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(α, β, γ) - Metric and its Properties

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In the present paper, we have introduced the concept of (α, β, γ) – metric and find some important tensors for (α, β, γ) – metric, where $\alpha = \{a_{ij}(x)\, y^i y^j\}^{1/2}$, 1- form $\beta = b_i(x)\, y^{\mathrm{i}}$ and cubic metric. $\gamma = \{a_{ijk}(x)\, y^i y^j y^k\}^{1/3}$. We have also considerd the hypersurface given the equation $b(x)$ = constant of the Finsler space with the(α, β, γ) – metric given by L= L $(\alpha,\, \beta,\, \gamma)$.

Keywords: Finsler Space with $(\alpha,\, \beta,\, \gamma)$ - metric; cubic metric; one form metric; angular metric tensor; fundamental tensor and reciprocal tensor.

1. Introduction

Matsumoto, M. in the year 1972 [5] , introduced the notion of $(\alpha,\, \beta)$ – metric and studied in detail. A Finsler metric L (x, y) is called an $(\alpha,\, \beta)$ – metric, if it is positively homogenous function of degree one in Riemannian metric $\alpha = \{a_{ij}(x) y^i y^j\}^{\frac{1}{2}}$ and 1-form $\beta = b_i(x)\, y^i$. The well-known examples of $(\alpha,\, \beta)$ – metric are Rander's metric $\alpha + \beta$ [10], Kropina metric $\frac{\alpha^2}{\beta}$ [2 3], generalized Kropina metric $\frac{\alpha^{m+1}}{\beta^m}$ $(m \neq 0, -1)$[1], and Matsumoto metric $\frac{\alpha^2}{\alpha - \beta}$ [7] etc, whose studies have greatly contributed to the growth of Finsler geometry.

Again in the year 1979, Matsumoto, M. [4] introduced the concept of cubic metric on a differentiable manifold with the locaal co–ordinates, defined by

$$L(x, y) = \{a_{ijk}(x) y^i y^j y^k\}^{1/3}.$$

where, $a_{ijk}(x)$ are components of a symmetric tensor field of (0, 3) -type depending on the position x alone, and a Finsler space with a cubic metric is called the *cubic Finsler space.*

After that several authors also studied the cubic Finsler spaces [3 5 12 13 14 15]. In the year 2011, Pandey, T. N. and Chaubey, V. K., [9] had introduced the concept of (γ, β) – metric and a number of propositions and theorems obtained, where $\gamma = \{a_{ijk}(x)\, y^i y^j y^k\}^{1/3}$ is a cubic metric and $\beta = b_i(x)\, y^i$ is a one-form metric.

After studying these valuable research papers, we have introduced the (α, β, γ)– metric, where $\alpha = \{a_{ij}(x)\, y^i y^j\}^{1/2}$, 1-form $\beta = b_i(x)\, y^i$ and cubic-metric, $\gamma = \{a_{ijk}(x)\, y^i y^j y^k\}^{1/3}$.In the year 1995, Matsumoto, M.,[6] had discussed the properties of special hypersurface of Rander space with $b_i(x) = (\partial_i b)$ being the gradient of a scalar function $b_i(x)$ and also consider a hypersurface which is given by $b(x) =$ constant.

In this paper we have also considered the hypersurface given by the equation $b(x)-$ constant, of the Finsler space with (α, β, γ)– metric.

2. *Basic tensors of $(\alpha,\ \beta,\ \gamma)$ – metric*

Definition : A Finsler metric $L(x, y)$ is called a $(\alpha,\ \beta,\ \gamma)$ – metric,when L is positively homogenous function $L\ (\alpha,\ \beta,\ \gamma)$ of first degree in the variables $\alpha,\ \beta$ and γ , where $\gamma = \{a_{ijk}(x)\, y^i y^j y^k\}^{1/3}$ is a cubic metric and $\beta = b_i(x)\, y^i$ is a one-form metric.

In this present paper we have used the following results

$$a_{ijk}(x)\, y^j y^k = a_i, \quad a_{ijk} y^k = a_{ij}, a^{ij} b_j = b^i,\ a_i a^i = \gamma^3$$

where, (a^{ij}) is the inverse matrix of (a_{ij}).

As for $(\alpha,\ \beta,\ \gamma)$ – metric,

$$L = L\ (\alpha,\ \beta,\ \gamma) \tag{2.1}$$

Where,

$$\alpha = \{a_{ij}(x) y^i y^j\}^{\frac{1}{2}} \qquad \beta = b_i(x)\, y^i \ \text{ and } \qquad \gamma = \{a_{ijk}(x)\, y^i y^j y^k\}^{1/3} \tag{2.2}$$

Differentiating (2.2), we get,

$$\frac{\partial \alpha}{\partial y^r} = \frac{y_r}{\alpha}\ , \qquad \text{where} \quad a_{ir} y^r = y_i, b_r = \frac{\partial \beta}{\partial y^r} \text{ and } \quad \frac{\partial \gamma}{\partial y^r} = \frac{a_r}{\gamma^2} \tag{2.3}$$

Again differentiating (2.1) with respect to y^i, we get,

$$l_i = \dot{\partial}_i L, \text{ where } \dot{\partial}_i L \ = \frac{\partial L}{\partial y^i}$$

$$l_i = \frac{L_\alpha}{\alpha} y_i + L_\beta b_i + \frac{L_\gamma}{\gamma^2} a_i \tag{2.4}$$

Further subscripts $\alpha,\ \beta,\ \gamma$ denote partial differentiations with respect to $\alpha,\ \beta,\ \gamma$ respectively.

Again differentiating (2.4) with respect to y^j, the angular metric tensor $h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ is given by

$$h_{ij} = P_0^{*} a_{ij}(x) + P_{-1} a_{ij}(x,y) + q_{-2}^{*} y_i y_j + q_{-1}^{*}(y_i b_j + y_j b_i) + q_{-3}^{*}(a_i y_j + a_j y_i)$$

$$+ q_{-2}(a_i b_j + a_j b_i) + q_{-4} a_i a_j + q_0 b_i b_j \qquad (2.5)$$

Where,

$$P_0^{*} = \frac{LL_\alpha}{\alpha}, \qquad P_{-1} = \frac{2LL_\gamma}{\gamma^2}, \qquad q_{-2}^{*} = \frac{L}{\alpha^2}\left(L_{\alpha\alpha} - \frac{L_\alpha}{\alpha}\right),$$

$$q_{-1}^{*} = \frac{LL_{\alpha\beta}}{\alpha}, \qquad q_{-3}^{*} = \frac{LL_{\alpha\gamma}}{\alpha\gamma^2}, \qquad q_{-2} = \frac{LL_{\beta\gamma}}{\gamma^2},$$

$$q_{-4} = \frac{L}{\gamma^4}\left(L_{\gamma\gamma} - \frac{2L_\gamma}{\gamma}\right), \qquad q_0 = LL_{\beta\beta},$$

In (2.5) the subscripts of coefficients $P_0^{*}, P_{-1},\ q_{-2}^{*},\ q_{-1}^{*}, q_{-3}^{*},\ \ q_{-2},\ \ q_{-4}$ and q_0 are used to indicate respective degrees of homogeneity.

Again ,

$$g_{ij} = h_{ij} + l_i l_j$$

$$g_{ij} = a_{ij}(x) P_0^{*} + P_{-1} a_{ij}(x,y) + \left(q_{-2}^{*} + \frac{L_\alpha^2}{\alpha^2}\right) y_i y_j$$

$$+ \left(q_{-1}^{*} + \frac{L_\alpha L_\beta}{\alpha}\right)(y_i b_j + y_j b_i) + \left(q_{-3}^{*} + \frac{L_\alpha L_\gamma}{\alpha\gamma^2}\right)(a_i y_j + a_j y_i)$$

$$+ \left(q_{-2} + \frac{L_\beta L_\gamma}{\gamma^2}\right)(a_i b_j + a_j b_i) + \left(q_{-4} + \frac{L_\gamma^2}{\gamma^4}\right) a_i a_j + \left(q_0 + L_\beta^2\right) b_i b_j$$

If ,

$$\left(q_{-2}^{*} + \frac{L_\alpha^2}{\alpha^2}\right) = P_{-2}^{*}, \qquad \left(q_{-1}^{*} + \frac{L_\alpha L_\beta}{\alpha}\right) = P_{-1}^{*},$$

$$\left(q_{-3}^{*} + \frac{L_\alpha L_\gamma}{\alpha\gamma^2}\right) = P_{-3}^{*}, \qquad \left(q_{-2} + \frac{L_\beta L_\gamma}{\gamma^2}\right) = P_{-2},$$

$$\left(q_{-4}+\frac{L_\gamma^2}{\gamma^4}\right)=P_{-4}, \qquad \left(q_0+L_\beta^2\right)=P_0,$$

then, we have,

$$g_{ij}=P_0^{*}a_{ij}\left(x\right)+P_{-1}a_{ij}\left(x,y\right)+P_{-2}^{*}y_iy_j+P_{-1}^{*}\left(y_ib_j+y_jb_i\right)$$
$$+P_{-3}^{*}\left(a_iy_j+a_jy_i\right)+P_{-2}\left(a_ib_j+a_jb_i\right)+P_{-4}a_ia_j+P_0b_ib_j$$

Since we know that $\frac{\partial\gamma}{\partial y^i}=\frac{a_i}{\gamma^2}$and from [11] $\frac{\partial\gamma}{\partial y^i}=\frac{y_i}{\gamma}$, then we get $a_i=\gamma y_i$
By using $a_i=\gamma\ y_i$, we find,

$$g_{ij}=P_0^{*}a_{ij}\left(x\right)+P_{-1}a_{ij}(x,y)+a_ia_j\left(P_{-2}^{*}\gamma^{-2}+2\frac{P_{-3}^{*}}{\gamma}+P_{-4}\right)$$

$$+\left(\frac{P_{-1}^{*}}{\gamma}+P_{-2}\right)\left(a_ib_j+a_jb_i\right)+P_0b_ib_j$$

where we put,

$$S_{-4}=P_{-2}^{*}\gamma^{-2}+2\frac{P_{-3}^{*}}{\gamma}+P_{-4}$$

$$S_{-2}=\frac{P_{-2}^{*}}{\gamma}+P_{-2}$$

then we have,

$$g_{ij}=P_0^{*}a_{ij}(x)+P_{-1}a_{ij}(x,y)+S_{-4}a_ia_j+S_{-2}\left(a_ib_j+a_jb_i\right)+P_0b_ib_j \tag{2.6}$$

We know that,

$$g^{hj}g_{ij}=\delta_i^h$$

Then ,the reciprocal tensor of g_{ij} is given by,

$$g^{ij}=\frac{a^{ij}}{J}-a^ia^j\ \frac{(\pi_{-1}S_{-4}-\tau_{-1}S_{-2})d}{J}\ -b^ib^j\frac{(\tau P_0-\pi S_{-2})d}{J}$$

$$-a^i b^j \frac{(\pi_{-1} S_{-2} - \tau_{-1} P_0)\, d}{J} - a^j b^i \frac{(\tau S_{-2} - \pi S_{-4})\, d}{J}$$

Where, J $= P_0^{*} + P_{-1}$, $\tau = P_0^{*} + P_{-1} + \gamma^3 S_{-4} + S_{-2}\beta$

$\pi = S_{-2}\gamma^3 + P_0\beta$, $\tau_{-1} = \beta S_{-4} + S_{-2} b^2$

$\pi_{-1} = P_0^{*} + P_{-1} + S_{-2}\beta + P_0 b^2$, d $= \frac{1}{\tau\pi_{-1} - \pi\tau_{-1}}$

$$g^{ij} = S_1 a^{ij} - S_2 a^i a^j - S_3 b^i b^j - S_4 \left(a^i b^j + a^j b^i\right) \tag{2.7}$$

Where,

$S_1 = \frac{1}{J}$, $S_2 = \frac{(\pi_{-1} S_{-4} - \tau_{-1} S_{-2})d}{J}$,

$S_3 = \frac{(\tau P_0 - \pi S_{-2})d}{J}$, $S_4 = \frac{(\pi_{-1} S_{-2} - \tau_{-1} P_0)d}{J} = \frac{(\tau S_{-2} - \pi S_{-4})d}{J}$

where,

$(\pi_{-1} S_{-2} - \tau_{-1} P_0) = (\tau S_{-2} - \pi S_{-4})$

$(\pi_{-1} S_{-2} - \tau_{-1} P_0) = (\tau S_{-2} - \pi S_{-4}) = \frac{P_0^{*} P_{-1}}{\gamma} + \frac{P_{-1} P_{-1}^{*}}{\gamma} + \beta(\frac{P_{-1}^{*}}{\gamma})^2 +$

$2\beta \frac{P_{-1}^{*} P_{-2}}{\gamma} + P_0^{*} P_{-2} + P_{-1} P_{-2} + \beta(P_{-2})^2 - \beta \frac{P_0 P_{-2}^{*}}{\gamma^2} - 2\beta \frac{P_0 P_{-3}^{*}}{\gamma} - \beta P_0 P_{-4}$

Theorem (2.1) The angular metric tensor h_{ij} , the fundamental tensor g_{ij} and its reciprocal tensor g^{ij} of *(α, β, γ)* – metric are given by equations (2.5) , (2.6) and (2.7) respectively.

3. The Hypersurfaces $\mathbf{F^{n-1}(c)}$

In this section we have considered a special $(\alpha,\ \beta,\ \gamma)$ – metric with a gradient $b_i\,(x) = \partial_i b$ for a scalar function $b\,(x)$ and consider a hypersurface $F^{n-1}\,(c)$which is given by the equation $b\,(x) = c$(constant).

Since the parametric equation of $F^{n-1}(c)$ is $x^i = x^i\,(u^\alpha)$, hence, $(\partial/\partial u^\alpha)b(x(u)) = 0 = b_i\,(x)\,X^i_\alpha$, where $b_i\,(x)$ are considered as covariant components of a normal vector field of $F^{n-1}\,(c)$. Therefore, along the $F^{n-1}\,(c)$, we have,

$$b_i X^i_\alpha = 0 \qquad \text{and} \qquad b_i y^i = 0 \tag{3.1}$$

In general, the induced metric $\underline{L}(u,\ v)$ given by,

$$\underline{L}(u,\ v) = L\left(\left(a_{\alpha\beta}\,(u)\,v^\alpha v^\beta\right)^{\frac{1}{2}},\left(a_{\alpha\beta\gamma}\,(u)\,v^\alpha v^\beta v^\gamma\right)^{\frac{1}{3}}\right),$$

(3.2)

where,

$$a_{\alpha\beta}\,(u) = a_{ij}\,(x\,(u))\,X^i_\alpha X^j_\beta \qquad \text{and} \qquad a_{\alpha\beta\gamma}\,(u) = a_{ijk}\,(x\,(u))\,X^i_\alpha X^j_\beta X^k_\gamma$$

By using equation (3.1) and (2.7), we have,

$g^{ij}b_i b_j = b^2(S_1 - S_3 b^2)$, where $b^2 = a^{ij}b_i b_j$

Hence we get,

$$b_i = b\sqrt{S_1 - b^2 S_3}N_i$$

(3.3)

Hence from (2.7) and (3.3) we get,

$$b^i = a^{ij}b_j = \frac{b}{\sqrt{S_1 - b^2 S_3}}N^i + \left(\frac{b^2 S_4}{S_1 - b^2 S_3}\right)a^i$$

(3.4)

Theorem (3.1). Let F^n be a Finsler space with $(\alpha,\ \beta,\ \gamma)$ – metric (2.1) and $b_i\,(x) = \partial_i b(x)$ and $F^{n-1}(c)$ be a hypersurface of F^n given by $b\,(x) = c$ (constant). If the Riemannian metric $a_{ij}\,(x)\,dx^i dx^j$ be positive definite and b_i is a non- zero field, then the induced metric of $F^{n-1}(c)$is a Riemannian metric given by (3.2) and relations (3.3) and (3.4) hold.

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