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Holographic Dark Energy with Modified Chaplygin Gas and Scalar Field in $(2+1)-$Dimensional Space-time

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In this paper, we generalized the work of Sheykhi (2011) and Ghose (2014) and establish the connection between Holographic Dark energy interacting with modified Chaplygin gas and then obtain the the evolution of holographic dark energy with corresponding equation of state. In the first part of the paper, we have generalized the work of Sheykhi (2011) by choosing Hubble radius as system's IR cut-off and construct the analytical form of the potentials as a function of scalar fields, namely $V = V(\phi)$ as well as this dynamics of the scalar fields as a function of time, namely $\phi = \phi(t)$ then we have implemented the connection between Holographic dark energy and scalar fields model including quintessence, tachyon, $K-$essence and dilaton energy density in a $(2+1)-$dimensional spacetime FRW universe. In the second part of the paper, we have generalized the work of Ghose (2014) and investigate holographic dark energy (HDE) correspondence of interacting Modified Chaplygin Gas (MCG) and obtained evolution of the HDE with corresponding equation of state. Considering the present value of the density parameter a stable configuration is found which accommodates Dark Energy (DE). We note a connection between DE and Phantom fields. It reveals that the DE might have evolved from a Phantom state in the past. We also obtained the stability of the model and analyzed the physical and geometrical interpretations of the cosmological model with reference to the $(2+1)-$dimensional spacetime.

Keywords. Dark Energy, Scalar Field, Modified Chaplygin gas.
2020 Mathematics Subject Classification 98.80.Cq, 98.80.-k, 96.36 +x, 04.50 kd.

1. Introduction.

The recent revolution coming from string theory and black hole theory has taught us many unexpected things about the nature of spacetime and its relation to matter, energy and entropy. Such a conceptual paradigm shift must eventually have serious implications for cosmology. The most native dark energy (DE) candidate is the

cosmological constant introduced by Einstein. Although cosmological constant is consistent with the cosmological observations, there exits a fine-tunning problem in particle physics. This fine -tuning problem in the greatest challenge in high energy physics. Therefore cosmologists and particle physicists have proposed some other DE models, for example quintessence [1], phantom [2], $K-$essence [3], Tachyon [4], Chaplygin gas [5] etc.
Holographic dark energy (HDE) models have got a lot of enthusiasm because they link the DE density to cosmic horizon, a global property of the universe, and have a close relationship to the spacetime foam. In the literature various models of HDE have been investigated via considering different system's IR cutoff. According to the holographic principle, the entropy of the system scales not with its volume, but with its surface area. Cohen [6] suggested that in quantum field theory a short distance cut-off is related to a long distance cut-off due to the limit set by formation of black hole, namely it is quantum zero point energy density caused by a short distance cut-off, the total energy in a region of size L should not exceed the mass of a black hole of the same size, thus $L^3\rho_D \leq LM_p^2$. Choosing the largest IR cut-off L which saturates the inequality, we obtain

$$\rho_D = 3c^2 M_p^2 L^{-2}, \tag{1}$$

where c is the unknown constant and $M_p = (8\pi G)^{-1/2}$ is the planck mass.
In $(N+1)-$ dimension with $N = n+3$, the mass of the Schwarzschild black hole is given by [7],

$$M = \frac{(N-1)\Omega_{N-1} r_H^{N-2}}{16\pi G_N},$$

where $8\pi G_N = M_*^{1-N}$.
If we can see the effect of extra dimensions then we have the relation

$$L^3\rho_D \sim \frac{(N-1)\Omega_{N-1} L^{N-2}}{16\pi G_N},$$

$$\Rightarrow \rho_D = \frac{d(N-1)\Omega_{N-1} L^{N-5}}{16\pi G_N},$$

where d is the unknown constant.
In $(N+1)-$dimension the dark energy as follows

$$\rho_D = \frac{d(N-1)\Omega_{N-1}}{2} M_*^{N-1} L^{N-5}. \tag{2}$$

If we choose the Hubble horizon as the IR cut-off, then we have

$$\rho_D = \frac{d(N-1)\Omega_{N-1}}{2} M_*^{N-1} H^{5-N}. \tag{3}$$

In $(2+1)-$ dimension holographic dark energy (HDE) reaches in the following form

$$\rho_D = \frac{d(N-1)\Omega_{N-1}}{2} M_*^{N-1} L^{-3}. \tag{4}$$

In the presence of interaction between DE and DM, the simple choice for IR cutoff be the Hubble radius, $L = H^{-1}$, which can be simultaneously drive accelerated expansion and solve the coincidence problem [8 9]. In the literature maximum author used $L = H^{-1}$ in the framework of $(3+1)-$dimension. Sharif [10] used $L = H^{-1}$ as system IR cutoff in the higher dimensions. In this paper, it is very difficult to construct the analytical form of the potentials $V = V(\phi)$ and dynamics of the scalar field $\phi = \phi(t)$ by choosing $L = H^{-1}$ as system IR cutoff in the framework of $(2+1)-$dimension so for the simplicity we choose Hubble radius $L^{3/2} = H^{-1}$ as system's IR cut-off, we are able to construct the analytical form of the potentials as a function of scalar field, namely $V = V(\phi)$ as well as the dynamics of the scalar fields as a function of time, namely $\phi = \phi(t)$. We are establishing the correspondence between holographic DE density, quintessence, tachyon, $K-$essence and dilaton energy density in the framework of $(2+1)-$dimensions theory of gravitation.
Recently Chaplygin gas (CG) is considered in the literature as one of the prospective candidate for DE which however was first introduced in 1904 in aerodynamics. Although it contains a positive energy density it is referred as an exotic fluid due to its negative nature of pressure. CG may be described by a complex scalar field originating from generalized Born–Infeld action. The equation of state for CG is given by: An interesting model to describe DE is CG [5 11]. Pure CG obeys equation of state (EoS) [12] of the form,

$$p = \frac{B}{\rho}, \tag{5}$$

where p and ρ are pressure and energy density respectively and B is positive constant.
Chaplygin Gas (CG) is not consistent with observational data [13]. Motivated by the desire to investigate the observational loopholes better and better the form of EoS of matter is later generalized by adding arbitrary constant with an exponent over the mass density, which is called as Generalized Chaplygin Gas (GCG).
The equation of state for the GCG is given by

$$p = -\frac{B}{\rho^{\alpha}}, \tag{6}$$

where $0 < \alpha \leq 1$.
Later on GCG is again modified through the addition of an ordinary matter field, which is termed in the literature as Modified Chaplygin Gas (MCG), claiming even better match with observational results. The MCG equation of state has two parts: the first term gives an ordinary fluid obeying a linear barotropic EoS, and the second term relates pressure to some negative power energy density. So here we are essentially dealing with a two-fluid model.
MCG obeys EoS of the form,

$$p = A\rho - \frac{B}{\rho^{\alpha}}, \tag{7}$$

where $0 < A < 1/3$, $0 < \alpha \leq 1$, $B > 0$ are positive constants.
Here A and B describes the features of dark energy models and Chaplygin gas respectively.
In the past few decades, there has been a large interest in $(2+1)-$dimensional gravity, particularly after the demonstration of the fact that it's quantum version is solvable by Witten [14] and contains black-hole solutions by Bañados [15]. Recently there has been much attention given to study gravitational theories in dimensions other than four. The reasons for this are many and varied; however, the principal motivation comes from string theory, grand unified theory, and quantum gravity. The unique status of Einstein's field equations in two space and one time dimensions provides the principal reason to focus on $(2+1)-$dimensions theory of gravitation. General relativity (GR) in $(2{+}1)-$dimensions are known to have a number of unique simplifying characteristics: there are no gravitational waves, no black holes within the absence of negative cosmological constant, the Weyl curvature is identically zero, and the theory's weak field limit does not correspond to Newtonian gravity in $2-$space aspects.

In the year 1990s, the interest was motivated by the discovery of the $(2+1)-$dimension stationary circularly symmetric black hole solution by [16] [17], which has certain characteristics intrinsic in black holes $(3{+}1)$. Although most of the studies in $(2+1)-$gravity are related to black hole physics, certain attention has been devoted to cosmology. Some Friedmann-Robertson-Walker (FRW) models were analyzed by [18,?] in $(2+1)-$dimensional spacetime.
In $(2+1)-$dimensional Einstein gravity, Cornish and Frankel [20] constructed solutions for isotropic dust-filled and radiation-dominated universes for $k = -1, 0, 1$. Saslaw [21] developed an interesting concept of a possible relationship between the homogeneity of the universe and the dimensionality of space ; if our universe went through a spatially two-dimensional stage, determined by a $(2+1)-$dimensional dust-filled model, it might be possible to account for its present large-scale homogeneity.

Cruz and Martinez [22] derived flat FRW model for a homogeneous scalar field minimally coupled to gravity in $(2+1)-$dimension. Wang and Abdalla [23] used $(2+1)$ FRW models to examine the cosmic holographic principle. Khadekar [24] find the holography in $(2{+}1)$-dimensional cosmological model with generalized equation of state and Khadekar and Gharad [25] were investigating $(2+1)$-dimensional cosmological viscous models with G and Λ variable. While Khadekar et al. [26] studied modified Chaplygin gas with bulk viscous cosmology in FRW $(2+1)-$dimensional spacetime and Islam et al. [27] investigated $(2{+}1)-$dimensional cosmological models in $f(R,T)$ gravity with $\Lambda(R,T)$.

Sheykhi [28] established the connection between the scalar field model of DE including quintessence, tachyon, $K-$essence and dilaton energy density by choosing Hubble radius $L = H^{-1}$ as system's IR cut-off for interacting HDE. Similarly, Ghose [29] investigated HDE model considering interacting GCG in the framework of $(4+1)-$dimensional spacetime and obtained the evolution of the modified HDE with corresponding EoS.

With the motivation of the above work in this paper we have generalized the work of Sheykhi [28] and Ghose [29] in the framework of $(2+1)-$dimensional spacetime. This paper is divided into two parts. In the first part of the paper we have generalized the work of Sheykhi [28] by choosing Hubble radius $L^{3/2} = H^{-1}$ as system's IR cut-off and construct the analytical form of the potentials as a function of scalar field, namely $V = V(\phi)$ as well as this dynamics of the scalar fields as a function of time, namely $\phi = \phi(t)$ then we have implemented the connection between HDE and scalar fields model including quintessence, tachyon, $K-$essence and dilaton energy density in a $(2+1)-$dimensional spacetime FRW universe.

In the second part of the paper, we have generalized the work of Ghose [29] and investigate holographic dark energy (HDE) correspondence of interacting Modified Chaplygin Gas (MCG) and obtained evolution of the HDE with corresponding equation of state. Considering the present value of the density parameter a stable configuration is found which accommodates Dark Energy (DE). We note a connection between DE and Phantom fields. It reveals that the DE might have evolved from a Phantom state in the past. We also obtained the stability of the model and analyzed the physical and geometrical interpretations of the cosmological model with reference to the $(2+1)-$dimensional spacetime.

This paper is organized as follows: section 2, deals with Einstein field equations for flat FRW models in $(2+1)-$dimension. We have obtained EoS parameter of HDE by choosing HDE of the form $\rho_D = 2c^2 M_p^2 L^{-3}$ with Hubble radius $L^{3/2} = H^{-1}$ as system's IR cut-off in section 3. In section 4, we have established the connection between scalar model of the dark energy namely, quintessence, tachyon, $K-$essence and dilaton energy density. As a results we have reconstructed the analytic form of potential $V = V(\phi)$ as well as the dynamics of the scalar field as a function of time i.e. $\phi = \phi(t)$.

In second part of the paper we have proposed the interacting Holographic Modified Chaplygin Gas (MCG) in $(2+1)-$dimensional spacetime in section 5. The stability of the model is discussed in section 6 and finally section 7, we have given a brief discussion and conclusion.

2. FRW Model and Friedmann equations

We consider $(2+1)-$dimensional FRW line element of the form

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right], \tag{8}$$

where, $a(t)$ denotes the scale factor.

The coordinates (t, r, θ) represent the co-moving coordinates and the constant k denotes the curvature of the space $k = 0, 1, -1$ for flat, closed and open universe

respectively.
The Einstein field equations in $(2+1)$-dimension spacetime is given by [20],

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\pi G T_{\mu\nu}, \tag{9}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar. In future, we consider $2\pi G = 1$.
The energy momentum tensor is given by,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \tag{10}$$

where ρ & p is the energy density and pressure respectively.
Also, u^μ is the velocity three vector with $g^{\mu\nu}u^\mu u_\nu = 1$.
The field equations (9) with the help of line element (8) in $(2+1)-$dimensions are given by,

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{\rho}{2}, \tag{11}$$

$$\frac{\ddot{a}}{a} = -p, \tag{12}$$

where "$(\cdot)$" denotes derivative with respect to cosmic time t.
The Hubble parameter is defined as $H = \frac{\dot{a}}{a}$.
We assume the law of conservation energy $(T^{\mu\nu}_{;\nu} = 0)$ in $(2+1)-$dimensional spacetime is given by,

$$\dot{\rho} + 2H(\rho + p) = 0. \tag{13}$$

By using the equation of state $p = \omega\rho$ in Eq. (13) reduce to,

$$\dot{\rho} + 2H(1 + \omega)\rho = 0. \tag{14}$$

In the following section, consider two types of fluids with total energy density as $\rho = \rho_m + \rho_D$, where ρ_m and ρ_D represents the energy density of dark matter and dark energy density consisting Cold Dark Matter (CDM) respectively with an equation of state parameter $\omega_m = 0$. For non interacting fluids, the conservation equations for ρ_D, ρ_m and p_D, p_m satisfy separately.

3. Holographic Dark Energy with Hubble Radius $\mathbf{L = H^{-2/3}}$ as an IR Cut-off

In terms of Hubble parameter the first Friedmann equation for flat universe in $(2+1)-$dimension can be rewritten as

$$H^2 = \frac{1}{2M_p^2}\Big(\rho_D + \rho_m\Big). \tag{15}$$

However, in the case of interacting DE models, the following equations results from the conservation equations for Dark matter (DM) and DE are

$$\dot{\rho_m} + 2H\rho_m = Q, \tag{16}$$

$$\dot{\rho_D} + 2H(1+\omega_D)\rho_D = -Q, \tag{17}$$

where $\omega_D = p_D/\rho_D$ is the equation of state (EoS) parameter of HDE, and Q stands for the interaction term.
It should be noted that the ideal interaction term must be motivated from the theory of quantum gravity.
In this paper we consider the simple form of interacting term $Q \propto H\rho_D$,

$$Q = 2b^2 H\rho_D, \tag{18}$$

where b^2 is a coupling constant.
By using Eq. (4) the Holographic Dark Energy density in $(2+1)-$dimension has the form

$$\rho_D = 2c^2 M_p^2 L^{-3},$$

In terms of Hubble radius the HDE has the form

$$\rho_D = 2c^2 M_p^2 H^2, \tag{19}$$

where Hubble radius $L = H^{-2/3}$ the holographic dark energy.
This Eq. (19) is analogous to the Eq. taken by Sheykhi [28] in $(3+1)-$dimensional spacetime.
After inserting Eq. (19) into Eq. (15), we get

$$u = \left(\frac{1-c^2}{c^2}\right), \tag{20}$$

where $\frac{\rho_m}{\rho_D} = u$ is the energy density ratio.
From Eq. (19) it is noted that the ratio of the energy densities is a constant.
Note that this condition also fulfilled in $(2+1)-$dimension for all time; otherwise the dark energy density would not even approximately be proportional to L^{-2}. By differentiating Eq. (19) w. r. to t, we get

$$\dot{\rho_D} = 4c^2 M_p^2 H\dot{H}. \tag{21}$$

Differentiating Eq. (15) w. r. to t, we get

$$\frac{1}{2M_p^2}\left(\dot{\rho_m} + \dot{\rho_D}\right) = 2H\dot{H}. \tag{22}$$

Adding Eq. (16) and (17) and put $u = \frac{\rho_m}{\rho_D}$ we get

$$\left(\dot{\rho_m} + \dot{\rho_D}\right) = -2H\rho_D\left[u + (1+\omega_D)\right], \tag{23}$$

put this value in Eq. (22), we get

$$\frac{-H\rho_D}{M_p^2}\left[u + 1 + \omega_D\right] = 2H\dot{H}. \tag{24}$$

put this value in Eq. (21), we get

$$\dot{\rho_D} = -2c^2 H\rho_D \Big[u + 1 + \omega_D\Big], \tag{25}$$

put this value in Eq. (17) after using the value of Q from Eq. (18), we get

$$\omega_D = \frac{-b^2}{1-c^2}. \tag{26}$$

Here the c and b are constants and hence the EoS parameter ω_D is also constant. In the absence of interaction term $b^2 = 0$, we get dust case with $\omega_D = 0$. We note that $1 - c^2 \neq 0$ in equation (26). If $1 - c^2 = 0$ then $c^2 < 1$ or $1 > c^2$ then $\omega_D < 0$. Beside the acceleration expansion $\Big(\omega_D < -\frac{1}{2}\Big)$ then from Eq. (26), we have

$$\frac{-b^2}{1-c^2} < \frac{-1}{2} \Rightarrow c^2 > 1 - 2b^2.$$

Thus, this model can describe the accelerated expansion if $1 - 2b^2 < c^2 < 1$ in $(2{+}1)-$dimensional spacetime. Moreover, ω_D can cross the phantom line ($\omega_D < -1$) provided $b^2 > 1 - c^2$. The analogous results discussed by Sheykhi [28].

4. Correspondence with Scalar field Model

In this section of the paper, we are implementing a correspondence between interacting HDE and various scalar field models of the type i.e. Holographic quintessence model, Holographic tachyon model, Holographic $K-$essence model, Holographic dilaton model by equating the equation of state for this model with the equations of state parameter of interacting HDE obtained Eq. (26).

4.1. *Reconstructing Holographic quintessence model*

In this part of the paper, we assume the quintessence scalar field model of DE to establish the correspondence between HDE and quintessence scalar field.
The energy density and pressure in $(2+1)-$dimension spacetime is given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{27}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{28}$$

where ϕ is the kinetic energy and $V(\phi)$ potential be the function of scalar field ϕ. From Eqs. (27) and (28) with $p_\phi = \omega_\phi \rho_\phi$, we get

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}.$$

After solving above Eqs. for ϕ and $V(\phi)$, we get

$$\dot{\phi}^2 = \Big(1 + \omega_\phi\Big)\rho_\phi, \tag{29}$$

$$V(\phi)=\Big(\frac{1-\omega_\phi}{2}\Big)\rho_\phi, \tag{30}$$

and

$$\omega_\phi=\frac{\dot{\phi}^2-2V(\phi)}{\dot{\phi}^2+2V(\phi)}. \tag{31}$$

For implementing the correspondence between HDE and quintessence scalar-field, we identify $\rho_\phi=\rho_D$ and $\omega_\phi=\omega_D$ then from Eqs. (19), (26) and (29) gives

$$\Rightarrow \dot{\phi}=\sqrt{2\left(1-\frac{b^2}{1-c^2}\right)}\times cM_pH,$$

After integrating and considering the constant of integration is zero i.e $\phi(a_o=1)=0$, we get

$$\phi(a)=cM_p\sqrt{2\left(1-\frac{b^2}{1-c^2}\right)}\times lna. \tag{32}$$

Hence, from Eq. (30), we have

$$V(\phi)=\frac{1}{2}\left(1+\frac{b^2}{1-c^2}\right)\times 2c^2M_p^2H^2. \tag{33}$$

Again from Eq. (25) with Eqs. (19) and (20), we have

$$\frac{\dot{H}}{H^2}=-\left[1-\frac{b^2c^2}{1-c^2}\right]. \tag{34}$$

Let us take $k=\left(1-\frac{b^2c^2}{1-c^2}\right)$ then the above equation reduces in the form of

$$\frac{\dot{H}}{H^2}=-k.$$

Integrating on both sides, we have

$$H=\frac{1}{kt}. \tag{35}$$

Again, integrating on both sides, we have

$$a(t)=t^{1/k}. \tag{36}$$

In this case Eq. (32) can be written as

$$\phi(t)=\frac{cM_p}{k}\sqrt{2\left(1-\frac{b^2}{1-c^2}\right)}logt, \tag{37}$$

and from Eq. (33), we get

$$V(\phi)=\frac{c^2M_p^2}{k^2}\left(1+\frac{b^2}{1-c^2}\right)\times\frac{1}{t^2}. \tag{38}$$

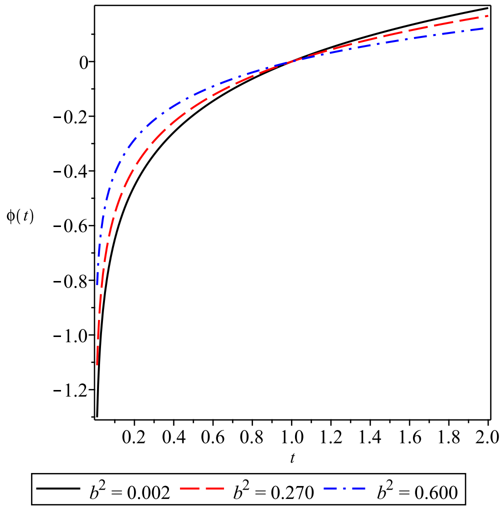

Fig. 1. The kinetic energy ϕ is shown against time for the different values of the constants $c = 0.1, B = 1, M_p^2 = 1, k = 0.5, b^2 = 0.200(black), 0.270(red), 0.600(blue)$.

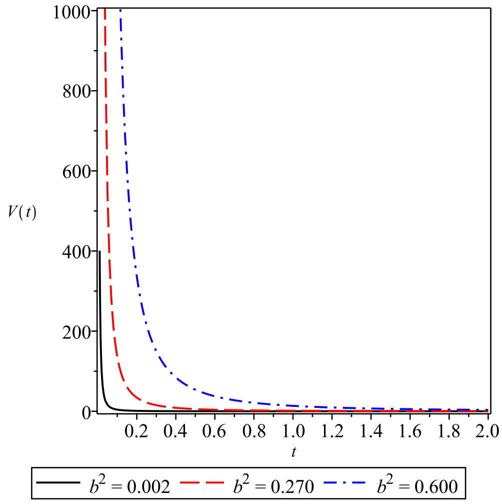

Fig. 2. The potential $V(\phi)$ is shown against time for the different values of the constants $c = 0.1, B = 1, M_p^2 = 1, k = 0.5, b^2 = 0.200(black), 0.270(red), 0.600(blue)$.

By using Eq. (37), we have

$$t^2 = exp\left[\frac{2k\phi(t)}{cM_p}\left(2 - \frac{2b^2}{1-c^2}\right)^{-1/2}\right], \tag{39}$$

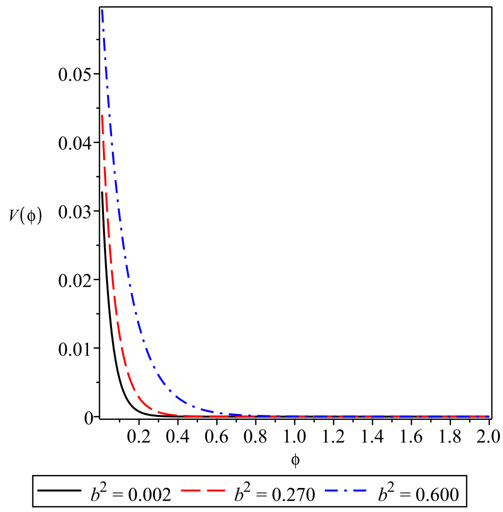

Fig. 3. The potential $V(\phi)$ is shown against time for the different values of the constants $c = 0.1, B = 1, M_p^2 = 1, k = 0.5, b^2 = 0.200(black), 0.270(red), 0.600(blue)$.

put this value in Eq. (38), we get

$$V(\phi) = \frac{c^2 M_p^2}{k^2}\left(1 + \frac{b^2}{1-c^2}\right) \times exp\left[\frac{-2k\phi(t)}{cM_p}\left(2 - \frac{2b^2}{1-c^2}\right)^{-1/2}\right], \tag{40}$$

From the scale factor Eq. (36) it is noted that the obtained potential leads to an accelerated universe at the present time. This expression is different than the expression obtained earlier by Sheykhi [28]. In addition to the fact that the exponential potential can give rise to an accelerated expansion in which the field energy density ρ_ϕ is proportional to the matter energy density ρ_m.

From the Eq. (26) we can easily implemented a correspondence between interacting HDE and scalar field model of the type tachyon, $K-$essence and dilaton energy density in a $(2+1)-$dimension FRW universe. It is observed that we get analogous results obtained earlier by Sheykhi [28] in the framework of $(3+1)-$ dimension spacetime. Hence, we are not discussed the same here in details. But we only discussed the graphical representation of the potential function of scalar field $V = V(\phi)$ and dynamics of the scalar field $\phi = \phi(t)$ in the concluding remark of the paper that was not discussed earlier by Sheykhi [28].

5. Interacting Holographic MCG Model

In this section we have generalized the work of Ghose [29] and obtained HDE model by considering interacting MCG in $(2+1)-$ dimensional spacetime. We consider the interaction quantity Q to be of the form $Q = \Gamma\rho_\Lambda$ and denote the ratio of the energy densities for the two fluids with r, i.e. $r = \frac{\rho_m}{\rho_D}$. For $Q > 0$, one define decay of MCG into CMD and Γ is the decay rate.

We define effective equation of state parameters as given by Setare [30],

$$\omega_D^{eff} = \omega_D + \frac{\Gamma}{2H}$$

and

$$\omega_m^{eff} = -\frac{1}{r}\frac{\Gamma}{2H}. \tag{41}$$

The conservation equations are given by

$$\dot{\rho_m} + 2H\rho\Big(1 + \omega_m^{eff}\Big) = 0, \tag{42}$$

$$\dot{\rho_D} + 2H\rho\Big(1 + \omega_D^{eff}\Big) = 0, \tag{43}$$

Density parameters are defined as

$$\Omega_m = \frac{\rho_m}{\rho_{cr}} = \frac{\rho_m}{2M_p^2 H^2}$$

and

$$\Omega_D = \frac{\rho_D}{\rho_{cr}} = \frac{\rho_D}{2M_p^2 H^2}, \tag{44}$$

where $\rho_{cr} = 2M_p^2 H^2$.
In terms of density parameters equation (15) becomes

$$1 = \frac{1}{2M_p^2 H^2}(\rho_D + \rho_m),$$

By using Eq. (44), we get

$$\Rightarrow \Omega_D + \Omega_m = 1. \tag{45}$$

Again, by using Eq. (15) and (44), we obtain

$$r = \frac{1 - \Omega_D}{\Omega_D}. \tag{46}$$

After inserting Eq. (7) into Eq. (13), we have

$$\rho_D = \left[\frac{B}{(1+A)} + \frac{C}{a^{2(1+\alpha)(1+A)}}\right]^{\frac{1}{1+\alpha}}. \tag{47}$$

where C is the constant of integration.
The EoS parameter ω_D is given by

$$\omega_D = \frac{p_D}{\rho_D} = \left[A - \frac{B}{\rho_D^{\alpha+1}}\right],$$

By using Eq. (47), we get

$$\omega_D = A - \frac{B}{\left[\frac{B}{(1+A)} + \frac{C}{a^{2(1+\alpha)(1+A)}}\right]}. \tag{48}$$

From Eq. (41), we have

$$\omega_D^{eff} = A - \frac{B}{\left[\frac{B}{(1+A)} + \frac{C}{a^{2(1+\alpha)(1+A)}}\right]} + \frac{\Gamma}{2H}. \tag{49}$$

We choose decay rate which is given as

$$\Gamma = 2b^2(1+r)H, \tag{50}$$

where b^2 is coupling constant.
By using Eqs. (50) and (26) into Eq. (49), we get

$$\omega_D^{eff} = b^2\left(\frac{1}{\Omega_D} - \frac{1}{1-c^2}\right). \tag{51}$$

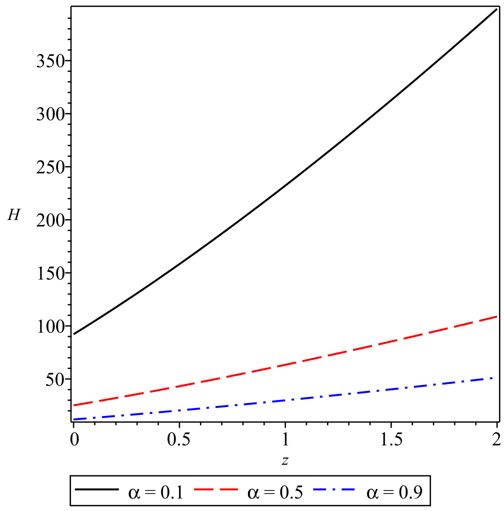

Fig. 4. Hubble parameter versus red shift the different values of the constants $A = 1/3, B = 1, \alpha = 0.1(black), \alpha = 0.5(red), \alpha = 0.9(blue)$.

If we establish the correspondence between the holographic dark energy and MCG energy density the by equating Eqs. (19) and (47), we have

$$2c^2M_p^2H^2 = \left[\frac{B}{(1+A)} + \frac{C}{a^{2(1+\alpha)(1+A)}}\right]^{1+\alpha},$$

$$C = a^{2(1+\alpha)(1+A)}\left[\left(2c^2M_p^2H^2\right)^{(1+\alpha)} - \frac{B}{(1+A)}\right]. \tag{52}$$

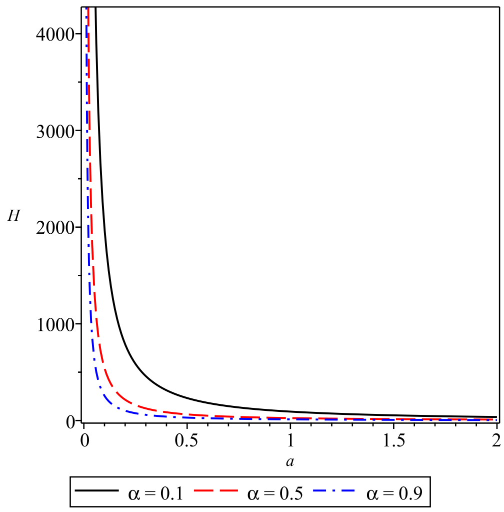

Fig. 5. Hubble parameter versus red shift the different values of the constants $A = 1/3, B = 1, \alpha = 0.1(black), \alpha = 0.5(red), \alpha = 0.9(blue)$.

Now, using the EoS parameter of holographic dark energy from Eqs. (26) and (48), we get

$$A - \frac{B}{\left[\frac{B}{(1+A)} + \frac{C}{a^{2(1+\alpha)(1+A)}}\right]} = \frac{-b^2}{(1-c^2)},$$

Substituting the value of C in the above equation we get the following

$$B = (2c^2 M_p^2 H^2)^{(1+\alpha)} \left[A + \frac{b^2}{(1-c^2)}\right]. \tag{53}$$

Substituting the value of B in Eq. (52) we get the value of C as

$$C = \frac{(2c^2 M_p^2 H^2 a^{2(1+A)})^{(1+\alpha)}}{1+A} \bigg(1 + \omega_D\bigg). \tag{54}$$

It is noted that from Eqs. (53) and (54) that B and C are time-dependent which was also obtained by Setare[30]. By using CG as an alternative to quintessence [5] obtained as $A = \Lambda(\Lambda + \rho_m)$. Shapiro [31] observed that this result leads to the fact that if Λ vary with time, B does not remain constant.

6. Squared Speed of Sound in CG and Stability of the Model

The squared speed of sound is defined as

$$v_g^2 = \frac{dp_D}{d\rho_D}. \tag{55}$$

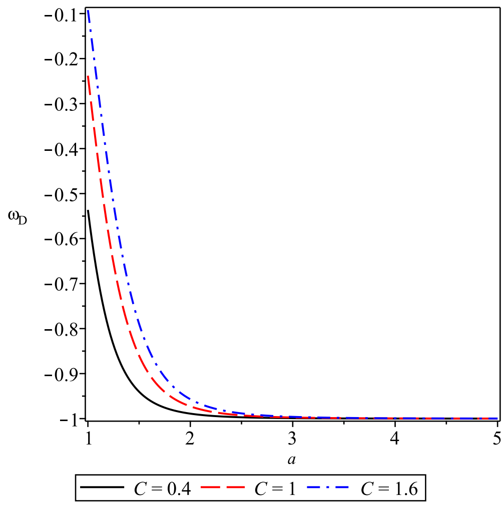

Fig. 6. EoS parameterω_D versus scale factor for the different values of the constants $A = 1/3, B = 1, \alpha = 0.5, C = 0.4(black), C = 1(red), C = 1.6(blue)$.

The MCG model is unstable if $v_g^2 < 0$. In our holographic MCG model, we get

$$v_D^2 = \frac{dp_D}{d\rho_D} = \frac{\dot{p_D}}{\dot{\rho_D}}. \tag{56}$$

In this case $\dot{p_D}$ is given by

$$\dot{p_D} = \dot{\omega}_D^{eff}\rho_D + \omega_D^{eff}\dot{\rho_D}, \tag{57}$$

where "$(\cdot)$" means differentiation with respect to cosmic time.
Divide Eq. (57) throughout by $\dot{\rho_D}$, we get the expression for squared speed as

$$v_D^2 = \omega_D^{eff} + \dot{\omega}_D^{eff}\frac{\rho_D}{\dot{\rho_D}}. \tag{58}$$

By using Eq. (51), we obtain

$$\dot{\omega}_D^{eff} = \frac{b^2}{\Omega_D^2}\dot{\Omega}_D, \tag{59}$$

where $\dot{\Omega}_D$ is determined from Eqs. (44) and (18), we get

$$\dot{\Omega}_D = 2c^2\frac{\dot{H}}{H}, \tag{60}$$

By using $\dot{\omega}_D^{eff}$ and $\dot{\Omega}_D$ in Eq. (59), we get

$$v_D^2 = b^2\left(\frac{1}{\Omega_D} - \frac{1}{1-c^2} + \frac{c^2}{\Omega_D^2}\right). \tag{61}$$

Using the observed value $\Omega_D \approx 0.73$ in Eq. (61), we get $v_D^2 = 0.10220$ which ensures that $v_D^2 > 0$ thus, it evident that the modified Chaplygin gas model is stable in the framework of $(2+1)-$dimensional spacetime.

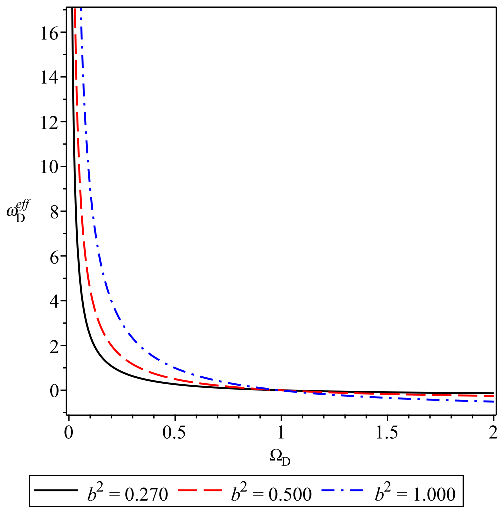

Fig. 7. Effective DE equation of state ω_D^{eff} versus Ω_D for the different values $c = 0.1, b^2 = 0.270(black), b^2 = 0.500(red), b^2 = 1.000(blue)$.

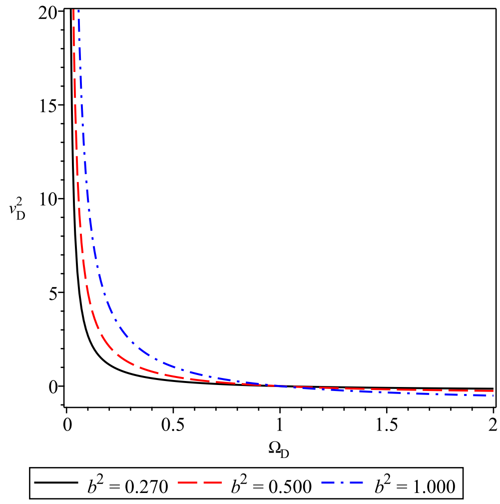

Fig. 8. Squared speed for MCG v_D^2 versus Ω_D for the different choice of b^2.

7. Discussion and Conclusion

This paper is divided into two parts in the first part of the paper we generalized the work of Sheykhi [28] by choosing the Hubble radius $L = H^{-2/3}$ as system's IR cutoff for interacting holographic dark energy and established a connection between the scalar-field model of dark energy including quintessence, tachyon, $K-$essence, and dilaton energy density and holographic energy density of the form $\rho_D = 2c^2 M_p^2 H^2$

in the framework of $(2+1)-$dimension theory of gravitation. As a result, we have reconstructed the analytical form of potentials, namely, $V = V(\phi)$ as well as the dynamics of the scalar fields as a function of time explicitly, namely, $\phi = \phi(t)$.

According to the evolutionary behavior of the interacting holographic dark energy model, we found that in the case of quintessence from the scale factor Eq. (36) it is noted that the obtained potential leads to an accelerated universe at the present time. Requiring $\ddot{a} > 0$ for the present time, leads to $k < 0$, which can be translated into $c^2 > 0$. Note that the condition $k < 1$ valid only for the late time where we have a dark energy universe. In general k depends on c, and for matter dominated epoch where c is no longer a constant, then k is also not a constant and varies with time. We have obtained the value of Hubble parameter is standard i.e. the age of the universe. In addition to the fact that the exponential potential can give rise to an accelerated expansion in which the energy density ρ_D is propositional to the matter energy density ρ_m. In the Fig. (1) shows that the kinetic energy ϕ increases as cosmic time t increases and the potential $V(\phi)$ decreases as the cosmic time t increases in fig. (2) while in the Fig. (3) we observed that the potential $V(\phi)$ decreases as kinetic energy ϕ increases for the holographic quintessence model.

For the holographic tachyon model with a equation of state parameter given by

$$\omega_T = \frac{p}{\rho} = \dot{\phi}^2 - 1. \tag{62}$$

To established the correspondence between HDE and tachyon field, we equate the value of ω_D from Eq. (26) with the value of ω_T from Eq. (62), we get

$$\phi = \left[1 - \frac{b^2}{1-c^2}\right]^{1/2} t, \tag{63}$$

Similarly, by using above value of ϕ, we get

$$V(\phi) = 2c^2 M_p^2 \times \frac{1}{k^2 t^2} \times \frac{b}{\sqrt{1-c^2}}. \tag{64}$$

From Eq. (63) we obtain tachyon potential in terms of the scalar field as

$$V(\phi) = \frac{2c^2 M_p^2}{k^2} \frac{b}{\sqrt{1-c^2}} \left(1 - \frac{b^2}{1-c^2}\right) \frac{1}{\phi^2(t)}. \tag{65}$$

It is observed that the evolution of tachyon model is given by $\phi(t)$ is propositional to the $'t'$ and the tachyon potential is inverse a square power law corresponding to the solution obtained early by Copeland [32]. From the Fig. (9) shows that the kinetic energy ϕ increases as cosmic time $'t'$ increases and the potential $V(\phi)$ decreases as the cosmic time t increases in fig. (10). while in the Fig. (11) shows that the potential $V(\phi)$ decreases as kinetic energy ϕ increases.

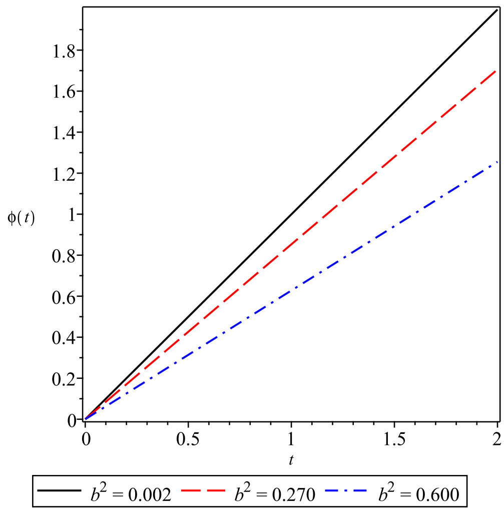

Fig. 9. The kinetic energy ϕ is shown against time for the different values of the constants $c = 0.1, B = 1, M_p^2 = 1, k = 0.5, b^2 = 0.200(black), 0.270(red), 0.600(blue)$.

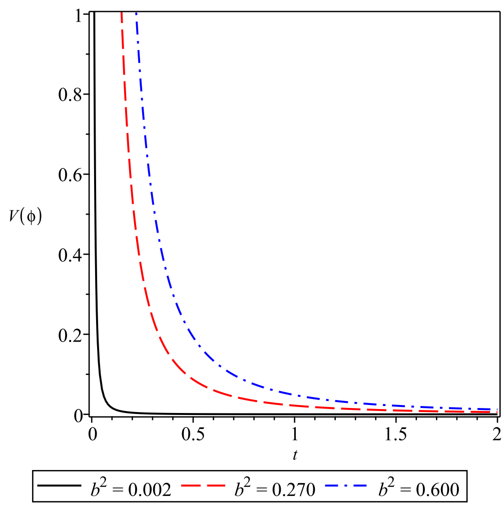

Fig. 10. The potential $V(\phi)$ is shown against time for the different values of the constants $c = 0.1, B = 1, M_p^2 = 1, k = 0.5, b^2 = 0.200(black), 0.270(red), 0.600(blue)$.

For the holographic $K-$essence model the potential with the equation of state parameter is given by

$$\omega_K = \frac{X - 1}{3X - 1}, \tag{66}$$

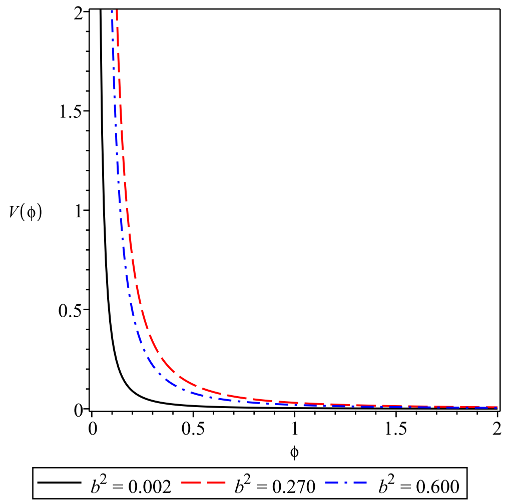

Fig. 11. The potential $V(\phi)$ is shown against time for the different values of the constants $c = 0.1, B = 1, M_p^2 = 1, k = 0.5, b^2 = 0.200(black), 0.270(red), 0.600(blue)$.

Equating the value of ω_D from Eq. (26) with the value of ω_K from Eq. (66), we get

$$X = \frac{1 + b^2 - c^2}{1 + 3b^2 - c^2}. \tag{67}$$

Here, X is a positive constant ($c^2 < 1$). The EoS parameter in Eq. (66) diverges for $X = 1/3$. Let us consider the cases with $X > 1/3$ and $X < 1/3$ separately. In the first case where $X > 1/3$, the condition $\omega_K < -1/3$ leads to $X < 2/3$. Thus, we should have $1/3 < X < 2/3$ in this case. We obtain the EoS of a cosmological constant ($\omega_K = -1$) for $X = 1/2$. In the second case where $X < 1/3$, we have $X - 1 < -2/3 < 0$, thus $\omega_K = \frac{X-1}{3X-1} > 0$ which means that we have no acceleration at all so this case is ruled out. As a result in $K-$essence model the accelerated universe can be achieved provided $1/3 < X < 2/3$ which translates into $1 - 3b^2 < c^2 < 1$. Combining Eq. (67) with $X = -\dot{\phi}^2/2$ given by Armendariz-Picon [33]

$$\dot{\phi}^2 = 2\left(\frac{1 + b^2 - c^2}{1 + 3b^2 - c^2}\right), \tag{68}$$

and thus we obtain the expression for the scalar field in the flat FRW background

$$\phi(t) = \left[2\left(\frac{1 + b^2 - c^2}{1 + 3b^2 - c^2}\right)\right]^{1/2} t, \tag{69}$$

where we have taken the integration constant ϕ_0 equal to zero.
By taking the correspondence between HDE and $K-$essence into this account. Equating $\rho(\phi, X) = f(\phi)(-X + 3X^2)$ with the Eq. (26) we have

$$f(\phi) = \frac{2c^2 M_p^2}{k^2}\left[\frac{1 + 3b^2 - c^2}{c^2 - 1}\right]\frac{1}{\phi^2(t)}. \tag{70}$$

For the holographic $K-$essence model the potential has a power law expansion and it is noted that from Eq. (69) then $\dot{\phi}$ is constant this means that kinetic energy of $K-$essence become constant through ϕ is not constant and evolves with time.

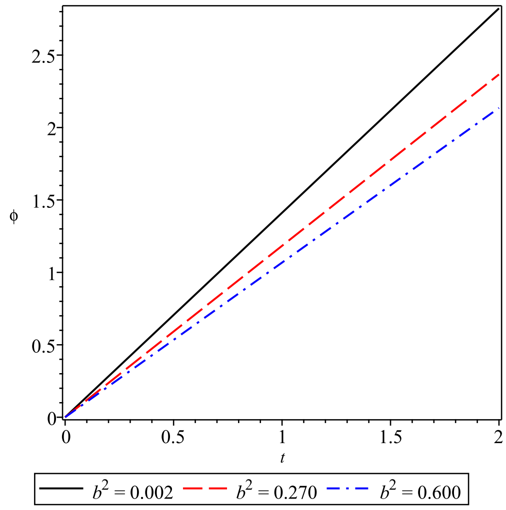

Fig. 12. The kinetic energy ϕ is shown against time for the different values of the constants $c = 0.1, B = 1, M_p^2 = 1, k = 0.5, b^2 = 0.200(black), 0.270(red), 0.600(blue)$.

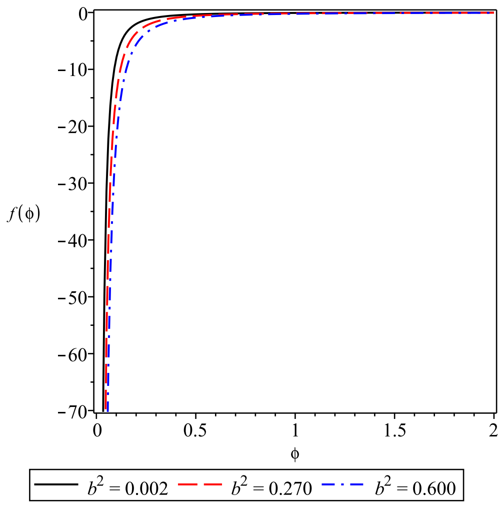

Fig. 13. The potential $V(\phi)$ is shown against time for the different values of the constants $c = 0.1, B = 1, M_p^2 = 1, k = 0.5, b^2 = 0.200(black), 0.270(red), 0.600(blue)$.

In the Fig. (12) it is observed that the kinetic energy ϕ of $K-$essence becomes constant as cosmic time $'t'$ increases while the potential $f(\phi)$ decreases as the cosmic time ϕ increases in fig.(13).

For the case of holographic dilaton field with the equation of state parameter is given as

$$\omega_d = \frac{1 - \alpha e^{\lambda\phi} X}{1 - 3\alpha e^{\lambda\phi} X}. \tag{71}$$

To establish the correspondence between HDE and dilaton field we equate the value of ω_D from Eq. (26) with the value of ω_d from Eq. (71), we get

$$\phi = \frac{2}{\lambda} ln\left[\frac{\lambda}{\sqrt{2\alpha}}\left(\frac{1+b^2-c^2}{1+3b^2-c^2}\right)^{1/2} t\right]. \tag{72}$$

We used the correspondence to find the form of scalar field and established the region for the constant in which we can except the scaling solution giving rise to accelerated expansion.

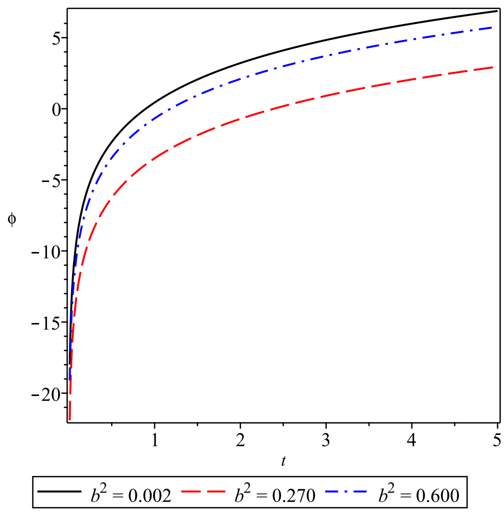

Fig. 14. The kinetic energy ϕ is shown against cosmic time for the different values of the constants $c = 0.1, B = 1, M_p^2 = 1, k = 0.5, b^2 = 0.002(black), b^2 = 0.270(red), b^2 = 0.600(blue)$.

The existence of scaling solutions for the dilaton has been studied by Piazza and Tsujikawa [34] and found that in this case the scaling solution corresponds to $Xe^{\lambda\phi} =$ constant which has the solution from above Eq. $\phi(t) \propto lnt$. We found the results by equating the EoS parameter of HDE and dilaton field are consistent which was obtained earlier by Piazza and Tsujikawa [34]. In this work for the simplicity we have taken c is constant. From the Fig. (14) shows that the kinetic energy ϕ of dilaton

field increases as cosmic time $'t'$ increases.
In the second part of the paper we have investigated the holographic dark energy of modified Chaplygin gas in the framework of $(2+1)-$dimensional spacetime. We have obtained the evolution of modified holographic dark energy with corresponding equation of state by considering the present value of the density parameter and found stable configuration which accommodate dark energy (DE). Here, we have noted a connection between dark energy and phantom fields. It reveals that the dark energy might have evolved from a phantom state in the past. We have also obtained the stability of the model and analyzed the physical and geometrical interpretations of the cosmological model in the framework of $(2+1)-$dimensional spacetime. In the Fig. (4) it is observed that $A = 1/3$ and $B = 1$ the behaviour of Hubble parameter H in terms of red shift is quite close to observations Thakur [35], while the Hubble parameter H decreases as the scale factor a increases in fig. (5).
From the Fig. (6) shows that $A = 1/3$ and $B = 1$ the behaviour of EoS parameter ω_D approaches -1 as scale factor $'a'$ increases while from the fig. (7) it is shown that the behaviour of effective DE equation of state parameter ω_D^{eff} decreases as EoS parameter ω_D increases. From the fig. (8) it is observed that the behaviour of squared speed for modified Chaplygin gas v_D^2 decreases as EoS parameter ω_D increases for the different choice of b^2.

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