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On The Normal Structure of a Hypersurface in a 2-Quasi Sasakian Manifold

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Quasi Sasakian manifold have been introduced by Blair [3]. The purpose of this paper is to study the existence of the normal hypersurface of 2–Quasi Sasakian manifold in the sense of Goldberg-Yano [6]. We obtain a characterization for these hypersurfaces and also a theorem of characterization for the normal structure on a contact totally umbilical hypersurface of 2–Quasi Sasakian manifold.

Keywords: Quasi Sasakian manifold; normal structure ;totally umbilical hypersurface.

1. Introduction

Let M be a real (2n+2) dimensional differential manifold endowed with an almost 2-contact metric structure (f,ξ_1,ξ_2,η_1,η_2,g)satisfying,where f is a tensor field of type (1,1),ξ_1,ξ_2 are vector field and η_1,η_2 are 1-form which satisfies, f^2=-I+$\eta_1\otimes\xi_1$ +$\eta_2\otimes\xi_2$

(1)

$\eta^1(\xi_1)$=$\eta^1(\xi_1)$=1
$f(\xi_1)$=$f(\xi_2)$=0
$\eta^1$of = $\eta^2$of=0
$\eta^1(\xi_2)$=$\eta^2(\xi_1)$=0
And g is an associated Riemannian metric on M that is a metric which satisfies g(fX, fY) = g(X, Y) $-\eta^1$(X).η^1(Y)-η^2(X).η^2(Y)
then we say that (f,ξ_1,ξ_2,η_1,η_2,g) is an almost 2-contact metric structure.In such a way we obtain an almost 2–contact metric manifold. Through out the paper, all manifold and maps are differentiable of class C^∞. We denote by F($\tilde{\text{M}}$) the algebra of the differentiable function on $\tilde{\text{M}}$ and by $\Gamma(E)$ .The F($\tilde{\text{M}}$)module of the sections

of a vector bundle E over M.

The Nijenhuis tensor field, denoted by N_f with respect to the tensor field f, is given by

$$N_f(X, Y) = [fX, fY] + f^2[X, Y] - f[fX, Y] - f[X, fY], \forall X, Y \epsilon \Gamma(T\tilde{M}) \tag{2}$$

The almost 2-contact metric manifold $\tilde{M}(f,\xi_1,\xi_2,\eta_1,\eta_2,g)$ is called normal if

$$N_f(X, Y) + 2c\eta_1(X,Y)\xi_1 + 2d\eta_2(X,Y)\xi_2 = 0, \forall X, Y \epsilon \Gamma(T\tilde{M})$$

According to [5], we say that an almost 2-contact metric manifold $\tilde{M}$ is a 2–Quasi Sasakian manifold if and only if ξ_1, ξ_2 are a killing vector field on $\tilde{M}$ and

$$(\tilde{\nabla}_X f)Y = g(f\tilde{\nabla}_X \xi_1 Y)\xi_1 - \eta_1(Y) f\tilde{\nabla}_X \xi_1 - \eta_2(Y) f\tilde{\nabla}_X \xi_2 \ , \forall X, Y \varepsilon \Gamma(T\tilde{M}) \tag{3}$$

where $\tilde{\nabla}$ is a Levi-Civita connection with respect to the metric g.
Next we define a tensor field F of type (1,1)by

$$FX = -\tilde{\nabla}_X \xi_1 - \tilde{\nabla}_X \xi_2 \ , \forall X \varepsilon \Gamma(T\tilde{M}) \tag{4}$$

Lemma 1: Let $\tilde{M}$ be a 2–Quasi Sasakian manifold. Then we have

(a) $g(FX,Y) + g(X,FY) = 0, \forall X, Y \varepsilon \Gamma(T\tilde{M})$

(b) $f \, 0 \, F = F \, 0 \, f$

(c) $F(\xi_1) = F(\xi_2)$

(d) $\eta_1 O \, F = \eta_2 O \, F = 0$

(e) $(\tilde{\nabla}_X f)Y = \eta_1(Y) fFX + \eta_2(Y) fFX - g(fFX,Y)\xi_1 - g(fFX,Y)\xi_2 \ , \forall X, Y \varepsilon \Gamma(T\tilde{M})$

(5)

Let $\tilde{M}$ be a 2–Quasi Sasakian manifold and M a hypersurface of $\tilde{M}$ such that ξ_1,ξ_2 are tangent to M. Denote by the same symbol g the induced metric on M and N the unit vector field normal to M. The normal vector bundle to M, denoted by $TM^{\perp}$,satisfies

$$\mathbf{T\tilde{M} = TM \bigoplus TM^{\perp}} \tag{6}$$

The Gauss and Weingarten formula are given by,
(a) $\tilde{\nabla}_X$Y=∇_XY+B(X,Y)N

(7)

(b) $\tilde{\nabla}_X$N=-AX , $\forall$ X,Y$\varepsilon\Gamma$(TM)
where ∇ is the 2–Quasi Sasakian manifold such that B(X, Y) = g(N,$\tilde{\nabla}_X$Y) and A is the shape operator with respect to the section N. Denoting by U = fN, from (1-f) we induce f(U,U) = 1. Moreover it is easy to see that U$\varepsilon\Gamma$(TM).Denote by D$^\perp$=SpanU the One Dimensional distribution,and by D the orthogonal complement of D$^\perp \bigoplus(\xi_1, \xi_2)$ in TM. Then we have
TM=D$\bigoplus$D$^\perp \bigoplus(\xi_1, \xi_2)$

(8)

It is easy to see that fD = D. According to [1] from (8) we deduce that M is a CR-sub manifold of $\tilde{\text{M}}$.

We say that M is contact totally umbilical if

h(X,Y) = g(fX,fY)H + η_1(X)h(Y,ξ_1) + η_1(Y)h(X, ξ_1) + η_2(X)h(Y,ξ_2)+ η_2(Y)h(X,ξ_2),
$\forall$ X,Y $\in \Gamma$(TM)

(9)

Where h(X, Y) = B(X, Y)N and H$\varepsilon\Gamma$(TM$^\perp$) is the mean curvature vector field of M, denoting by "P" the projection morphism of TM to D and using (8), we deduce

X = PX + a(X)U + η_1(X)ξ_1 + η_2(X)ξ_2, $\forall$ X$\varepsilon\Gamma$ (TM)

(10)

Where a is a 1-form on M defined by,

$$\text{a(X)=g(X,U), X}\varepsilon\Gamma\text{ (TM)}$$

From (10) and (1-a)we infer

$$\text{fX=tX-a(X)N ,X}\varepsilon\Gamma\text{ (TM)}$$

where t is the tensor field defined by,

$$\text{tX=fPX , X}\varepsilon\Gamma\text{ (TM)}$$

Next from [5] we recall the following:

Lemma 2: Let M be a hypersurface of a 2–Quasi Sasakian manifold $\tilde{M}$, Then we have

(a) FU = fAξ_1 + fBξ_2

(12)

(b) FN = Aξ_1 + Bξ_2

(c) [U, ξ_1, ξ_2] = 0

By straight forward calculation we get.

Preposition 1:

Let M be a hypersurface of 2–Quasi Sasakian manifold $\tilde{M}$,Then we have

(a) tU = 0

(13)

(b) $t_1\xi_1$ + $t_2\xi_2$=0
(c) tX=-X+a(X)U+η_1(X)ξ_1+η_1(X)ξ_2
(d) g(TX,Y)+g(X,TY)=0 ,$\forall$ X,Y $\varepsilon\Gamma$(TM)

Using (6) and (12-b) we infer,
FX=αX - η_1(AX)N - η_2(BX)N ,

$\forall X\varepsilon\Gamma(T\tilde{M})$(14)

Where α is a tensor field of type (1,1) on M.
Theorem 1: Let M be a hypersurface of a 2–Quasi Sasakian manifold $\tilde{M}$. Then the covariant derivative of tensor t, a, b,η_1,η_2 are given by

(a) (∇_Xt)Y = η_1(Y)[tα(X)- η_1(AX)U] + η_2(Y)[tβ(X)-η_2(BX)U]-a(Y)AX - b(Y)BX + g(FX, fY)ξ_1 + g(FX, fY)ξ_2 + B(X,Y)U,

(b) (∇_Xa)Y = B(X,tY) + η_1(Y)η_1(AtX) + η_2(Y)η_2(BtX)

(c) ($\nabla_X\eta_1$)Y = g(Y,$\nabla_X\xi_1$) + g(Y,$\nabla_X\xi_2$), $\forall$ X,Y $\epsilon\Gamma$(TM)

2. Characterizations of normal structure on hypersurfaces of a 2-Quasi Sasakian Manifold:

The purpose of this section is to study the notion of normal structure in sense of Goldberg-Yano [6] and to establish a necessary and sufficient condition for the existence of this structure on a hypersurface of 2–Quasi Sasakian manifold tangent to ξ_1, ξ_2. First we define the tensor field of type (1, 2) as follows

S(X, Y) = N_t(X, Y) + 2da(X,Y)U + 2db(X,Y)V + 2dη_1(X,Y)ξ_1 + 2dη_2(X, Y)ξ_2,

$\forall$ X,Y $\varepsilon\Gamma$(TM)

where N_t is the Nijenhuis tensor with respect to the tensor field t. Next we state the following.

Theorem2:

On a hypersurface M of a 2–Quasi Sasakian manifold $\tilde{M}$ the tensor field S is given by ,

S(X,Y) = a(X)(AtY-tAY)+ a(Y)(AtX-tAX) + b(X)(BtY-tBY)- b(Y)(BtX-tBX) + (a$\bigwedge\eta_1$)(X,Y)tAξ_1 + (b$\bigwedge\eta_2$)(X,Y)tBξ_2 + [a(X)η_1(AtY)-a(Y)η_1(AtX)]ξ_1+[b(X)η_2(BtY)-b(Y)η_2(BtX)]ξ_2 , $\forall$ X,Y$\varepsilon\Gamma$(TM)

(16)

Proof : From (15a) and the fact that ∇ is a torsion free connection on M, we infer

N_t(X,Y) = ∇_{tX}tY -∇_{tY}tX + t[(∇_yt)X- (∇_Xt)Y]

= η_1(Y)[tαtX - η_1(AtX)U] + η_2(Y)[tαtX - η_2(BtX)U] + g(FY,ftX)ξ_1+ g(FY,ftX)ξ_2-a(Y)AtX + b(Y) BtX + c(tX, Y)U - η_1(X)[tαtγ-η_1(AtY)U]-η_2(X)[tαtγ- η_2(BtY)U] -g(ftY, FX)ξ_1- g(ftY,FX)ξ_2+ a(X)AtY + b(X)BtY - c(X,tY)U + tη_1(X)tαY - η_1(Y)tαX + η_2(X)tαY -η_2(Y)tαXa(Y)AX + b(Y)BX - a(X)AY - b(Y)BY

N_t(X,Y) = a(X)(AtY - tAY)-a(Y)(AtX - tAX) + b(X)(BtY - tBY)-b(Y)(BtX - tBX) + η_1(Y)(tαtX - $t^2\alpha$X) - η_1(X)(tαtY - $t^2\alpha$Y) + η_2(Y)(tβtX - $t^2\beta$X)-η_2(X)(tβtY - $t^2\beta$Y) + g(ftX, FY) - g(ftY, FX) ξ_1 + g(ftX, FY) - g(ftY, FX)ξ_2 + c(tX,Y) - c(X, tY) + η_1(X)η_1(AtY) + η_2(X)η_2(BtY) - η_1(Y)η_1(AtX) + η_2(Y)η_2(BtX)U, $\forall$ X,Y $\varepsilon\Gamma$(TM)

(17)

On the other hand (15b), we deduce

2da(X,Y) = (∇_Xa)Y - (∇_Ya)X

= c(tY,X)-η_1(Y)η_1(AtX) + η_2(Y)η_2(BtX)- c(tX, Y)- η_1(X)η_1(AtY)-η_2(X)η_2(BtY).

(18)

From (11), (12b), (13c), we infer that

g(ftX,FY) - g(ftY,FX) = g(t^2X,FY) - g(t^2Y,FX)

= g(FX,Y) - g(X,FY) + a(X)g(U,FY) - a(Y)g(U,FX)

= -2dη_1(X,Y)- 2eη_2(X,Y)+ a(Y)g(X,fAξ_1) + b(Y)g(X,fBξ_1) - a(X)g(Y,fAξ_1) - b(X)g(Y,fBξ_1)

= -2dη_1(X,Y) - 2eη_2(X,Y) + a(X)η_1 + b(X)η_2 - a(Y)η_1(AtX) - b(Y)η_2(BtX)

(19)

Next by using (11) and (14) we get
tα tX - $t^2\alpha$X = (fαtX - ftαX)^T

= m[a(X)fAξ_1 + b(X)fBξ_2 - η_1(AX)N - η_2(BX)N]^T

(20)

where X^T denote the tangential part of X, the relation (16) follows from (17) - (19). The proof is complete.

Definition 1: The hyper surface M of a 2–Quasi Sasakian manifold M is normal in the sence of Goldberg-Yano [6] if **S = 0**.

Now we give a characterization for a normal hypersurface of 2–Quasi Sasakian manifold $\tilde{\text{M}}$.

Theorem 3: Let M be a hypersurface of a 2–Quasi Sasakian manifold / M. Then M is normal in sence Goldberg-Yano (or shortly Normal) if and only if

$$\text{AtX} = \text{tAX} ,\forall \text{ X } \varepsilon\Gamma\text{D}$$

(21)

Proof: First let X,Y $\varepsilon\Gamma$(D$\bigoplus$\{ξ_1 + ξ_2\}) then **a(X) = a(Y) = 0** and from (16) we obtain S(X,Y) = 0. If we consider X = ξ_1 + ξ_2, Y = U in (16) then we get

S(U,ξ_1,ξ_2) = (tAξ_1 + tBξ_2) - (tAξ_1 + tBξ_2) = 0.

Finally, if X$\varepsilon\Gamma$(D) and Y = U from (16) we deduce

S(X,U) = tAX - η_1(AtX)ξ_1 - η_2(BtX)ξ_2, $\forall$ X $\varepsilon\Gamma$(D)

(22)

If (21) is true, then from (22) it follows that S = 0. Then from (22) we deduce that tAX + η_1(AtX)ξ_1 + η_2(BtX)ξ_2 = tAX, $\forall$X $\varepsilon\Gamma$(D).

By direct calculation using (13-b) we obtain

$$\eta_1(\mathrm{AtX}) + \eta_2(\mathrm{BtX}) = 0$$

and from (22) we obtain (21). The proof is complete. From Theorem (3) we deduce

Corollary1: The hyper surface M of a 2–Quasi Sasakian manifold $\tilde{\mathrm{M}}$ is normal iff

c(X,tY) + c(tX,Y) = 0, $\forall$ X $\varepsilon\Gamma$(D), Y$\varepsilon\Gamma$(TM).

Corollary2: If the hypersurface M of a 2–Quasi Sasakian manifold $\tilde{\mathrm{M}}$ is normal, then we have

a. FX = αX

b. ∇_XU $\varepsilon\Gamma$(D)

c. $\nabla_X\xi_1$ + $\nabla_X\xi_2\varepsilon\Gamma$(D), $\forall$X$\varepsilon\Gamma$(D)

Proof: From (4),(14),(21) we deduce the assertion (a) and (c).For **Y = U**, from (15-a), we infer that

$$\nabla_X \mathrm{U} = \text{-tAX} + \eta_1(\mathrm{AtX})\xi_1 + \eta_2(\mathrm{BtX})\xi_2,\ \forall \mathrm{X}\varepsilon\Gamma(\mathrm{TM}) \tag{23}$$

Which proves assertion (c). The proof is complete.

Next we obtain the following

Theorem4:

The hyper surface M of a 2–Quasi Sasakian manifold is normal if and only if U is a killing vector field.

Proof:

Using (23),we deduce

$$\mathrm{g}(\nabla_X \mathrm{U},\mathrm{Y}) + \mathrm{g}(\nabla_Y \mathrm{U}, \mathrm{X}) = \mathrm{g}(\mathrm{AtX} - \mathrm{tAX} + \eta_1(\mathrm{AtX})\xi_1 + \eta_2(\mathrm{BtX})\xi_2 - \eta_1(\mathrm{X})\mathrm{tA}\xi_1 - \eta_2(\mathrm{X})\mathrm{tB}\xi_2, \mathrm{Y}),\ \forall\ \mathrm{X},\mathrm{Y}\varepsilon\Gamma(\mathrm{TM}) \tag{24}$$

Suppose that U is a killing vector field then from (24) it follows

$$\mathrm{AtX} - \mathrm{tAX} + \eta_1(\mathrm{AtX})\xi_1 + \eta_2(\mathrm{BtX})\xi_2 = 0,\ \forall \mathrm{X}\ \varepsilon\Gamma(\mathrm{D}) \tag{25}$$

Now from (25) we obtain η_1(AtX) + η_2(BtX) = 0, $\forall$X $\varepsilon\Gamma$(D) and using (25) we deduce (21).

Conversely by using (24) we infer that

g(∇_XU,Y) + g(∇_YU,X) = 0, $\forall$ X $\varepsilon\Gamma$(D), $\forall$ Y$\varepsilon\Gamma$(TM)

(26)

Next since ∇ is a metric connection, from (13) and (23) we infer that
$g(\nabla_U U,X) = a(AtX) = 0, \forall X\varepsilon\Gamma(TM)$

(27)

By using (4),(5a),(12a),(12c), we get
$g(\nabla_X U,\xi_1,\xi_2)+g(\nabla_{\xi_1} U + \nabla_{\xi_2} U , X)= - g(U,\nabla_X \xi_1+\nabla_X \xi_2)+ g(X,\nabla_U \xi_1+\nabla_U \xi_2)$
$= 2a(FX)$

$= 2[\eta_1 \ (AtX) + \eta_2(BtX)] = 0 \ \forall \ X \ \varepsilon\Gamma(TM)$

(28)

From (26) - (28) it follows that U is a killing vector field.

References

1. A. Bejancu N. Papaghiue, Semi-invariant submanifold of a Sasakian Manifold, An. stiint.Univ "Al.I.Cuza"Iasi, Supl.XVII, I 1-a(1981), 163-170.

2. A. Bejancu N. Papaghiue, Normal Semi-invariant submanifold of a Sasakian Manifold, Math. Bectnik, 35(1983), 3445-355.

3. D. E. Blair, The theory of quasi Sasakian structures, Thesis University of Illinois, 1966.

4. D. E. Blair, Contact manifold in a Riemannian Geometry, lecture notes in Math. 509(1976) Springer, Berlin.

5. C. Calin, Contribution to geometry of CR-submanifolds, Thesis University "Al. I Cuza", Iasi Romania , 1998.

6. S. I. Goldberg and K. Yano, On normal globally framed f-manifolds, **Tôhoku** Math. J., 22(1970), 362-370.

7. S. KAnemaki, Quasi Sasakian manifolds, **Tôhoku** Math. J., 29(1977), 227-233.

8. Z. Olszak, Curvature properties of quasi Sasakian manifolds, Tensor(N. S.), 38(1982), 19-23.

9. M. Saxena, Invarient Sub Manifold of a manifold satisfing fλ (2ν+3,3)–STR., J. Tensor Society of India, 21(2003), 29-36.

10. A. Oubina, New class of almost contact metric structure,Publ Math. Debrecen 32(1985), 187-193