On Ricci Tensors of a Finsler Space with Special $(\alpha, \beta)-$Metric

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Abstract

In the present paper, we find the Ricci tensor of a Finsler space of a special (α, β)-metric $F = \mu_1 \alpha + \mu_2 \beta + \mu_3 \frac{\beta^2}{\alpha}$, (where μ_1, μ_2 and μ_3 are constants) and $\alpha = \sqrt{a_{ij} y^i y^j}$ be a Riemannian metric and β be a 1-form. Further, we prove that if α is a positive (negative) sectional curvature and F is of α-parallel Ricci curvature with constant Killing 1-form β, then (M, F) is a Riemannian Einstein space.

1. Introduction

Finsler space with (α, β)-metrics were introduced in 1972 by M. Matsumoto [9]. The study of Finsler spaces with (α, β)-metrics is a very important aspect of Finsler geometry and its applications (see [2, 5]). An (α, β)-metric is a scalar function on TM defined by

$$F = \alpha \phi(\frac{\beta}{\alpha}), \ s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a C^∞ on (b_0, b_0) with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form in the manifold M. Therefore, (M, F) is called the associated Riemannian manifold. A Finsler space is a manifold M equipped with a family of smoothly varying Minkowski norms; one on each tangent space, Riemannian metrics are examples of Finsler norms that are induced from an inner-product.

Some of the examples of (α, β)-metrics are the Randers metric, Matsumoto metric and Berwald metric, etc., Randers metric and its Ricci tensor are related via their applications in physics. The well-known Ricci tensor was introduced in 1904 by G. Ricci. Nine years later Ricci's work was used to formulate Einstein's theory of gravitation [1] Einstein metrics are defined in the next section but, roughly, we will say a Finsler metric F is Einstein if the average of its flag curvatures at a flag pole y is a function of position x alone, rather than the a priori position x and ag pole y. C. Robles [13] investigated Randers Einstein metrics in her thesis in 2003. She obtained the necessary and sufficient conditions for Randers metric to be Einstein and by using Einstein Zermelo navigation description, she proved the pair (h, W) of a Riemannian metric and an appropriate vector field W has been founded in [6].

Let $H_{ij} = H^k_{ikj}$; denote the canonical section of the vector bundle $\pi^* TM$ and the vertical derivation with respect to y^i by v and $\dot{\partial}_i$ respectively. For an (α, β)-metric $F = \alpha\phi(\frac{\beta}{\alpha})$, by using the geodesic coefficient of α, we can introduce a new geometry quantity. Let us denote the Levi-Civita connection of α by $\widetilde{\nabla}$. We define the Ricci tensor $\bar{H}$ and $\widetilde{H}$ on $\pi^* TM$ as follows:

$$\bar{H}_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j H(v, v), \widetilde{H}(X, Y) = \widetilde{\nabla}_{\widehat{v}} H(\bar{X}, Y), \ \ X = \pi^*(\widehat{X}), \ \ Y = \pi^*(\widehat{Y}),$$

where $\widehat{X}, \widehat{Y} \in TM_0$ and $\widehat{v}$ is the geodesic spray associated with α. The curvature $\widetilde{H}$ is closely related to the Ricci curvature and its related to (α, β)-metrics, especially to the associated Riemannian manifold (M, α). In this paper we investigate an (α, β)-metric of α-parallel Ricci curvature and we prove the following main theorem:

Theorem 1.1. *Let $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$,(where μ_1, μ_2 and μ_3 are constants) be a Finsler metric on a connected manifold M of dimension n. Suppose that α is a positive(negative)sectional curvature and Ricci tensor $\widetilde{H}$=0 ($H(v, v) = 0$) and β is a constant Killing one form. Then (M, F) is a Riemannian Einstein space.*

2.   Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM be the tangent space at $x \in M$ and $TM = \cup_{x \in M} T_x M$ be the tangent bundle of M. Each element of TM has the form $(x; y)$ where $x \in M$ and $y \in T_x M$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by, $\pi(x; y) = x$. The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber π_v^*TM at $v \in TM_0$ is just $T_x M$, where $\pi(v) = x$. Then

$$\pi^*TM = \{(x, y, v) | y \in T_x M_0, v \in T_x M\}.$$

A Finsler metric on a manifold M is a function [9, 10] $F : TM \to [0, \infty)$ which has the following properties:

i) F is C^∞ on TM_0,

ii) $F(x; \lambda y) = \lambda F(x, y), \lambda > 0$,

iii) For any tangent vector $y \in T_x M$, the vertical Hessian of $\frac{F^2}{2}$ given by

$$g_{ij} = \left(\frac{1}{2} F^2\right) y^i y^j,$$

is positive definite.

Every Finsler metric F induces a spray [7]:

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

is defined by

$$G^i(x, y) = \frac{1}{4} g^{il}(x, y) \{2 \frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y)\} y^j y^k,$$

where the matrix (g^{ij}) means the inverse of matrix (g_{ij}) and the coefficients G_j^i, G_{jk}^i and hv-curvature G_{jkl}^i of the Berwald connection can be derived from the spray G^i as follows:

$$G_j^i = \frac{\partial G^i}{\partial y^j}, \ G_{jk}^i = \frac{\partial G_j^i}{\partial y^k}; G_{jkl}^i = \frac{\partial G_{jk}^i}{\partial y^l}.$$

When $F = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric, $K_k^i = R_{jkl}^i(x) y^j y^l$, where $R_{jkl}^i(x)$ denote the coefficients of the usual Riemannian curvature tensor. Thus, the Ricci scalar function of F is given by

$$\rho = \frac{1}{F^2} K_i^i, H(v, v) = K_i^i.$$

Therefore, the Ricci scalar function is positive homogeneous of degree 0 in y. This means $\rho(x,y)$ depends on the direction of the ag pole y, but not its length.

$$H_{ij} = \frac{1}{2}\dot\partial_i\dot\partial_j H(v,v).$$

A Finsler manifold (M,F) is called an Einstein space if there exists a differentiable function c defined on M such that $H(v,v) = cF^2$. The Ricci identity for a tensor W_{jm} of π^*TM is given by the following formula [11].

$$D_k D_l W_{jm} - D_l D_k W_{jm} = -W_{rm}H^r_{jkl} - W_{jr}H^r_{mkl} - \frac{\partial W_j m}{\partial y^r}H^r_{0kl},$$

where D_k denotes the horizontal covariant derivative with respect to $\{\frac{\delta}{\delta x^k}\}$ in the Berwald connection. Let (M,F) be an n-dimensional Finsler space. For every $x \in M$, assume $S_x M = \{y \in T_x M \setminus F(x,y) = 1\}$. $S_x M$ is called the indicatrix of F at $x \in M$ and is a compact hyper surface of $T_x M$, for every $x \in M$. Let $v : S_x M \to T_x M$ be its canonical embedding, where $\| v \| = 1$ and (t,U) be a coordinate system on $S_x M$. Then, $S_x M$ is represented locally by $v^i = v^t(t^\alpha)$. $(\alpha = 1,2,...,(n-1))$.One can easily show that:

$$\frac{\partial}{\partial v^i} = F\frac{\partial}{\partial y^i}.$$

The $(n-1)$ vectors (v^i_α) from a basis for the tangent space of $S_x M$ in each point, where $v^i_\alpha = \frac{\partial v^i}{\partial t^\alpha}$. Put $\partial_\alpha = \frac{\partial}{\partial t^\alpha}$ then we can easily obtain:

$$\partial_\alpha = Fv^i_\alpha\frac{\partial}{\partial y^i}$$

Let $g = g_{ij}(x,y)y^iy^j$ is a Riemannian metric on $T_x M$. Inducing g in $S_x M$,one gets the Riemannian metric $\bar{g} = \bar{g}_{\alpha\beta}dt^\alpha dt^\beta$, where $\bar{g}_{\alpha\beta} = v^i_\alpha v^i_\beta g_{ij}$. The canonical unit vertical vector field $V(x,y) = y^i\frac{\partial}{\partial y^i}$ together the $(n-1)$ vectors, ∂_α from the local basis for $T_x M$, $B = u^1, u^2, ..., u^n$ where, $u^\alpha = (v^i_\alpha)$ and $u^n = V$. We conclude that $g(V,\partial_\alpha) = 0$, that is $y_i v^i_\alpha = 0$.

Let (M,F) be an n-dimentional Finsler space equipped with an (α,β)-metric F, where

$$\alpha(x,y) = \sqrt{a_{ij}(x)y^iy^j}, \quad \beta(x,y) = b_i(x)y^i.$$

M. Matsumato ([2, 9]) prove that, the spray G^i of Finsler space with (α,β)-metrics are given by $2G^i = \lambda^i_{00} + 2B^i$,

where

$$
\begin{aligned}
B^i &= \frac{E}{\alpha}y^i + \frac{\alpha F_\beta}{F_\alpha}s_0^i - \frac{\alpha F_{\alpha\alpha}}{F_\alpha}C\frac{y^i}{\alpha} - \frac{\alpha}{\beta}b^i, \\
E &= (\beta\frac{F_\beta}{F})C, \\
C &= \alpha\beta(r_{00}F_\alpha - 2\alpha s_0 F_\beta)/2(\beta^2 F_\alpha + \alpha\lambda^2 F_{\alpha\alpha}), \\
b^i &= a^{ir}b_r, b^2 = b^r b_r, \lambda^2 = b^2\alpha^2 - \beta^2, \\
r_{ij} &= \frac{1}{2}(\widetilde{\nabla}_j b_i + \widetilde{\nabla}_i b_j), s_{ij} = \frac{1}{2}(\widetilde{\nabla}_j b_i - \widetilde{\nabla}_i b_j), \\
s_j^i &= a^{ih}s_{hj}, s_j = b_i s_j^i.
\end{aligned}
$$

The matrix (a^{ij}) means the inverse of matrix (a_{ij}). The function γ_{jk}^i stands for the Christoffel symbols in the space (M, α), and the suffix 0 means transvecting length with respect to α, equivalently

$$
r_{ij} = 0 \; and \; s_i = 0.
$$

In an n-dimentional coordinate neighborhood U, we consider a linear partial differential equation of second order,

$$
F(\varphi) = g^{ik}\frac{\partial^2\varphi}{\partial x^i x^k} + h^i\frac{\partial\varphi}{\partial x^i}
$$

where $g^{ik}(x)$ and $h^i(x)$ are continuous function of point x in U, and quadratic form $g^{jk}Z_j Z_k$ is supposed to be positive definite everywhere n U. Then we call L an elliptic differential operator.

Strong Maximum Principle: In coordinate neighborhood U, if a function $\varphi(p)$ of class C^2 satisfies

$$
F(\varphi) > 0,
$$

where $\varphi : M \to R^n$, and if there exists a fixed point p_0 in U such that $\varphi(p) \le \varphi(p_0)$ the we have $\varphi(p) = \varphi(p_0)$,. If φ have absolute maximum in U, then φ is constant on U.

Now, we consider the (α, β)-metrics where α is of positive (negative) sectional curvature. Let $(\frac{\delta}{\delta x^i})$ and $(\frac{\delta}{\delta x^i})$ be the natural locally horizontal basis of TM_0 with respect to F and αrespectively. To prove the main theorem, we use the following proposition, proved by [12]:

2.1. **Proposition.** Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α, β)-metric on a connected manifold M. Suppose that α is of positive (negative) sectional curvature. Then, we have $H(v, v) = c\alpha^2 \; c \in R$, if and only if $\bar{H} = 0$.

3.   Einstein Criterion for Special $(\alpha, \beta)-$Metric

In this section, we consider the Finsler space with special (α, β)-metric is of the form $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$. In [13], we obtained the following relation between $H(v, v)$ and $R(v, v)$ for special (α, β)-metrics with constant Killing 1-form β:

$$H(v, v) = R(v, v) - \frac{2\alpha^3}{(\alpha - 2\beta)^3}s_0^i s_{io} + \frac{2\alpha^2}{(\alpha - 2\beta)}\nabla_i s_0^i - \frac{\alpha^4}{(\alpha - 2\beta)^2}s^{ij}s_{ij}$$

Let $H(v, v) = c\alpha^2$, where $c \in R$. we obtain (3.2)

$$0 = R(v, v) - \frac{2\alpha^3}{(\alpha - 2\beta)^3}s_0^i s_{io} + \frac{2\alpha^2}{(\alpha - 2\beta)}\nabla_i s_0^i - \frac{\alpha^4}{(\alpha - 2\beta)^2}s^{ij}s_{ij} - c\alpha^2$$

$$0 = R(v, v)[\alpha^3 - 8\beta^3 - 6\alpha^2\beta + 12\alpha\beta^2] - 2\alpha^3 s_0^i s_{i0} + [2\alpha^4 + 8\alpha^2\beta^2 - 8\alpha^2\beta^2$$
$$-8\alpha^3\beta]\widetilde{\nabla}_i s_0^i + [-\alpha^5 + 2\alpha^4\beta]s^{ij}s_{ij} - c\alpha^2[\alpha^3 - 8\beta^3 - 6\alpha^2\beta + 12\alpha\beta^2]$$

Multiplying (3.2) by $(\alpha - 2\beta)^3$ removes y from the denominators and we can derive the following identity:

$$Rat + Irrat = 0.$$

where Rat and Irrat are polynomials of degree 6 and 5 in y respectively and are given as follows:

$$\begin{aligned}
Rat &= (\mu_1^3\alpha^6 - \mu_3^3\beta^6 - 3\mu_1^2\mu_3\alpha^4\beta^2 + 3\mu_1\mu_3^2\alpha^2\beta^4)R(v, v) + (4\mu_1^2\mu_3\alpha^6\beta \\
&\quad + 4\mu_3^3\alpha^2\beta^5 - 8\mu_1\mu_3^2\alpha^4\beta^3)\nabla_i s_0^i - (\mu_1\mu_2^2\alpha^8 - \mu_2^2\mu_3\alpha^6\beta^2 + 4\mu_1\mu_3^2\alpha^6\beta^2 \\
&\quad - 4\mu_3^3\alpha^4\beta^4)s^{ij}s_{ij} - c\alpha^2(\mu_1^3\alpha^6 - \mu_3^3\beta^6 - 3\mu_1^2\mu_3\alpha^4\beta^2 + 3\mu_1\mu_3^2\alpha^2\beta^4) \\
&\quad + [(2\mu_1\mu_2^2 - 4\mu_1^2\mu_3)\alpha^6 - 6\mu_2^2\mu_3\alpha^4\beta^2 - 12\mu_3^3\alpha^2\beta^4]s_0^i s_i 0, \\
Irrat &= (-16\mu_2\mu_3^2\alpha^2\beta^3)s_0^i s_{i0} + [2\mu_1^2\mu_2\alpha^2\beta^4 + 2\mu_1^2\mu_2\alpha^6 - 4\mu_1\mu_2\mu_3\alpha^4\beta^2]\nabla_i s_0^i \\
&\quad + [4\mu_1\mu_2\mu_3\alpha^6\beta - 4\mu_2\mu_3^2\alpha^4\beta]s^{ij}s_{ij}.
\end{aligned}$$

We have the following lemma:

Lemma 3.1. *Let* $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ *be a metric with constant Killing from* β, *and* $H(v, v) = c\alpha^2$ *for some constants* $c \in R$. *Then* (M, F) *is a Riemannian Einstein space.*

Proof: We know that α can never be a polynomial in y. Otherwise the quadratic $\alpha^2 = a_{ij}(x)y^i y^j$ would have been factored into two linear terms. Its zero set would then consist of a hyper-plan, contradicting the positive definiteness of a_{ij}. Now suppose the polynomial Rat were not zero. The above equation would imply that it is the product of polynomial Irrat with a non-polynomial factor α

This is not possible. So Rat must vanish and, since α is positive at all $y \neq 0$ e see that Irrat must be zero as well. Notice that Rat=0 shows that α^2 divides $\beta^6 R(v, V)$. Since α^2 is an irreducible degree two polynomial in y, and β^6 factors into six linear terms, it must be the case that α^2 divides $R(v, v)$, that is, (M, α) is an Einstein space. Therefore, $(v, v) = k\alpha^2$, where the function k must be a constant by the Riemannian Schur's Lemma for the case $n > 2$. But, we can easily reform $Rat = 0$ as the following relation:

$$-\mu_3^2 k\beta^6 + 4\mu_3^3 \beta^5 \nabla_i s_0^i + c\mu_3^3 \beta^6 - 12\mu_3^3 \beta^4 s_0^i s_{i0}$$
$$= \alpha^2 [3\mu_1 \mu_3^2 \beta^4 k - 8\mu_1 \mu_3^2 \beta^3 \nabla_i s_0^i + 4\mu_3^3 \beta^4 s^{ij} s_{ij}],$$

which results in, α^2 divides β. From the irreducibility of α^2, it shows that, $\beta = 0$ and F is a Riemannian Einstein metric.

4. Proof of the main Theorem

Theorem 4.1. Let $F = \mu_1 \alpha + \mu_2 \beta + \mu_3 \frac{\beta^2}{\alpha}$ be a Finsler metric on a connected manifold M of dimension n. Suppose that α is a positive(negative) sectional curvature and Ricci tensor $H = 0$, $(H(v, v) = 0)$ and β is a constant Killing 1-form. Then (M, F) is a Riemannian Einstein space.

Proof: By proposition 2.1 says that, α is of positive (negative) sectional curvature. Then, $H(v, v) = c\alpha^2$, where c is a non-zero constant and by lemma 3.1 it shows that if F is Einstein then, it is Riemannian Einstein space, that means:if F is Einstein if and only if Rat=0 and Irrat=0 are holds. Again from proof of lemma, we know that α^2 divides $R(v, v)$, then their is a function k defined on M and $F = \alpha$, it implies that, $R(v, v) = k\alpha^2$. Hence (M, α) is an Einstein space. Therefore, we conclude that main theorem $(M, F = \frac{(\alpha+\beta)^2}{\alpha} + \beta)$ is a Riemannian Einstein space.

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