Journal of The Tensor Society (J.T.S.)
Vol. 15 (2021), page 45 - 52
https://doi.org/10.56424/jts.v15i01.10614

A remark on CR-structures

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Received : August 13, 2021
Accepted : October 10, 2021
Published : January 27, 2022

The present paper aims to establish the relationship between CR-structure and the quadratic structure and find some basic results. Integrability conditions and certain theorems on CR-structure and the quadratic structure are discussed.

2020 Mathematics Subject Classification: 53C15.
Keywords: CR-structures, Almost complex structure, Nijenhuis tensor, Integrability.

1. Introduction

The tangent bundle of submanifolds in the differential geometry is very fascinating. There are many types of submanifolds regarding the almost complex structure of the ambient manifold, but we consider main three, namely, given as holomorphic submanifolds, totally real submanifolds, and CR-(Cauchy-Riemannian) submanifolds. Bejancu [1] has delibrate CR-submanifold of a Kahlerian manifold in which a new class of submanifolds of the complex manifold was start off. Bejancu has launched the concept of CR-submanifold and produce its basic effects. Numerous investigators made treasures contributions to CR-submanifolds including Bejancu [2], Blair and Chen [3], Chen [4], Dragomir at el [5], Yao and Kon [15] and Khan [8 9 13]. In this paper, we study the integrability conditions and Nijenhuis tensor on CR-structures and the quadratic structure.

Let us suppose the quadratic equation $x^2 + x + 1 = 0$. The set of solutions denoted by $x = \frac{1}{2}(1 \pm \sqrt{3}i)$. In n-dimensional manifold M, consider a tensor field $F(\neq 0)$ of the type (1, 1) and of class C^∞ on M such that

$$F^2 + F + I = 0 \tag{1}$$

such structure on M is called the quadratic structure of rank r. If the rank of F is constant and $r = r(F)$, then M is called the quadratic manifold.

The projection operators are given by

$$l = -(F^2 + F), \quad m = I + F^2 + F \tag{2}$$

where I denotes the identity operator on M.

Proposition 1.1 Let M be the quadratic manifold. Then

$$l + m = I, \quad l^2 = l, \quad and \quad m^2 = m. \tag{3}$$

Proof: Making use of equation (2), the theorem can be easily proved.

For $F \neq 0$ satisfying equation (1), there exist complementary distributions D_l and D_m corresponding to the projection operators l and m respectively. If the $rank(F) = constant$ and $r = r(F)$ on M, then $dim D_l = r$ and $\dim D_m = n - r$ [7, 8].

Proposition 1.2 Let M be the quadratic manifold. Then

$$Fl = lF = F, \quad Fm = mF = 0 \tag{4}$$

$$F^2 + F = -l, \quad (F^2 + F)l = -l, \quad (F^2 + F)m = 0. \tag{5}$$

Thus $(F^2 + F)^{\frac{1}{2}}$ acts on D_l as an almost complex structure and on D_m as a null operator.

Proof: Making use of equation (1), the theorem can be easily proved.

2. Nijenhuis tensor

Definition 2.1 If X, Y are two vector fields in M, then their Lie bracket $[X, Y]$ is given by [7]

$$[X, Y] = XY - YX. \tag{6}$$

The Nijenhuis tensor $N(X, Y)$ of F satisfying equation (1) in M is expressed as follows

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y], \tag{7}$$

for every vector field X, Y on M. Now, state the following proposition [14] for later use:

Proposition 2.1 A necessary and sufficient condition for the quadratic structure F to be integrable is that $N(X, Y) = 0$ for any two vector fields X and Y on M.

3. CR-structure

Suppose M be a differentiable manifold and T_cM is complexified tangent bundle. A CR-structure on M is a complex subbundle H of T_cM such that $H_p \cap \bar{H}_p = 0$ and H is involutive, i.e., for complex vector fields X and Y in H, $[X, Y]$ is in H. In such case we can say M is a CR-manifold. Let F be the quadratic integrable structure satisfying equation (1) of rank $r = 2m$ on M. We define complex subbundle H of T_cM by $H_p = \{X - \sqrt{-1}FX, X \in \chi(D_l)\}$, where $\chi(D_l)$ is the $\wp(D_m)$ module of all differentiable sections of D_l. Then $Re(H) = D_l$ and $H_p \cap \bar{H}_p = 0$, where $\bar{H}_p$ represents the complex conjugate of H [7].

Proposition 3.1 Let $P, Q \in H$, then we have

$$[P, Q] = [X, Y] - [FX, FY] - \sqrt{-1}([X, FY] + [FX, Y]), \tag{8}$$

for every vector field X, Y on M.

Proof: Consider $P = X - \sqrt{-1}FX$ and $Q = Y - \sqrt{-1}FY$. Then we have

$$\begin{aligned}[P, Q] &= [X - \sqrt{-1}FX, Y - \sqrt{-1}FY] \\ &= [X, Y] - [FX, FY] - \sqrt{-1}([X, FY] + [FX, Y]).\end{aligned}$$

Proposition 3.2 If the quadratic structure satisfying equation (1) is integrable, then we have

$$(F + I)([FX, FY] + F^2[X, Y]) = -l([FX, Y] + [X, FY]), \tag{9}$$

for every vector field X, Y on M.

Proof: Since $N(X, Y) = 0$, from (7) we get

$$[FX, FY] + F^2[X, Y] = F([FX, Y] + [X, FY]) \tag{10}$$

multiplying equation (10) by $F + I$, we get

$$(F + I)([FX, FY] + F^2[X, Y]) = (F^2 + F)([FX, Y] + [X, FY]) \tag{11}$$

making use of equation (2), we obtain

$$(F + I)([FX, FY] + F^2[X, Y]) = -l([FX, Y] + [X, FY]) \tag{12}$$

This completes the proof.

Theorem 3.3 The following conditions are equivalent

$$\begin{aligned} i. &\quad mN(X, Y) = 0, \\ ii. &\quad m[FX, FY] = 0, \\ iii. &\quad mN((F^2 + F)X, Y) = 0, \\ iv. &\quad m(F^2 + F)FX, FY] = 0, \\ v. &\quad m[(F^2 + F)lFX, FY] = 0. \end{aligned}$$

where X and Y are vector fields.

Proof: We have to show that

$(i) \Leftrightarrow (ii)$

Let $mN(X, Y) = 0$. Then in the perspective of equation (7), we have

$$mN(X, Y) = m[FX, FY] - mF[FX, Y] - mF[X, FY] + mF^2[X, Y],$$

since $mF = 0$, we have

$$mN(X, Y) = m[FX, FY],$$

since $mN(X,Y) = 0$, then

$$m[FX, FY] = 0,$$

Thus,

$$mN(X,Y) = 0 \Leftrightarrow m[FX, FY] = 0.$$

$(ii) \Leftrightarrow (iii)$

Let $m[FX, FY] = 0$. Then in the perspective of equation (7), we have

$$\begin{aligned} mN((F^2+F)X,Y) &= m[(F^2+F)FX, FY] - mF[(F^2+F)FX, Y] \\ &\quad - mF[(F^2+F)X, FY] + mF^2[(F^2+F)X, Y], \end{aligned}$$

since, $(F^2 + F) = -l, \quad mF = 0$, then above equation becomes

$$mN((F^2+F)X,Y) = m[-lFX, FY] = -m[FX, FY] as \;\; lF = F,$$

since $m[FX, FY] = 0$, then we have

$$mN((F^2+F)X,Y) = 0,$$

Thus,

$$m[FX, FY] = 0 \Leftrightarrow mN((F^2+F)X,Y) = 0.$$

$(iii) \Leftrightarrow (iv)$

Let $mN((F^2+F)X,Y) = 0$ and using $mF = 0$. Then we have

$$mN((F^2+F)X,Y) = m[(F^2+F)FX, FY],$$

since,

$$m[(F^2+F)FX, FY] = 0$$

Thus,

$$mN((F^2+F)X,Y) = 0 \Longrightarrow m[(F^2+F)FX, FY] = 0.$$

$(iv) \Leftrightarrow (v)$

Let $m[(F^2+F)FX, FY] = 0$ and using $lF = F, \quad Fm = 0$. Then we have

$$\begin{aligned} mN((F^2+F)lX,Y) &= m[(F^2+F)lFX, FY], \\ &= m[(F^2+F)FX, FY] \quad as\; lF = F, \\ So \;\; m[(F^2+F)lFX, FY] &= m[(F^2+F)FX, FY], \\ As \quad m[(F^2+F)FX, FY] &= 0, \end{aligned}$$

Thus,

$$m[(F^2+F)FX, FY] = 0 \Longrightarrow m[(F^2+F)lFX, FY] = 0.$$

$(v) \Leftrightarrow (i)$ Let $m[(F^2+F)lFX, FY] = 0$.

Making use of (2) and (3), we obtain

$$m[-l^2FX, FY] = 0,$$

$$-m[lFX, FY] = 0'$$

$$m[FX, FY] = 0, \quad as \quad lF = F,$$

Since $mN(X, Y) = m[FX, FY] \;\; as \;\; m[FX, FY] = 0,$

$$mN(X, Y) = 0,$$

Thus, $m[(F^2 + F)lFX, FY] = 0 \Longrightarrow mN(X, Y) = 0.$

This completes the proof.

Proposition 3.4 If $(F^2 + F)^{\frac{1}{2}}$ acts on D_l as an almost complex structure, then

$$m[(F^2 + F)lX, FY] = m[-X, FY] = 0. \tag{13}$$

for every vector field X, Y on M.

Proof: From equation (5), we know that $(F^2 + F)^{\frac{1}{2}}$ acts on D_l as an almost complex structure then equation (13) follows in an obvious manner. To prove that $m[(F^2 + F)lX, FY] = 0$, by using the formula $[X, Y] = XY - YX$ where X, Y are C^∞ vector fields and equation (5), we obtain equation (13).

Proposition 3.5 For $X, Y \in \chi(D_l)$, we have

$$l([X, FY] + [FX, Y]) = [X, FY] + [FX, Y].$$

Proof: By Definition (2.1), we have

$$[X, Y] = XY - YX$$

Now,

$$\begin{aligned} l[X, FY] &= l(XFY - FYX) \\ &= X(lFY) - (lF)YX \\ &= XFY - FYX, \quad as \quad lF = F, \quad using \quad equation (4) \\ &= [X, FY] \end{aligned}$$

Similarly, $l[FX, Y] = [FX, Y],$

Hence,

$$l([X, FY] + [FX, Y]) = [X, FY] + [FX, Y].$$

Theorem 3.6 The integrable quadratic structure satisfying equation (1) on M defines a CR-structure H on it such that $ReH \equiv D_l$.

Proof: To prove the quadratic equation satisfies equation (1) defines CR-structure on M it suffices to prove $[P, Q] \in \chi(D_l)$.

From equation (8), we have

$$[P, Q] = [X, Y] - [FX, FY] - \sqrt{-1}([X, FY] + [FX, Y]). \tag{14}$$

Now,

$$\begin{aligned} [P, Q] - \sqrt{-1}F[P, Q] = {} & [X, Y] - [FX, FY] - \sqrt{-1}([X, FY] + [FX, Y]) \\ & - \sqrt{-1}F([X, Y] - [FX, FY]) - F([X, FY] + [FX, Y]). \end{aligned}$$

Making use of Theorem (3.5) and equation (9), we obtain

$$[P,Q]-\sqrt{-1}F[P,Q]=[X,Y]-[FX,FY]-\sqrt{-1}(F^2+F)([FX,FY]+F^2[X,Y]) \\ -\sqrt{-1}F([X,Y]-[FX,FY]-\sqrt{-1}(F^2+F)([FX,FY]+F^2[X,Y])),$$

by definition of CR-structure, we have

$$[P,Q]\in\chi(D_l).$$

This completes the proof.

Definition 3.1 Let $\tilde{K}$ be the complementary distribution of $Re(H)$ to TM. We define a morphism of vector bundles $F : TM \to TM$ given by $F(X) = 0$ for all $X \in \chi(\tilde{K})$, such that

$$F(X)=\frac{1}{2}\sqrt{-1}(P-\bar{P}) \tag{15}$$

where $P = X + \sqrt{-1}Y \in \chi(H_p)$ and $\bar{P}$ is a complex conjugate of P [7].

Corollary 3.1 If $P = X + \sqrt{-1}Y$ and $\bar{P} = X - \sqrt{-1}Y$ belong to H_p and $F(X) = \frac{1}{2}\sqrt{-1}(P-\bar{P}), F(Y) = \frac{1}{2}(P+\bar{P})$ and $F(-Y) = -\frac{1}{2}(P+\bar{P})$, then $F(X) = -Y, F^2(X) = -X$ and $F(-Y) = -X$.

Proof: On using Definition (3.1), we have

$$\begin{aligned} F(X) &= \frac{1}{2}\sqrt{-1}(X+\sqrt{-1}Y-(X-\sqrt{-1}Y)) \\ &= \frac{1}{2}\sqrt{-1}(2\sqrt{-1}Y) \\ F(X) &= -Y. \end{aligned} \tag{16}$$

On operating F both sides of equation (16), we obtain

$$F(F(X)) = F(-Y). \tag{17}$$

But

$$F(Y)=\frac{1}{2}(X+\sqrt{-1}Y+X-\sqrt{-1}Y),$$

which on simplifying gives

$$F(Y) = X.$$

Also,

$$\begin{aligned} F(-Y) &= -\frac{1}{2}(X-\sqrt{-1}Y+X+\sqrt{-1}Y) \\ &= -X. \end{aligned} \tag{18}$$

Combining equations (17) and (18), we get

$$F^2(X) = -X.$$

Theorem 3.7 If M has a CR-structure H, then we have $F^2 + F + I = 0$ and consequently the quadratic structure is defined on M such that the distributions D_l and D_m coincide with $Re(H)$ and $\tilde{K}$ respectively.

Proof: Suppose M has a CR-structure on M. Then in view of Definition (3.1) and Corollary (3.1) we can write [7]

$$F(X) = -Y;. \tag{19}$$

Operating equation (19) by $F + I$, we get

$$\begin{aligned}
(F+I)F(X) &= (F+I)(-Y) \\
&= F(-Y) - Y = -X - Y \\
&= -X + F(X), \quad as \quad FX = -Y \\
&= F^2(X) + F(X), \quad as \quad F^2X = -X \\
&= -X, \quad as \quad F^2 + F + I = 0 \\
(F+I)F(X) &= -X, \\
(F+I)F &= -I,
\end{aligned}$$

Hence,

$$F^2 + F + I = 0.$$

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