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Some Curvature Identities on Nearly Kähler Manifolds

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In this paper we have studied and obtained expressions of some curvature identities on nearly Kähler manifold which is con-circularly flat and projectively flat. Also we got interesting results on 6-dimensional nearly Kähler manifold with an example.

Key words : Kähler manifold, nearly Kähler manifold, con-circularly flat spaces, projectively flat spaces, Einstein manifold.
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1. Introduction

An almost Hermite manifold (M,g,F) is said to be nearly Kähler manifold if $(\nabla_X F)(X) = 0$ is satisfies for all vector fields X on M, where ∇ denotes the Livi-Civita connection associated with the metric g. A nearly Kähler manifold is called strict if $\nabla_X(F) \neq 0$ for any non-vanishing vector field $X \in TM$, where TM denotes the tangent bundle of M. On the other hand, Nagy proved in [11, 12] that, in the compact case, his study amounts to that of quaternion-Kähler manifolds with positive scalar curvature [13] and nearly Kähler manifolds of dimension 6. Thus our focus on the study of such manifolds of dimension 6 can be justified by his results.

Definition : Let M be an almost Hermite manifold with almost complex structure F and Riemannian metric g. Then

$$F^2 = -I, \quad g(F(X), F(Y)) = g(X, Y),$$

for all vector fields X and Y on M. We denote by ∇ the operator of covariant differentiation with respect to g. If the almost complex structure F on M satisfies

$$(\nabla_X F)(Y) + (\nabla_Y F)(X) = 0, \tag{1}$$

for any vector fields X and Y on M, then the manifold M is called a nearly Kähler manifold or an almost Tachibana manifold.

Putting X for Y in (1.1) we get

$$(\nabla_X F)(X) = 0.$$

If in an almost Tachibana manifold, Nijenhuis tensor vanishes, then it is called a Tachibana manifold.

2. Preliminaries

In this section, we explain our notation and write down some important curvature identities. Let (M,g,F) be a connected almost Hermitian manifold. Then we have $g(FX, FY) = g(X, Y)$ for all X and Y in TM. Throughout this paper we shall assume that (M,g,F) is nearly Kähler, that is $(\nabla_X F)(X) = 0$ for all $X \in TM$.
Let R denote the curvature tensor defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for any vector fields X and Y in TM. Let $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ denote the value of the curvature tensor for every X, Y, Z and W in TM. Then we have the following identities [1,2,3]:

$$(\nabla_X F)(Y) + (\nabla_{FX} F)(FY) = 0; \tag{2}$$

$$(\nabla_X F)(FY) + F((\nabla_X F)(Y)) = 0; \tag{3}$$

$$R(W, X, Y, Z) - R(W, X, FY, FZ) = g((\nabla_W F)(X), (\nabla_Y F)(Z)), \tag{4}$$

$$\textit{and} \quad R(W, X, Y, Z) = R(FW, FX, FY, FZ). \tag{5}$$

We now define linear transformations R_1 and R_1^* by

$$Ric(X, Y) = g(R_1(X), Y) = \sum_{i=1}^{2n} R(X, e_i, Y, e_i) \quad \textit{and}$$

$$Ric^*(X, Y) = g(R_1^*(X), Y) = \frac{1}{2}\sum_{i=1}^{2n} R(X, FY, e_i, Fe_i)$$

respectively, where $\{e_1, ..., e_{2n}\}$ denotes a local orthonormal frame field on M. We shall call Ric the *Ricci* tensor of the metric and Ric^* the *Ricci** tensor respectively. Now note that $Ric - Ric^*$ is given by the formula

$$(Ric - Ric^*)(X, Y) = \sum_{i=1}^{2n} g((\nabla_X F)e_i, (\nabla_Y F)e_i)$$

for all vector fields X, Y on M [5]. Furthermore, Gray [3] proved that

$$\sum_{i,j=1}^{2n}(Ric - Ric^*)(e_i, e_j)(R(X, e_i, Y, e_j) - 5R(X, e_i, FY, Fe_j)) = 0.$$

So using the above results we have proved

Theorem 2.1. A necessary and sufficient condition for an almost Hermite manifold to be an almost nearly Kähler manifold is

$$\nabla_X F(Y) + \nabla_Y F(X) = F(\nabla_X Y) + F(\nabla_Y X).$$

Proof : First we suppose that an almost Hermite manifold is an almost nearly Kähler manifold. Then

$$(\nabla_X F)(Y) + (\nabla_Y F)(X) = 0$$

$$or,\ \nabla_X F(Y) - F(\nabla_X Y) + \nabla_Y F(X) - F(\nabla_Y X) = 0,$$

$$or,\ \nabla_X F(Y) + \nabla_Y F(X) = F(\nabla_X Y) + F(\nabla_Y X).$$

Conversely, we suppose that

$$\nabla_X F(Y) + \nabla_Y F(X) = F(\nabla_X Y) + F(\nabla_Y X)$$

$$or,\ \nabla_X F(Y) - F(\nabla_X Y) + \nabla_Y F(X) - F(\nabla_Y X) = 0,$$

$$or,\ (\nabla_X F)(Y) + (\nabla_Y F)(X) = 0.$$

Hence the manifold is an almost nearly Kähler manifold.

Proposition 2.1.[5] (i) For a nearly Kähler manifold

$N(X, Y) = 2M(X, Y) = -4F((\nabla_X F)(Y)) = 4F((\nabla_Y F)(X)) = 4F((\nabla_{F(X)} F)F(Y))$,

where $M(X, Y) = \nabla_{F(X)} F(Y) - \nabla_X Y - F(\nabla_{F(X)} Y) - F(\nabla_X F(Y))$.

(ii) If M is nearly Kähler manifold then $N(X, Y) = F(\nabla_X F)Y$,

where $4N(X, Y) = [X, Y] - [FX, FY] + F[FX, Y] + F[X, FY]$.

Theorem 2.2. If the Nijenhuis tensor N of a nearly Kähler manifold M vanishes, then M is Kähler manifold.

Proof : From the Proposition (2.1) we obtain

$$N(X,Y) = -4F((\nabla_X F)(Y)).$$

If $N(X,Y) = 0$, then $F((\nabla_X F)(Y)) = 0$. That is, $F^2(\nabla_X F)Y = 0$.
Hence $(\nabla_X F)(Y) = 0$.
Therefore the manifold is a Kähler manifold.

Theorem 2.3. On a nearly Kähler manifold $divF = 0$.

Proof : On a nearly Kähler manifold we have

$$(\nabla_X F)(Y) + (\nabla_Y F)(X) = 0.$$

Now contracting X and Y we have

$$(\nabla_X F)(X) = 0.$$

That is, $divF = 0$.

3. Curvature identities on nearly Kähler manifold

In this section we prove some curvature identities for a nearly Kähler manifold.

Theorem 3.1. For a con-circularly flat nearly Kähler manifold the following relation holds

$$\begin{aligned} &2g(F(R(X,Y)Z,W)) + g[(\nabla_X F)(\nabla_Y Z),W] - g[(\nabla_Y F)(\nabla_X Z),W] \\ &= \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned}$$

Proof : In an n-dimensional Riemannian manifold the con-circular curvature tensor is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{6}$$

so (3.1) can be written as

$$\begin{aligned} \widetilde{C}(X,Y,Z,W) = {} & \widetilde{R}(X,Y,Z,W) \\ & - \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \end{aligned} \tag{7}$$

where

$$\widetilde{C}(X,Y,Z,W) = g(C(X,Y)Z,W), \widetilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$$

and r is the scalar curvature. Now for con-circularly flat manifold, we have $\widetilde{C}(X,Y,Z,W) = 0$. Hence from (3.2) we get

$$\widetilde{R}(X,Y,Z,W) = \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \tag{8}$$

Now putting $Z = F(Z)$ in (3.3) we get

$$g(\nabla_X \nabla_Y F(Z), W) - g(\nabla_Y \nabla_X F(Z), W) - g(\nabla_{[X,Y]} F(Z), W) = \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \quad (9)$$

By using

$$\nabla_X F(Y) = (\nabla_X F)Y + F(\nabla_X Y)$$

and nearly Kähler condition

$$(\nabla_X F)(Y) + (\nabla_Y F)(X) = 0$$

we have

$$\begin{aligned} &-g[\nabla_X(\nabla_Z F)Y, W] + g[(\nabla_X F)(\nabla_Y Z), W] + g(F(\nabla_X \nabla_Y Z), W) \\ &+g[\nabla_Y(\nabla_Z F)X, W] - g[(\nabla_Y F)(\nabla_X Z), W] - g(F(\nabla_Y \nabla_X Z), W) \\ &-g[(\nabla_{[X,Y]} F)Z, W] - g(F(\nabla_{[X,Y]} Z), W) \\ &= \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned}$$

this implies

$$\begin{aligned} &2g(F(R(X,Y)Z, W)) + g[(\nabla_X F)(\nabla_Y Z), W] - g[(\nabla_Y F)(\nabla_X Z), W] \\ &= \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned}$$

Theorem 3.2. For a con-circularly flat nearly Kähler manifold the following expression holds

$$\sum_{i=1}^{2n} g((\nabla_{e_i} F)(e_i), (\nabla_{e_i} F)(e_i)) = 0.$$

Proof : we know in a nearly Kähler manifold the curvature tensor $\widetilde{R}$ satisfies,

$$\widetilde{R}(X,Y,X,Y) = \widetilde{R}(X,Y,F(X),F(Y)) + g((\nabla_X F)(Y), (\nabla_X F)(Y)),$$

where $\widetilde{R}(X,Y,X,Y) = g(R(X,Y)X,Y)$.

Also for con-circularly flat manifold, we have $\widetilde{C}(X,Y,Z,W) = 0$. So

$$\widetilde{R}(X,Y,Z,W) = \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \tag{10}$$

Now
from (3.5) and putting $X = Y = e_i$, $1 \leq i \leq 2n$ and summing over i
we obtain

$$\sum_{i=1}^{2n} g((\nabla_{e_i}F)(e_i),(\nabla_{e_i}F)(e_i)) = 0.$$

Note: For a conformally flat, projectivly flat, con-harmonic flat and Bochner flat nearly Kähler manifold the following relations holds

$$\sum_{i=1}^{2n} g((\nabla_{e_i}F)(e_i),(\nabla_{e_i}F)(e_i)) = 0.$$

Theorem 3.3. If a nearly Kähler manifold M is of constant holomorphic sectional curvature c at every point P in M and con-circularly flat, then

$$\sum_{i=1}^{2n} g((\nabla_X F)(Y),(\nabla_{e_i}F)(e_i)) = 0.$$

Proof : We know that in a nearly Kähler manifold M of constant holomorphic sectional curvature c at every point P in M, the Riemannian curvature tensor of M is of the form

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = \;& \frac{c}{4}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)] \\ , \;& +g(X,F(W))g(Y,F(Z)) \\ & -g(X,F(Z))g(Y,F(W)) \\ & -2g(X,F(Y))g(Z,F(W))] \\ & +\frac{1}{4}[g((\nabla_X F)W,(\nabla_Y F)Z) - g((\nabla_X F)Z,(\nabla_Y F)W) \\ & -2g((\nabla_X F)Y,(\nabla_Z F)W)].\end{aligned}$$

Also for con-circularly flat manifold, we have $\widetilde{C}(X,Y,Z,W) = 0$. So

$$\widetilde{R}(X,Y,Z,W) = \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \tag{11}$$

Now
from (3.6) and putting $Z = W = e_i$, $1 \leq i \leq 2n$ and summing over i
we have

$$\sum_{i=1}^{2n} g((\nabla_X F)(Y), (\nabla_{e_i} F)(e_i)) = 0.$$

Note: For a conformally flat, projectivly flat, con-harmonic flat and Bochner flat nearly Kähler manifold M is of constant holomorphic sectional curvature c at every point P in M, then the following expression holds

$$\sum_{i=1}^{2n} g((\nabla_X F)(Y), (\nabla_{e_i} F)(e_i)) = 0.$$

4. Curvature identities in 6 - dimensional nearly Kähler manifolds

In a lower dimensions, the nearly Kähler manifolds are widely determined. If M is nearly Kähler manifold with dim$M \leq 4$, then M is Kähler manifold. If dim$M = 6$, then we have the following [2,3,6,14].

Proposition 4.1. [10] Let (M,g,F) be a 6-dimensional, strict, nearly Kähler manifold. Then we have

(i) ∇F has constant type; that is,

$$g((\nabla_X F)(Y), (\nabla_X F)(Y)) = \frac{r}{30}(g(X, X)g(Y, Y) - g(X, Y)^2 - g(FX, Y)^2)$$

for all vector fields X and Y,
(ii) the first Chern class of (M, F) vanishes, and
(iii)M is Einstein manifold;

$$Ric = \frac{r}{6}g, Ric^* = \frac{r}{30}g.$$

Furthermore, from this proposition we have the following lemma [2,3,14].

Lemma 4.1. For vector fields W, X, Y and Z, we have

$$\begin{aligned} g((\nabla_W F)(X), (\nabla_Y F)(Z)) = \frac{r}{30}[&g(W, Y)g(X, Z) - g(W, Z)g(X, Y) \\ &-g(W, FY)g(X, FZ) + g(W, FZ)g(X, FY)] \end{aligned}$$

and

$$g((\nabla_W \nabla_Z X), Y) = \frac{r}{30}(g(W, Z)g(FX, Y) - g(W, X)g(FZ, Y) + g(W, Y)g(FZ, X)).$$

We can easily verify that for a 6-dimensional nearly Kähler manifold the con-circular curvature tensor takes the form

$$\widetilde{C}(X, Y, Z, W) = \widetilde{R}(X, Y, Z, W) - \frac{r}{30}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].$$

The dimension of the manifold can be verified by using Lemma (4.1) and concircularly flatness conditions.

We also deduce the following result

Result 4.1. *For a projectively flat 6-dimensional nearly Kähler manifold Ricci curvature tensor is* $S(X,Y) = \frac{r}{6}g(X,Y)$*. So the manifold is an Einstein manifold.*

5. Example of nearly Kähler manifold

A 6-dimensional unit sphere S^6 has an almost complex structure F defined by the vector cross product in the space of purely imaginary Cayley numbers. This almost complex structure is not integrable and satisfies $(\nabla_X F)(X) = 0$, for any vector field X on S^6. Hence S^6 is a nearly Kähler manifold which is not Kähler.

A structure on an n-dimensional manifold M given by a non-null tensor field f satisfies $f^3 + f = 0$, is called an f-structure. Then the rank of f is a constant, say r. If $r = n$, then the f-structure gives an almost complex structure of the manifold M. In this case n is even.

The results can be verified in the above example.

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