{,k}\sigma_{,ij}) \\ &- \{2\sigma_{,ij} + (n-2)\sigma_{,i}\sigma_{,j}\}\bar{A}_k]. \end{aligned} \qquad (6.15)$$

We may assume that $\bar{A}_k = A_k$, then (6.15) yields

$$\begin{aligned} \{2\sigma_{,ij} + (n-2)\sigma_{,i}\sigma_{,j}\}\bar{A}_k &= 2(\sigma_{ij,k} + \sigma_{jk,i} + \sigma_{ki,j}) \\ &+ 2n(\sigma_{ij}\sigma_{,k} + \sigma_{jk}\sigma_{,i} + \sigma_{ki}\sigma_{,j} + 3\sigma_{,i}\sigma_{,j}\sigma_{,k}). \end{aligned} \tag{6.16}$$

Again if the 1-form $\bar{A}_k$ is of the form (6.16) then from (6.15) we get $\bar{A}_k = A_k$, that is, the 1-form A_k of the space is invariant. This leads to the following:

Theorem 6.1. If a cyclic Ricci-recurrent space is transformed into another cyclic Ricci-recurrent space then the associated 1-form of the space is invariant if and only if the 1-form of the space satisfies the relation (6.16).

Contracting (6.12) over i and k we obtain

$$2\bar{K}_{,j} = \bar{K}\bar{A}_j. \tag{6.17}$$

From (6.11) we have

$$\bar{K} = e^{-2\sigma}K + 2(n-1)\bar{g}^{ik}\sigma_{ik} + n(n-1)\bar{g}^{ik}\sigma_{,i}\sigma_{,k} \tag{6.18}$$

and hence

$$\begin{aligned} \bar{K}_{,j} &= -2e^{-2\sigma}\sigma_{,j}K + e^{-2\sigma}K_{,j} + 2(n-1)\bar{g}^{ik}\sigma_{ik,j} \\ &+ n(n-1)\bar{g}^{ik}[\sigma_{,ij}\sigma_{,k} + \sigma_{,i}\sigma_{,kj}]. \end{aligned} \tag{6.19}$$

In view of (6.19), (6.17) yields

$$\begin{aligned} &-4e^{-2\sigma}\sigma_{,j}K + 2e^{-2\sigma}K_{,j} + 4(n-1)\bar{g}^{ik}\sigma_{ik,j} \\ &+2n(n-1)\bar{g}^{ik}[\sigma_{,ij}\sigma_{,k} + \sigma_{,i}\sigma_{,kj}] = \bar{K}\bar{A}_j. \end{aligned} \tag{6.20}$$

This leads to the following:

Theorem 6.2. If a cyclic Ricci-recurrent space is transformed into another cyclic Ricci-recurrent space then the associated 1-form of the space satisfies the relation *(6.20)*.

7. Example of Cyclic Ricci-recurrent Space

This section deals with an interesting example of cyclic Ricci-recurrent space.

Example 7.1. Let $\mathbb{R}^n (n > 3)$ be endowed with the following metric

$$g_{ij}dx^i dx^j = \phi(dx^i)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1 dx^n, \tag{7.1}$$

$$\phi = (Ak_{\lambda\mu} + Dc_{\lambda\mu})x^\lambda x^\mu, \tag{7.2}$$

where $[k_{\lambda\mu}]$ is a symmetric and non-singular matrix consisting of constants, $[c_{\lambda\mu}]$ is a symmetric matrix of constants satisfying rank$c_{\lambda\mu} > 1$ and $k^{\lambda\mu}c_{\lambda\mu} = 0$ with $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$ and A, D are functions of x^1 only such that $0 \neq A \neq$ constant, $0 \neq D \neq$ constant. Then $\mathbb{R}^n$ with above metric is conformally recurrent and Ricci-recurrent with vanishing scalar curvature. Moreover it is also cyclic Ricci-recurrent and its Ricci tensor is a Codazzi one.

Proof. In the above metric the only component of the Ricci tensor, Weyl conformal curvature tensor and their covariant derivatives not identically zero are those related to

$$S_{11} = (n-2)A, \quad C_{1\lambda\mu 1} = Dc_{\lambda\mu}, \tag{7.3}$$

$$S_{11,j} = (n-2)A_{,j}, \quad C_{1\lambda\mu 1,j} = D_{,j}c_{\lambda\mu}. \tag{7.4}$$

Moreover as one can easily verify, in the metric (7.1) we have $g^{11} = 0$. Hence the scalar curvature $K = g^{ij}S_{ij} = g^{11}S_{11} = 0$. The assertion is now a consequence of (7.3), (7.4) and (1.1). This completes the proof.

The above example shows that there exists a subclass of cyclic Ricci-recurrent metrics with vanishing scalar curvature. Thus we can state the following:

Theorem 7.1. There exists a cyclic Ricci-recurrent space with vanishing scalar curvature.

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