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A study on conformal Ricci soliton on almost $C(\alpha)$-manifolds admitting W_2 curvature tensor

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We study the conformal Ricci solitons on $C(\alpha)$-manifolds. It is shown that if a $C(\alpha)$-manifold with a conformal Ricci soliton satisfies certain curvature conditions, then the manifold is either an Einstein manifold or η-Einstein manifold.

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1. Introduction

In recent years the pioneering works of R. Hamilton [8] and G. Perelman [14] towards the solution of the Poincare conjecture in dimension 3 have produced a flourishing activity in the research of self similar solutions or solitons of the Ricci flow. The study of the geometry of solitons, in particular their classification in dimension 3, has been essential in providing a positive answer to the conjecture; however, in higher dimension and in the complete possibly noncompact case, the understanding of the geometry and the classification of solitons seems to remain a desired goal for a not too proximate future. In the generic case a soliton structure on the Riemannian manifold (M, g) is the choice of a smooth vector field X on M and a real constant λ satisfying the structural requirement

$$Ric(g) + \frac{1}{2}L_X g = \lambda g, \tag{1}$$

where $Ric(g)$ is the Ricci tensor of the metric g and $L_X g$ is the Lie derivative in the direction of X. Here λ refer to as the soliton constant. The soliton is called expanding, steady and shrinking if, respectively, $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$. When X is the gradient of a potential $f \in C^\infty(M)$, the soliton is called a gradient Ricci soliton and the equation (1) takes the form

$$Ric(g) + \nabla f = \lambda g. \tag{2}$$

Equations (1) and (2) are considered as perturbations of the Einstein equation

$$Ric(g) = \lambda g \tag{3}$$

In case X and ∇f are Killing vector fields. When $X = 0$ or f is constant, we call the underlying Einstein manifold a trivial Ricci soliton.
In [7] A.E.Fischer introduced a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The resulting equations are named the conformal Ricci flow equations because of the role that conformal geometry plays in constraining the scalar curvature and because these equations are the vector field sum of a conformal flow equation and a Ricci flow equation. These new equations are given by

$$\frac{\partial g}{\partial t} + 2\left(Ric(g) + \frac{1}{n}g\right) = -\rho g,$$
$$R(g) = -1.$$

Where $R(g)$ is the scalar curvature of the manifold and ρ is scalar non-dynamical field and n is the dimension of manifold.The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics and because of this analogy, the timedependent scalar field ρ is called a conformal pressure and as for the real physical pressure in fluid mechanics that serves to maintain the incompressibility of the fluid, the conformal pressure serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint.
The conformal Ricci soliton equation is given by [12]

$$L_X g + 2S = \left[2\lambda - \left(\rho + \frac{2}{n}\right)\right] g. \tag{4}$$

The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

If in the conformal Ricci soliton equation the tangent vector X is gradient of a smooth function f the equation will be called conformal gradient Ricci soliton equation and written as

$$\nabla\nabla f + 2S = \left[2\lambda - \left(\rho + \frac{2}{n}\right)\right] g. \tag{5}$$

In the present paper we shall study the conformal Ricci soliton in almost $C(\alpha)$ manifolds satisfying the certain semi-symmetric and pseudo-symmetric conditions. In sections.4 and 5 we discuss the conditions $R \cdot W_2 = 0$ and $R \cdot W_2 = L_{W_2} Q(g, W_2)$

(W_2-semi symmetric and W_2 pseudo symmetric respectively), here W_2 is the W_2-curvature tensor, R is the derivation of the tensor algebra of the tangent space of the manifold and L_{W_2} is some smooth function on M. In sections.6 and 7 we discuss the conditions $W_2 \cdot R = 0$ and $W_2 \cdot R = L_R Q(g, R)$, here W_2 acts as the derivation of tensor algebra of the tangent space of the manifold and L_R is smooth function on M.

2. Preliminaries

Let M be n-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a tensor field of type $(1, 1)$, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \tag{6}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in TM \tag{7}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{8}$$

$$g(\phi X, X) = 0, \tag{9}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y). \tag{10}$$

If an almost contact Riemannian manifold M satisfies the condition

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{11}$$

for some smooth functions a and b on M, then M is said to be an η-Einstein manifold. If, in particular, a=0 then this manifold will be called a special type of η-Einstein manifold.

Theorem 1. *An almost contact manifold is called an almost $C(\alpha)$ manifold if the Riemannian curvature R satisfies the following relation* [11]

$$R(X, Y)Z = R(\phi X, \phi Y)Z - \alpha[g(Y, Z)X - g(X, Z)Y - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y] \tag{12}$$

where, $X, Y, Z \in TM$ and α is a real number.

Theorem 2. *An almost $C(\alpha)$-manifold is Sasakian, co-Kahler and Kenmotsu according as $\alpha = 1$, 0, -1 respectively.*

From (12) we have the following

$$R(X, Y)\xi = R(\phi X, \phi Y)\xi - \alpha[\eta(Y)X - \eta(X)Y], \tag{13}$$

$$R(\xi, Y)Z = -\alpha[g(Y, Z)\xi - \eta(Z)Y], \tag{14}$$

$$R(\xi, Y)\xi = -\alpha[\eta(Y)\xi - Y], \tag{15}$$

$$R(\xi, \xi)Z = 0. \tag{16}$$

3. Conformal Ricci soliton in almost $C(\alpha)$ manifolds.

The conformal Ricci flow equation on M is defined by the equation [4],

$$\frac{\partial g}{\partial t} + 2\left(Ric(g) + \frac{g}{n}\right) = -\rho g. \tag{17}$$

where $R(g) = -1$, ρ is a scalar non-dynamical field(time dependent scalar field) and n is the dimension of manifold.
The notion of conformal Ricci soliton is given by

$$(L_V g)(X,Y) + 2S(X,Y) = \left[2\lambda - \left(\rho + \frac{2}{n}\right)\right] g(X,Y). \tag{18}$$

This equation is the generalization of the Ricci soliton equation and it also satisfies the Ricci flow equation.
Now by using the definition of Lie derivative we can find the value of $L_\xi g$ that is given by

$$(L_\xi g)(X,Y) = g(-\phi X, Y) + g(X, -\phi Y) = 0 \tag{19}$$

By virtue of (19) in (18) we get

$$S(X,Y) = \sigma g(X,Y), \tag{20}$$

where $\sigma = \frac{1}{2}\left[2\lambda - \left(\rho + \frac{2}{n}\right)\right]$. If we put $X = Y = e_i$ in (20) where e_i is an orthonormal basis, and summing over i, we get $S = \sigma n$. But for conformal Ricci flow $R(g) = -1$, which yields the value of λ

$$\lambda = \frac{\rho}{2}. \tag{21}$$

We can consequently state the following

Theorem 3. *An almost $C(\alpha)$ manifolds admitting conformal Ricci soliton is an Einstein manifold and the scalar λ of the conformal Ricci soliton is equal to $\frac{\rho}{2}$.*

4. Conformal Ricci soliton in almost $C(\alpha)$ manifolds satisfying the condition $R \cdot W_2 = 0$.

The W_2-curvature tensor is given by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX]. \tag{22}$$

By virtue of (12), (20) and (22) we can get the following

$$W_2(\xi,Y)Z = \left(-\alpha - \frac{\sigma}{n-1}\right)[g(Y,Z)\xi - \eta(Z)Y], \tag{23}$$

$$W_2(\xi,Y)\xi = \left(-\alpha - \frac{\sigma}{n-1}\right)[\eta(Y)\xi - Y]. \tag{24}$$

Let us consider $(R(X,Y)\cdot W_2)(U,V)Z = 0$

$$R(X,Y)W_2(U,V)Z - W_2(R(X,Y)U,V)Z - W_2(U,R(X,Y)V)Z - W_2(U,V)R(X,Y)Z = 0. \tag{25}$$

Put $X = U = \xi$ in (25) and using (14), (15) and (16) we get

$$\alpha W_2(Y,V)Z = \alpha\left(-\alpha - \frac{\sigma}{n-1}\right)[g(V,Z)Y - g(Y,Z)V]. \tag{26}$$

Taking inner product of (26) with T we can write

$$W_2(Y,V,Z,T) = \left(-\alpha - \frac{\sigma}{n-1}\right)[g(V,Z)g(Y,T) - g(Y,Z)g(V,T)]. \tag{27}$$

Using (12) and (22) in (27) and contracting over Y and T we can get

$$S(\phi V,Z) = \left[\alpha(n-2) + \frac{r}{n-1} + \left(-\alpha - \frac{\sigma}{n-1}\right)(n-1)\right]g(V,Z) + \alpha\eta(V)\eta(Z) - \frac{1}{n-1}S(V,Z). \tag{28}$$

Interchanging V and Z in (28)

$$-S(\phi V,Z) = \left[\alpha(n-2) + \frac{r}{n-1} + \left(-\alpha - \frac{\sigma}{n-1}\right)(n-1)\right]g(V,Z) + \alpha\eta(V)\eta(Z) - \frac{1}{n-1}S(V,Z). \tag{29}$$

Now add (28) and (29)

$$S(V,Z) = \left(-\alpha + \frac{r}{n-1} - \sigma\right)(n-1)g(V,Z) + \alpha(n-1)\eta(V)\eta(Z). \tag{30}$$

We state the following

Theorem 4. *An almost $C(\alpha)$ manifold satisfying the condition $R\cdot W_2 = 0$ is an η-Einstein manifold.*

For conformal Ricci flow $R(g) = -1$, using (20) in (30) and on contraction over V and Z we get the value of λ and it is given by

$$\lambda = -\alpha\frac{(n-1)}{n} + \frac{\rho}{2}. \tag{31}$$

Thus we state the following

Theorem 5. *Ricci soliton in almost $C(\alpha)$ manifolds satisfying the condition $R\cdot W_2 = 0$ admitting conformal Ricci soliton is*

(1) shrinking if $\rho < \frac{2\alpha(n-1)}{n}$
(2) steady if $\rho = \frac{2\alpha(n-1)}{n}$
(3) expanding if $\rho > \frac{2\alpha(n-1)}{n}$

5. Conformal Ricci soliton in almost $C(\alpha)$ manifolds satisfying the condition $W_2 \cdot R = 0$.

Let us consider $(W_2(X,Y) \cdot R)(U,V)Z = 0$

$$W_2(X,Y)R(U,V)Z - R(W_2(X,Y)U,V)Z - R(U,W_2(X,Y)V)Z - R(U,V)W_2(X,Y)Z = 0. \tag{32}$$

Putting $X = U = \xi$ in (32) and using (23) and (24) we get

$$\left(-\alpha - \frac{\sigma}{n-1}\right)[R(Y,V)Z + \alpha(g(V,Z)Y - g(Y,Z)V)] = 0. \tag{33}$$

Since $\left(-\alpha - \frac{\sigma}{n-1}\right) \neq 0$ and taking inner product of (33) with T we can write

$$R(Y,V,Z,T) = \alpha[g(Y,Z)g(V,T) - g(V,Z)g(Y,T)]. \tag{34}$$

Putting $Y = T = e_i$ in (34), where e_i is an orthonormal basis and taking summation $i = 1, 2......n$ we get

$$S(V,Z) = -\alpha(n-1)g(V,Z). \tag{35}$$

We state the following

Theorem 6. *An almost $C(\alpha)$ manifold satisfying the condition $W_2 \cdot R = 0$ is an Einstein manifold.*

For conformal Ricci flow $R(g) = -1$, using (20) in (35) and on contraction over V and Z we get the value of λ and it is given by

$$\lambda = \frac{1}{2}\left(\rho + \frac{2}{n}\right) - \alpha(n-1). \tag{36}$$

Theorem 7. *Ricci soliton in almost $C(\alpha)$ manifolds satisfying the condition $W_2 \cdot R = 0$ admitting conformal Ricci soliton is*

(1) shrinking if $\rho < 2\alpha(n-1) - \frac{2}{n}$
(2) steady if $\rho = 2\alpha(n-1) - \frac{2}{n}$
(3) expanding if $\rho > 2\alpha(n-1) - \frac{2}{n}$

6. Conformal Ricci soliton in almost $C(\alpha)$ manifolds satisfying the condition $R \cdot W_2 = L_{W_2} Q(g, W_2)$

Let us consider $R \cdot W_2 = L_{W_2} Q(g, W_2)$

$$\begin{aligned} &R(X,Y)W_2(U,V)Z - W_2(R(X,Y)U,V)Z - W_2(U,R(X,Y)V)Z - W_2(U,V)R(X,Y)Z \\ &= L_{W_2}[(X \wedge Y)W_2(U,V)Z - W_2((X \wedge Y)U,V)Z - W_2(U,(X \wedge Y)V)Z \\ &-W_2(U,V)R(X \wedge Y)Z \end{aligned} \tag{37}$$

Put $X = U = \xi$ in (37) using (14), (15) and (16) also by using the definition of endomorphism that is $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ we get

$$(L_{W_2} - \alpha)[W_2(Y,V)Z - \left(-\alpha - \frac{\sigma}{n-1}\right)(g(V,Z)Y - g(Y,Z)V)]. \tag{38}$$

Since $L_{W_2} \neq \alpha$, taking inner product of (38) one can get

$$W_2(Y,V)Z = \left(-\alpha - \frac{\sigma}{n-1}\right)[g(V,Z)Y - g(Y,Z)V]. \tag{39}$$

Taking inner product of (39) with T we can write

$$W_2(Y,V,Z,T) = \left(-\alpha - \frac{\sigma}{n-1}\right)[g(V,Z)g(Y,T) - g(Y,Z)g(V,T)]. \tag{40}$$

Using (12) and (22) in (40) and contracting over Y and T we can get

$$\begin{aligned} S(\phi V,Z) = &\left[\alpha(n-2) + \frac{r}{n-1} + \left(-\alpha - \frac{\sigma}{n-1}\right)(n-1)\right] g(V,Z) + \alpha\eta(V)\eta(Z) \\ &- \frac{1}{n-1}S(V,Z). \end{aligned} \tag{41}$$

Interchanging V and Z in (41)

$$\begin{aligned} -S(\phi V,Z) = &\left[\alpha(n-2) + \frac{r}{n-1} + \left(-\alpha - \frac{\sigma}{n-1}\right)(n-1)\right] g(V,Z) + \alpha\eta(V)\eta(Z) \\ &- \frac{1}{n-1}S(V,Z). \end{aligned} \tag{42}$$

Now add (41) and (42)

$$S(V,Z) = \left(-\alpha + \frac{r}{n-1} - \sigma\right)(n-1)g(V,Z) + \alpha(n-1)\eta(V)\eta(Z). \tag{43}$$

We state the following

Theorem 8. *An almost $C(\alpha)$ manifold satisfying the condition $R \cdot W_2 = L_{W_2}Q(g,W_2)$ is an η-Einstein manifold $L_{W_2} \neq \alpha$.*

For conformal Ricci flow $R(g) = -1$, using (20) in (43) and on contraction over V and Z we get the value of λ and it is given by

$$\lambda = -\alpha\frac{(n-1)}{n} + \frac{\rho}{2}. \tag{44}$$

Theorem 9. *Ricci soliton in almost $C(\alpha)$ manifolds satisfying the condition $R \cdot W_2 = L_{W_2}Q(g,W_2)$ admitting conformal Ricci soliton is*

(1) shrinking if $\rho < 2\alpha\frac{(n-1)}{n}$
(2) steady if $\rho = 2\alpha\frac{(n-1)}{n}$
(3) expanding if $\rho > 2\alpha\frac{(n-1)}{n}$

7. Conformal Ricci soliton in almost $C(\alpha)$ manifolds satisfying the condition $W_2 \cdot R = L_R Q(g, R)$

Let us consider $W_2 \cdot R = L_R Q(g, R)$

$$W_2(X,Y)R(U,V)Z - R(W_2(X,Y)U,V)Z - R(U,W_2(X,Y)V)Z - R(U,V)W_2(X,Y)Z$$
$$= L_R[(X \wedge Y)R(U,V)Z - R((X \wedge Y)U,V)Z - R(U,(X \wedge Y)V)Z$$
$$-R(U,V)R(X \wedge Y)Z \tag{45}$$

Put $X = U = \xi$ in (45) and using (23) and (24) also by using the definition of endomorphism that is $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ we get

$$\left[L_R - \left(-\alpha - \frac{\sigma}{n-1}\right)\right][R(Y,V)Z + \alpha(g(V,Z)Y - g(Y,Z)V)] = 0. \tag{46}$$

Since $L_R \neq \left(-\alpha - \frac{\sigma}{n-1}\right)$ and taking inner product of (46) with T we can write

$$R(Y,V,Z,T) = \alpha[g(Y,Z)g(V,T) - g(V,Z)g(Y,T)]. \tag{47}$$

Putting $Y = T = e_i$ in (47), where e_i is an orthonormal basis and taking summation $i = 1, 2......n$ we get

$$S(V,Z) = -\alpha(n-1)g(V,Z). \tag{48}$$

We state the following

Theorem 10. *An almost $C(\alpha)$ manifold satisfying the condition $W_2 \cdot R = L_R Q(g,R)$ is an Einstein manifold provide $L_R \neq \left(-\alpha - \frac{\sigma}{n-1}\right)$.*

For conformal Ricci flow $R(g) = -1$, using (20) in (48) and on contraction over V and Z we get the value of λ and it is given by

$$\lambda = \frac{1}{2}\left(\rho + \frac{2}{n}\right) - \alpha(n-1) \tag{49}$$

Theorem 11. *Ricci soliton in almost $C(\alpha)$ manifolds satisfying the condition $W_2 \cdot R = L_R Q(g,R)$ admitting conformal Ricci soliton is*

(1) shrinking if $\rho < 2\alpha(n-1) - \frac{2}{n}$
(2) steady if $\rho = 2\alpha(n-1) - \frac{2}{n}$
(3) expanding if $\rho > 2\alpha(n-1) - \frac{2}{n}$

References

1. Ali Akbar, *Some Results on Almost $C(\lambda)$ manifolds,* Internationl Journal of Mathematical Sciences Engineering and Applications(IJMSEA), 7(1)(2013).
2. Ali Akbar and Avijit Sarkar, *On the Conharmonic and Concircular curvature tensors of almost $C(\lambda)$ manifolds,* International Journal of Advanced Mathematical Sciences, 1(3)(2013).
3. Ali Akbar and A. Sarkar, *Almost $C(\lambda)$ manifolds admitting W_2 curvarture tensor,* Jornal of Rajasthan Academy of Physical Sciences, ISSN vol. 13, (2014).

4. Ashoka S.R. Bagewadi C.S. and Ingalahalli G. *Curvature tensor of almost $C(\lambda)$ manifolds,* Malaya J. Mat. 2(1)(2014).
5. Călin C. and Crasmareanu M. *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds,* Bulletin of the Malaysian Mathematical Sciences Society, vol. 33, no.3, pp. 367-368, 2010.
6. Eisenhart L.P. *Symmetric tensors of the second order whose first covariant serivates are zero,* Trans. Amer. Math. Soc. 25, no-2 (1923), 297-306.
7. Fischer A. E. *An introduction to conformal Ricci flow,* Class. Quantum Grav.21(2004), S171 - S218.
8. Hamilton R. *Three manifold with positive Ricci curvature,* J. Differential Geom.17(2),(1982),256-306.
9. Hamilton R. *Four - manifolds with positive curvature operator,* J.Differential Geom. 24(2),(1986),153-179.
10. Ingalahalli G. and Bagewadi C. S. *Ricci solitons in α-Sasakain manifolds,* ISRN Geometry, vol. 2012, Article ID 421384, 14 pages, 2012.
11. Janssens D. and Vanhecke L. *Almost contact structures and curvature tensors,* Kath. Univ. Leuven, 1977.
12. Basu N. and Bhattacharyya N. *Conformal Ricci soliton in Kenmotsu manifold,* Global Journal of Advanced Research on Classical and Modern Geometries, Vol.4, (2015), Issue 1, pp.15-21.
13. Perelman G. *The entropy formula for the Ricci flow and its geometric applications,* arxiv.org/abs/math/0211159v1.
14. Perelman G. *Ricci flow with surgery on three manifolds,* arxiv.org/abs/math/0303109.
15. Tripathi M. M. *Ricci solitons in contact metric manifolds,* http://arxiv.org/abs/0801.4222.