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Metallic structure on Lagrangian manifold

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In this paper, the author convention with the Lagrange vertical structure on the vertical space $T_V(E)$ endowed with a non null (1,1) tensor field F_V satisfying metallic structure $F^2 - \alpha F - \beta I = 0$. The horizontal subspace $T_H(E)$ is applied on the same structure. Next, some theorems are proved and obtained conditions under which the distribution L and M are ∇-parallel, $\bar{\nabla}$ anti half parallel when $\nabla = \bar{\nabla}$. Lastly, certain theorems on geodesics on the Lagrange manifold are deduced.

Keywords: Metallic structure; Lagrangian manifold; vertical space.

1. Introduction

Let M and E be two differentiable manifolds of dimension n and $2n$ respectively and (E, π, M) be vector bundles with $\pi(E) = M$. The local coordinate systems $(x^1, x^2, ..., x^n)$ about x in M and $(y^1, y^2, ..., y^n)$ about y in E. The induced coordinates in $\pi^{-1}(U)$ are $(x^i, y^\alpha), 1 \leq i \leq n, 1 \leq \alpha \leq n$ [8] where U is a coordinate neighborhood in M. The canonical basis for tangent space $T_u(E)$ at $u \in \pi^{-1}(U)$ is $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha}\right\}$ or simply $\{\partial_i, \partial_\alpha\}$ where $\partial_i = \frac{\partial}{\partial x^i}$ etc. If (x^h, x^{α^1}) be coordinates of a point in the interesting region $\pi^{-1}(U) \cap \pi^{-1}(U)$, we can write [15]

$$x^{i^1} = x^{i^1}(x^i) \tag{1}$$

$$y^{\alpha^1} = \frac{\partial x^{\alpha^1}}{\partial x^\alpha} y^\alpha \tag{2}$$

and another canonical basis in the intersecting region are given by

$$\partial_{i^1} = \frac{\partial x^i}{\partial x^{i^1}} \partial_i \tag{3}$$

$$\partial_{\alpha^1} = \frac{\partial y^\alpha}{\partial y^{\alpha^1}} \partial_\alpha \tag{4}$$

The tangent space of E is denoted by $T(E)$ and spanned by $\{\partial_i, \partial_\alpha\}$ and its subspaces by $T_V(E)$ and $T_H(E)$ spanned by $\{\partial_\alpha\}$ and $\{\partial_i\}$ respectively [11]. Obviously

$$dimT_V(E) = dimT_H(E) = n$$

Let us suppose that the Riemannian material structure on $T(E)$ is given by

$$G = g_{ij}(x^i, y^\alpha)dx^i \otimes dx^j + g_{ab}(x^i, y^\alpha)\delta y^\alpha \otimes \delta y^b \tag{5}$$

where $g_{ij}(x^i, y^\alpha) = g_{ij}(x^i)$, $g_{ab} = \frac{1}{2}\partial_a\partial_b(x^i, y^\alpha)$ and $L(x^i, y^\alpha)$ the Lagrange function. We call such a manifold as Lagrangian manifold [4].

If $X \in T(E)$, we can write

$$X = \bar{X}^i\partial_i + X^\alpha\partial_\alpha \tag{6}$$

The automorphism $P : \chi(T(E)) \to \chi(T(E))$ defined by

$$PX = \bar{X}^i\partial_i + X^\alpha\partial_\alpha \tag{7}$$

is a natural almost product structure on $T(E)$ i.e. $P^2 = I, I$ unit tensor field. If v and h are the projection morphisms of $T(E)$ onto $T_V(E)$ and $T_H(E)$ respectively, then

$$P_0 h = v_0 P \tag{8}$$

2. Metallic Structure

Let $T_V(E)$ be the vertical space and there exists a non-null tensor field F_v of type (1,1) satisfying

$$F_v^2 - \alpha F_v - \beta I = 0 \tag{9}$$

where α, β are positive integers, we say that $T_V(E)$ admits metallic structure [16]. In this case rank $(F_v) = r$ which is constant every where. Let us call F_v as Lagrange vertical structure on $T_V(E)$

Theorem 1. *If Lagrange vertical structure F_v is defined on the vertical space $T_V(E)$, it is possible to define similar structure on the horizontal subspace $T_H(E)$ with the help of the almost product structure of $T(E)$.*

Proof: Let us put

$$F_h = PF_vP \tag{10}$$

then F_h is a tensor field of type (1,1) on $T_H(E)$. Also

$$F_h^2 = (PF_vP)(PF_vP) = PF_v^2P$$

as P is an almost product structure on $T(E)$.

Similarly $F_h^3 = PF_v^3P$ and so on. Thus, we have by virtue of (9)

$$F_h^2 - \alpha F_h - \beta I = P(F_v^2 - \alpha F_v - \beta I)P = 0 \tag{11}$$

Thus, F_h gives metallic structure on $T_H(E)$.

Theorem 2. *If Lagrange vertical structure F_v of rank r be defined on $T_V(E)$, the similar type of structure can be defined on the enveloping space $T(E)$ with the help of projection morphism of $T(E)$.*

Proof: Since Lagrange structure F_v is defined on $T_V(E)$, the Lagrange horizontal structure F_h is induced on $T_H(E)$ by theorem (2.1). If v and h are projection morphisms of $T_V(E)$ and $T_H(E)$ on $T(E)$, let us put

$$F = F_v h + F_v v \tag{12}$$

As $hv = vh = 0$ and $h^2 = h, v^2 = v$, we have

$$F^2 = F_h^2 h + F_v^2 v$$

Thus

$$\begin{aligned} F^2 - \alpha F - \beta I &= (F_h^2 - \alpha F_h - \beta I)h \\ &+ (F_v^2 - \alpha F_v - \beta I)v \\ &= 0 \end{aligned} \tag{13}$$

Making use of equations (9) and (11).

Hence

$$F^2 - \alpha F - \beta I = 0$$

Since $rank(F_v) = rank(F_h) = r$, hence $rank(F) = 2r$.

let us define tensor fields l and m of type (1,1) on $T(E)$ with metallic structure of rank $2r$ as follows

$$\begin{aligned} l &= \frac{(F^2 - \alpha F}{\beta} \\ m &= I - \frac{(F^2 - \alpha F}{\beta} \end{aligned} \tag{14}$$

Then it is easy to show that

$$l + m = I$$

$$l^2 = l, m^2 = m, lm = ml = 0, \tag{15}$$

$$Fl = lF = F, Fm = mF = 0. \tag{16}$$

This implies that the Hence the operators l and m when applied to the tangent space are complementary projection operators [3,?,?].

3. Parallelism of distributions

Let E be $2n$-dimensional Lagrangian manifold with metallic structure on $T(E)$ then there exist complementary distributions L and M corresponding to complementary projection operators l and m. Let $\bar{\nabla}$ and $\tilde{\nabla}$ be defined as follows

$$\bar{\nabla}_X Y = l\nabla_X(lY) + m\nabla_X(mY) \tag{17}$$

and

$$\tilde{\nabla}_X Y = l\nabla_{lX}(lY) + m\nabla_{mX}(mY) + l[mX, lY] + m[lX, mY] \tag{18}$$

It can be shown easily that $\bar{\nabla}$ and $\tilde{\nabla}$ are linear connections on E.

Definition 3.1 The distribution L is called ∇-parallel if for all $X \in L, Y \in T(E)$ the vector field $\nabla_Y X \in L$.

Definition 3.2 The distribution L will be said ∇-half parallel if for all $X \in L, Y \in T(E)$, $(\Delta F)(X, Y) \in L$ where

$$(\Delta F)(X, Y) = F\nabla_X Y - F\nabla_Y X - \nabla_{FX} Y + \nabla_Y(FX) \tag{19}$$

Definition 3.3 The distribution L is called ∇-anti half parallel if for all $X \in L, Y \in T(E), (\Delta F)(X, Y) \in M$ [5].

Now we prove the following theorems.

Theorem 3. *On the metallic structure manifold, the distribution L and M are $\bar{\nabla}$ as well as $\tilde{\nabla}$ parallel.*

Proof: Since $lm = ml = 0$, hence from (17) and (18), we have

$$m\bar{\nabla}_X Y = m\nabla_X(mY)$$

If $Y \in L, mY = 0$ so $m\bar{\nabla}_X Y = 0$ Therefore $\bar{\nabla}_X Y \in L$. Hence for $Y \in L, X \in T(E)$ $\Rightarrow \bar{\nabla}_X Y \in L$. So L is $\bar{\nabla}$-parallel.

Similarly for $X \in T(E), Y \in L$

$\tilde{\nabla}_X Y = m\nabla_{mX}(mY) + m[lX, mY] = 0$ as $mY = 0$.

So $\tilde{\nabla}_X Y \in L$. Hence L is $\tilde{\nabla}$-parallel.

In a similar manner, $\bar{\nabla}$ and $\tilde{\nabla}$ parallelism of M can also be proved.

Theorem 4. *On the metallic structure manifold, the distribution L and M are ∇-parallel if and only if $\bar{\nabla}$ and $\tilde{\nabla}$ are equal.*

Proof: If L and M are ∇-parallel then $\forall X, Y \in T(E)$,

$$m\nabla_X(lY) = 0, l\nabla_X(mY) = 0.$$

Therefore, since $l + m = I$,

$$\nabla_X(lY) = l\nabla_X(lY)$$

and

$$\nabla_X(mY) = m\nabla_X(mY)$$

So,

$$\nabla_X Y = l\nabla_X(lY) + m\nabla_X(mY) = \bar{\nabla}_X Y$$

Hence $\nabla = \bar{\nabla}$

The converse of the theorem proved easily.

Theorem 5. *On the metallic structure manifold, E, the distribution M is $\bar{\nabla}$-anti half parallel if for all $X \in M, Y \in T(E)$*

$$m\bar{\nabla}_Y(FX) = m\nabla_{FX} mY.$$

Proof: Since $Fm = mF = 0$, using the equation (19) for connection$\bar{\nabla}$, we have

$$m(\Delta F)(X, Y) = m\bar{\nabla}_Y FX - m\bar{\nabla}_{FX} Y \tag{20}$$

In view of the equation (17), we have

$$\bar{\nabla}_{FX} Y = l\nabla_{FX}(lY) + m\nabla_{FX}(mY)$$

$m\bar{\nabla}_{FX} Y = m\nabla_{FX}(mY)$ as $lm = 0, m^2 = m$

$$m(\Delta F)(X, Y) = m\bar{\nabla}_Y FX - m\bar{\nabla}_{FX} Y$$

As $(\Delta F)(X, Y) \in L$ so $m(\Delta F)(X, Y) = 0$. Thus

$$m\bar{\nabla}_Y(FX) = m\nabla_{FX}(mY),$$

which proves the theorem.

4. Geodesics on the Lagrangian manifold

Let γ be a curve in E with tangent T. Then γ is called geodesic with respect to connection∇ if $\nabla_T T = 0$ [8].

Theorem 6. *A curve γ will be geodesic with respect to connection $\bar{\nabla}$ if the vector fields $\nabla_T T - \nabla_T(mT) \in M$ and $\nabla_T(mT) \in L$.*

Proof: Since γ will be geodesic with respect to connection $\bar{\nabla}$, hence $\nabla_T T = 0$. On making use of the equation (17), the above equation assumes the following form

$$l\nabla_T(lT) + m\nabla_T(mT) = 0.$$

Since $l + m = I$ we can write the above equation as

$$l\nabla_T(I - m)T + m\nabla_T(mT) = 0$$

or

$$l\nabla_T T - l\nabla_T(mT) + m\nabla_T(mT) = 0$$

Therefore $l(\nabla_T T - \nabla_T(mT))$ and $m\nabla_T(mT) = 0$.

Hence $\nabla_T T - \nabla_T(mT) \in M$ and $\nabla_T(mT) \in L$, which proves the theorem.

Theorem 7. *The (1,1) tensor field l and m are always covariantly constants with respect to connection $\bar{\nabla}$.*

Proof: $\forall X, Y \in T(E)$, we have

$$(\bar{\nabla}_X l)(Y) = \bar{\nabla}_X(lY) - l\bar{\nabla}_X Y. \tag{21}$$

Making use of equation (17), we get

$$(\bar{\nabla}_X l)(Y) = l\nabla_X(l^2 Y) + m\nabla_X(mlY) - l\{l\nabla_X lY + m\nabla_X mY)\}$$

Since $l^2 = l, m^2 = m, lm = ml = 0$, we get

$$(\bar{\nabla}_X l)(Y) = l\nabla_X(lY) - l\nabla_X lY = 0.$$

So, l is covariantly constant. The fact that m is covariantly constant can be proved analogously.

References

1. Ayhan, I., Geodesics on the tangent sphere bundle of 3-sphere, International Electronic Journal of Geometry, 6 (2), 100-109 (2013).
2. Ayhan, I., L-dual lifted tensor fields between the tangent and cotangent bundles of a Lagrange manifold, International Journal of Physical and Mathematical Sciences, 4(1), 86–93 (2013).
3. Khan, M. N. I., Novel theorems for the frame bundle endowed with metallic structures on an almost contact metric manifold, Chaos, Solitons & Fractals, 146(3), online 2 April 2021. https://doi.org/10.1016/j.chaos.2021.110872.
4. Nikic, J., Distribution's parallelism and geodesics in the $F(3, \epsilon)$-structure Lagrangian manifold. Novi Sad J. Math. 27(2), 117-125 (1997).
5. Nikic, J., Comic, I, The Recurrent and Metric Connection andf-structures in Gauge Spaces of Second Order, FACTA UNIVERSITATIS Series: Mechanics, Automatic Control and Robotics, 5(1), 91-98 (2006).
6. Goldberg, S. I. and Yano, K., Polynomial structures on manifolds, Kodai Math Sem Rep., 22, 199-218 (1970).
7. Yano, K., On a structure defined by a tensor field f of type (1,1) satisfying $F^3 + f = 0$. Tensor N. S. 14, 99-109 (1963).
8. Yano, K. and Ishihara, S., Tangent and Cotangent Bundles, Marcel Dekker Inc., New York, 1973.
9. Das, Lovejoy, Nivas, R and Khan, M. N. I. and , On submanifolds of codimension 2 immersed in a Hsu–quarternion manifold, Acta Mathematica Academiae Paedagogicae Nyí regyháziensis. New Series, 25(1), 129-135 (2009).
10. Khan, M. N. I. and Das, Lovejoy, Parallelism of distributions and geodesics on $F(\pm a^2, \pm b^2)$-structure Lagrangian manifold, Facta Universitatis, Series: Mathematics and Informatics, 36(1), 165-163 (2021).
11. Das, Lovejoy, Fiberings on almost r-contact manifolds. Publications Mathematicae, Debrecen, Hongrie. 43(1-2), 1-7 (1993).

12. Das, Lovejoy, On CR-structure and F-structure satisfying $F^K + (-)^{K+1} F = 0$,. Rocky Mountain Journal of Mathematics. 36(3), 885-892 (2006).
13. Khan, M. N. I., Tangent bundle endowed with quarter-symmetric non-metric connection on an almost Hermitian manifold, Facta Universitatis, Series: Mathematics and Informatics, 35(1), 167-178 (2020).
14. Miron, R., Hrimiuc, H., Shimada, H., and Sabau, V. S., The Geometry of Hamilton and Lagrange Spaces, Kluwer Academic, New York, NY, USA, 2001.
15. Nagy, P. T., Geodesics on the tangent sphere bundle of a Riemann manifold, Geometriae, Dedicata 7, 233-243 (1978).
16. Khan, M. N. I., Complete and horizontal lifts of Metallic structures, International Journal of Mathematics and Computer Science, 15(4), 983–992 (2020).
17. Khan, M. N. I., On generalized Ricci-Recurrent Lorentzian Para-Sasakian manifold, Proceeding of National Academy of Sciences, 75(II), 147-148 (2005).
18. Helgason, S., Differential geometry, Lie groups and symmetric spaces, Academic Press, New York (1978).
19. Warner, F.W., Foundations of Differentiable Manifolds and Lie Groups. Graduate Texts in Mathematics. Springer, Berlin (2010).
20. Polyakor, N. D., Differential Geometry of f-structure manifolds, Journal of Soviet Mathematics, 29, 1571-1592 (1985).
21. Sasaki, S., Geodesic on the tangent sphere bundles over space forms. J. Reine Angew. Math. 288, 106-120 (1976).