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On the metallic structure on differentiable manifolds

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This paper aims to study the metallic structure on a differentiable manifold M and obtain the metallic structure that acts on complementary distribution $D_{\mathcal{L}}$ as an almost product structure and complementary distribution $D_{\mathcal{M}}$ as the null operator. Some calculations on the Nijenhuis tensor of the metallic structure on M are determined.

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1. Introduction

The study of the metallic means family was started by Kapparaff [14,15,16] and Spinadel [24,25]. The Gold mean, Silver mean, Bronze mean, etc. are members of the metallic means family. Let $x^2 - ax - b, \forall a, b \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, be polynomial of degree 2 and the set of positive solutions of $x^2 - ax - b = 0$ is represented $\sigma_{a,b} = \frac{1}{2}[a + \sqrt{a^2 + 4b}]$ and said to be the metallic means family. However, the polynomial structures on the differentiable manifold were developed by goldberg, Yano and petridis [9,?]. Andreou, Yano and Al-Aqeel [1,2,26] studied a structure defined by a tensor field f satisfying $f^5 + f = 0$ and established integrability conditions.

The polynomial structure of degree 2 stisfying $\phi^2 - a\phi - bI = 0, \forall a, b \in \mathbb{N}$ and ϕ is a tensor field of type (1,1), is known as metallic structure on a differentiable manifold M [3,11]. Recently, Chaudhary and Blaga [6] studied the metallic structures and derived generalized Wintgen inequality for slant submanifolds in metallic Riemannian space forms. Lifts of metallic structure on the frame bundle and tangent bundle have been studied by Khan [18,20]. The geometry of the metallic structure has been studied by numerous researchers [5,8,12,13].

This paper aims to study the metallic structure ϕ satisfying $\phi^2 - a\phi - bI = 0, \forall a, b \in \mathbb{N}$ on M and obtain the metallic structure that acts on complementary distribution $D_{\mathcal{L}}$ as an almost product structure and complementary distribution $D_{\mathcal{M}}$ as the null operator. Some calculations on the Nijenhuis tensor of the metallic structure on M are determined.

2. Metallic structures

Let M be an n-dimensional differentiable manifold. A tensor field ϕ of type (1,1) is called the metallic structure on M that satisfies [18]

$$\phi^2 - a\phi - bI = 0, \forall a, b \in \mathbb{N}, \tag{1}$$

where I is the unit vector field.

Proposition 1. [18] *Let $\mathcal{L}$ and $\mathcal{M}$ be the projection tensors given as*

$$\mathcal{L} = \frac{\phi^2 - a\phi}{b}, \tag{2}$$

and

$$\mathcal{M} = I - \left(\frac{\phi^2 - a\phi}{b}\right). \tag{3}$$

Then

$$\begin{gathered} \mathcal{L} + \mathcal{M} = 0, \\ \mathcal{L}^2 = \mathcal{L}, \quad \mathcal{M}^2 = \mathcal{M}, \quad \mathcal{L}\mathcal{M} = \mathcal{M}\mathcal{L} = 0, \\ \phi\mathcal{L} = \mathcal{L}\phi = \phi, \quad \phi\mathcal{M} = \mathcal{M}\phi = 0. \end{gathered} \tag{4}$$

where I denotes the identity operator in M.

Let $D_{\mathcal{L}}$ and $D_{\mathcal{M}}$ be the complementary distributions analogous to the projection tensors $\mathcal{L}$ and $\mathcal{M}$, respectively in M. If the rank of ϕ is r, then dimension of $D_{\mathcal{L}}$ is r and the dimension of $D_{\mathcal{M}}$ is $(n-r)$, where the dimension of M is n.

Theorem 2. *Let M be an n-dimensional differentiable manifold equipped with the metallic structure defined in (1). Then*

$$\frac{\phi^2 - a\phi}{b} = \mathcal{L}, \quad \frac{\phi^2 - a\phi}{b}\mathcal{L} = \mathcal{L}, \quad \frac{\phi^2 - a\phi}{b}\mathcal{M} = 0. \tag{5}$$

Thus $\sqrt{\frac{\phi^2 - a\phi}{b}}$ acts on $D_{\mathcal{L}}$ as an almost product structure and on $D_{\mathcal{M}}$ as a null operator.

Proof. The proof is obvious.

Theorem 3. *Let M be an n-dimensional differentiable manifold equipped with the metallic structure ϕ defined in (1). If ϕ is of maximal rank, ϕ remains a metallic structure on M.*

Proof. Let the rank of ϕ be maximal then $r = n$. Therefore $dimL = n$ and $dimM = 0$ i.e. $\mathcal{M} = 0$ and $\mathcal{L} = I$. Thus in the view of (2), we obtain

$$\mathcal{L} = \frac{\phi^2 - a\phi}{b} = I,$$
$$\phi^2 - a\phi - bI = 0. \tag{6}$$

Hence ϕ is a metallic structure on M.

Theorem 4. *Let M be an n-dimensional differentiable manifold equipped with the metallic structure defined in (1). A tensor field of type (1,1) J defined by*

$$J = 2\frac{\phi^2 - a\phi}{b} - I \tag{7}$$

gives an almost product structure.

Proof. From (7), we have

$$J^2 = 4\left(\frac{\phi^2 - a\phi}{b}\right) - 4\left(\frac{\phi^2 - a\phi}{b}\right) + I,$$
$$J^2 = I. \tag{8}$$

Hence J is an almost product structure on M.

Theorem 5. *Let p and q be the tensor fields p and q on M as*

$$p = \mathcal{M} + \sqrt{\frac{\phi^2 - a\phi}{b}}, \tag{9}$$

and

$$p = \mathcal{M} - \sqrt{\frac{\phi^2 - a\phi}{b}}, \tag{10}$$

Then

$$pq = qp = \mathcal{M} - I. \tag{11}$$

Proof. The proof is obvious.

3. Nijenhuis tensor

The Nijenhuis tensor $N(X, Y)$ of ϕ satisfying equation (1) in M is expressed as follows

$$N(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y], \tag{12}$$

for every vector field X, Y on M.

Theorem 3.3 The conditions listed below are equivalent.

$$\begin{aligned} i. \qquad & \mathcal{M}N(X,Y) = 0, \\ ii. \qquad & \mathcal{M}[\phi X, \phi Y] = 0, \\ iii. \quad & \mathcal{M}N\left(\frac{\phi^2 - a\phi}{b}X, Y\right) = 0, \\ iv. \quad & \mathcal{M}\left[\frac{\phi^2 - a\phi}{b}\phi X, \phi Y\right] = 0, \\ v. \quad & \mathcal{M}\left[\frac{\phi^2 - a\phi}{b}\mathcal{L}\phi X, \phi Y\right] = 0, \end{aligned}$$

where X and Y are vector fields.

Proof: We need demonstrate that

$(i) \Leftrightarrow (ii)$

Let $\mathcal{M}N(X,Y) = 0$. By using (12), the obtain equation is

$$\mathcal{M}N(X,Y) = \mathcal{M}[\phi X, \phi Y] - \mathcal{M}\phi[\phi X, Y] - \mathcal{M}\phi[X, \phi Y] + \mathcal{M}\phi^2[X,Y],$$

since $\mathcal{M}\phi = 0$, we have

$$\mathcal{M}N(X,Y) = \mathcal{M}[\phi X, \phi Y],$$

since $\mathcal{M}N(X,Y) = 0$, then

$$\mathcal{M}[\phi X, \phi Y] = 0,$$

Thus,

$$\mathcal{M}N(X,Y) = 0 \Leftrightarrow \mathcal{M}[\phi X, \phi Y] = 0.$$

$(ii) \Leftrightarrow (iii)$

Let $\mathcal{M}[\phi X, \phi Y] = 0$. By using (12), the obtain equation is

$$\begin{aligned} \mathcal{M}N(\frac{\phi^2 - a\phi}{b}X, Y) = & \mathcal{M}[\frac{\phi^2 - a\phi}{b}\phi X, \phi Y] - \mathcal{M}\phi[\frac{\phi^2 - a\phi}{b}\phi X, Y] \\ & - \mathcal{M}\phi[\frac{\phi^2 - a\phi}{b}X, \phi Y] \\ & + \mathcal{M}\phi^2[\frac{\phi^2 - a\phi}{b}X, Y], \end{aligned} \tag{13}$$

since, $\frac{\phi^2 - a\phi}{b} = \mathcal{L}$, $\mathcal{M}\phi = 0$, then (13) becomes

$$\mathcal{M}N(\frac{\phi^2 - a\phi}{b}X, Y) = \mathcal{M}[\mathcal{L}\phi X, \phi Y] = \mathcal{M}[\phi X, \phi Y] as \ \ \mathcal{L}\phi = \phi,$$

since $\mathcal{M}[\phi X, \phi Y] = 0$, then we have

$$\mathcal{M}N(\frac{\phi^2 - a\phi}{b}X, Y) = 0,$$

Thus,

$$\mathcal{M}[\phi X, \phi Y] = 0 \Leftrightarrow \mathcal{M}N(\frac{\phi^2 - a\phi}{b}X, Y) = 0.$$

$(iii) \Leftrightarrow (iv)$

Let $\mathcal{M}N(\frac{\phi^2-a\phi}{b}X,Y)=0$ and using $\mathcal{M}\phi=0$. Then the obtain equation is

$$\mathcal{M}N(\frac{\phi^2-a\phi}{b}X,Y)=\mathcal{M}[\frac{\phi^2-a\phi}{b}\phi X,\phi Y],$$

since,

$$\mathcal{M}[\frac{\phi^2-a\phi}{b}\phi X,\phi Y]=0$$

Thus,

$$\mathcal{M}N(\frac{\phi^2-a\phi}{b}X,Y)=0 \Longrightarrow \mathcal{M}[\frac{\phi^2-a\phi}{b}\phi X,\phi Y]=0.$$

$(iv) \Leftrightarrow (v)$

Let $\mathcal{M}[\frac{\phi^2-a\phi}{b}\phi X,\phi Y]=0$ and using $\mathcal{L}\phi=\phi, \quad \phi\mathcal{M}=0$. Then the obtain equation is

$$\begin{aligned}\mathcal{M}N(\frac{\phi^2-a\phi}{b}\mathcal{L}X,Y)&=\mathcal{M}[\frac{\phi^2-a\phi}{b}\mathcal{L}\phi X,\phi Y],\\&=\mathcal{M}[\frac{\phi^2-a\phi}{b}\phi X,\phi Y] \quad as\ \mathcal{L}\phi=\phi,\\ So\ \ \mathcal{M}[\frac{\phi^2-a\phi}{b}\mathcal{L}\phi X,\phi Y]&=m[\frac{\phi^2-a\phi}{b}\phi X,\phi Y],\\ As\quad \mathcal{M}[\frac{\phi^2-a\phi}{b}\phi X,\phi Y]&=0,\end{aligned}$$

Thus,

$$\mathcal{M}[\frac{\phi^2-a\phi}{b}\phi X,\phi Y]=0 \Longrightarrow \mathcal{M}[\frac{\phi^2-a\phi}{b}\mathcal{L}\phi X,\phi Y]=0.$$

$(v) \Leftrightarrow (i)$ Let $\mathcal{M}[\frac{\phi^2-a\phi}{b}\mathcal{L}\phi X,\phi Y]=0.$

By using (4), the obtain equations are

$$\mathcal{M}[\mathcal{L}^2\phi X,\phi Y]=0,$$

$$\mathcal{M}[\mathcal{L}\phi X,\phi Y]=0,$$

$$\mathcal{M}[\phi X,\phi Y]=0, \quad as\ \ \mathcal{L}\phi=\phi,$$

Since $\mathcal{M}N(X,Y)=\mathcal{M}[\phi X,\phi Y] \ \ as\ \ \mathcal{M}[\phi X,\phi Y]=0,$

$$\mathcal{M}N(X,Y)=0,$$

Thus, $\mathcal{M}[\frac{\phi^2-a\phi}{b}\mathcal{L}\phi X,\phi Y]=0 \Longrightarrow \mathcal{M}N(X,Y)=0.$

This completes the proof.

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