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Pseudo projective symmetric manifolds and Gray's decomposition

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In this paper we characterize pseudo projective symmetric manifolds endowed with the Gray's Decomposition. Moreover, we study conformally flat pseudo projective symmetric manifolds and establish that it is a manifold of quasi constant curvature.

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1. Introduction

In this paper, due to its fascinating role in differential geometry, we are concentred on symmetric manifolds. These manifolds were introduced by E. Cartan [3], in the late twenties as a generalization of manifolds with constant curvature. Since then, many abstractions and extensions of this curvature symmetry are studied. Semi-symmetric, Ricci symmetric, pseudo symmetric and many others are different such generalizations.

The projective curvature tensor, apart from the conformal curvature tensor, is an important tensor in differential geometry. Let N be a Riemannian manifold of dimension n. If any coordinate neighbourhood of N has a one-to-one correspondence with a domain in Euclidian space, and any geodesic of the Riemannian manifold corresponds to a straight line in Euclidean space, N is said to be locally projectively flat. For $n \geq 3$, N is locally projectively flat if and only if the well circulated projective curvature tensor $\mathcal{P}$ vanishes. Here $\mathcal{P}$ is defined by [13]

$$\mathcal{P}(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-1}\{S(Y,Z)g(X,U) - S(X,Z)g(Y,U)\}, \quad (1)$$

for $X, Y, Z, U \in \chi(N)$, where R is the curvature tensor and S indicates the Ricci tensor. In reality, N is projectively flat (that is, $\mathcal{P} = 0$) if and only if the manifold has constant curvature (pp. 84-85 of [22]). As a result, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In a Riemannian manifold (N^n, g) $(n > 3)$ the conformal curvature tensor is given by

$$\begin{aligned} C(U,V)W = R(U,V)W - \frac{1}{n-2}[&g(V,W)QU - g(U,W)QV \\ &+S(V,W)U - S(U,W)V] \\ +\frac{r}{(n-1)(n-2)}[&g(V,W)U - g(U,W)V], \end{aligned} \tag{2}$$

r being the scalar curvature, Q is the Ricci operator defined by $g(QY, Z) = S(Y, Z)$.

Let ∇ be the Levi-Civita connection of (N^n, g). If $\nabla R = 0$, then N^n, the Riemannian manifold is called locally symmetric [3]. The above local symmetry condition is equivalent to the case that $F(P)$, the local geodesic symmetry is an isometry [14], for each point $P \in N$. A very natural generalization of the class of Riemannian locally symmetric manifolds is the class of manifolds of constant curvature. Many geometers in several ways have weakened the interesting idea of locally symmetric manifolds over the past few decades, for more details see (Ref [1,4,5,15,17,20]).

According to Chaki [5], for a non-zero 1-form A the $(N^n, g), (n > 2)$, a non-flat Riemannian manifold is called pseudo symmetric if its curvature tensor obeys

$$\begin{aligned} (\nabla_V R)(X,Y,Z,U) = &2A(V)R(X,Y,Z,U) + A(X)R(V,Y,Z,U) \\ &+A(Y)R(X,V,Z,U) + A(Z)R(X,Y,V,U) \\ &+A(U)R(X,Y,Z,V), \end{aligned} \tag{3}$$

where ρ indicates the vector field defined by

$$g(X, \rho) = A(X), \tag{4}$$

for all X. The 1-form A is called the associated 1-form of N^n. If $A = 0$, then N^n reduces to a symmetric manifold in the Cartan sense. A pseudo symmetric manifold of dimension n is usually denoted by $(PS)_n$. The existence of such a manifold was ensured by Sen and Chaki [16] to study hypersurfaces of a conformally flat space of class one.

A non-flat Riemannian manifold is called pseudo projective symmetric [6] if the projective curvature tensor obeys

$$\begin{aligned} (\nabla_V \mathcal{P})(X,Y,Z,U) = &2A(V)\mathcal{P}(X,Y,Z,U) + A(X)\mathcal{P}(V,Y,Z,U) \\ &+A(Y)\mathcal{P}(X,V,Z,U) + A(Z)\mathcal{P}(X,Y,V,U) \\ &+A(U)\mathcal{P}(X,Y,Z,V). \end{aligned} \tag{5}$$

Throughout the paper we assume that ρ is a unit vector, that is, $A(\rho) = g(\rho, \rho) = 1$.

A non-flat Riemannian manifold obeying the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{6}$$

where a, b are real numbers and A being the 1-form, is named a quasi-Einstein manifold [8].

Gray [10] has presented the $\mathcal{O}(n)$-invariant orthogonal irreducible decomposition of the space of all tensors of type-$(0,3)$ obeying only the identities of the gradient of the Ricci tensor ∇S. This decomposition introduced the seven classes of Einstein-like manifolds whose Ricci tensors fulfill the defining condition of each subspace.

According to this investigation, in most of the cases, the pseudo projective symmetric manifold reduces to a quasi-Einstein manifold under certain restriction on the curvature tensor.

Hence, the present paper is organized as follows:

After brief discussion on pseudo projective symmetric manifolds in section 2 we examine all seven cases of the Gray's decomposition. Finally, we study conformally flat pseudo projective symmetric manifolds and establish that it is a manifold of quasi constant curvature.

2. Pseudo projective symmetric manifolds

Let

$$\mathcal{P}(X,U) = \sum_{i=1}^{n} \mathcal{P}(X, e_i, e_i, U), \tag{7}$$

where $\{e_i\}_{i=1}^{n}$ is an orthonormal basis of the tangent space at each point on the manifold N.

Then we acquire from (1)

$$\mathcal{P}(X,U) = \frac{n}{n-1} S(X,U) - \frac{r}{n-1} g(X,U), \tag{8}$$

where S denotes the Ricci tensor and r is the scalar curvature. Now, putting $Y = Z = e_i$ in (5), we get

$$\begin{aligned}(\nabla_V \mathcal{P})(X,U) = {} & 2A(V)\mathcal{P}(X,U) + A(X)\mathcal{P}(V,U) \\ & + \mathcal{P}(X,V,\rho,U) + \mathcal{P}(X,\rho,V,U) \\ & + A(U)\mathcal{P}(X,V),\end{aligned} \tag{9}$$

It is known that in a pseudo projective symmetric manifold [6],

$$r = constant \ \ and \ \ S(X,\rho) = \frac{r}{n} g(X,\rho). \tag{10}$$

Using (1), (8), (10) in (9), we infer that

$$\begin{aligned}(\nabla_V S)(X,U) = &\frac{2}{n}[nS(X,U) - rg(X,U)]A(V)\\ &+\frac{1}{n}[nS(U,V) - rg(U,V)]A(X)\\ &\frac{1}{n}[nS(X,V) - rg(X,V)]A(U)\\ &+\frac{n-1}{n}[R(X,V,\rho,U) + R(X,\rho,V,U)]\\ &-\frac{1}{n}[\frac{2r}{n}g(X,U)A(V) - \frac{r}{n}g(V,U)A(X) - S(X,V)A(U)]. \quad (11)\end{aligned}$$

3. Gray's Decompositions

In [10], Gray proposed that the gradient of the Ricci tensor, ∇S can be decomposed into $\mathcal{O}(n)$-invariant terms, (for more details, we refer [2,11]). This covariant derivative can be expressed as follows [12]:

$$\begin{aligned}(\nabla_Z S)(V,W) = &\tilde{R}(Z,V)W + \alpha(Z)g(V,W)\\ &+\beta(V)g(Z,W) + \beta(W)g(V,Z), \quad (12)\end{aligned}$$

for all vector fields Z, V, W, where

$$\alpha(Z) = \frac{n}{(n-1)(n+2)}\nabla_Z r, \quad \beta(Z) = \frac{n-2}{2(n-1)(n+2)}\nabla_Z r, \quad (13)$$

with $\tilde{R}(Z,V)W = \tilde{R}(Z,W)V$ is the trace-less tensor that can be written as a sum of its orthogonal components.

$$\begin{aligned}\tilde{R}(Z,V)W = &\frac{1}{3}[\tilde{R}(Z,V)W + \tilde{R}(V,W)Z + \tilde{R}(W,Z)V] \quad (14)\\ &+\frac{1}{3}[\tilde{R}(Z,V)W - \tilde{R}(V,Z)W] + \frac{1}{3}[\tilde{R}(Z,V)W - \tilde{R}(W,Z)V].\end{aligned}$$

The decompositions (12) and (14) provide $\mathcal{O}(n)$-invariant subspace, characterized by invariant equations that are linear in $(\nabla_Z S)(V,W)$.

Therefore, the relation between ∇S and the divergence of the Weyl conformal curvature tensor C are connected by the equation [12]

$$(\mathrm{div\ C})(\mathrm{Z,V})\mathrm{W} = \frac{\mathrm{n}-3}{\mathrm{n}-2}[\tilde{\mathrm{R}}(\mathrm{Z,V})\mathrm{W} - \tilde{\mathrm{R}}(\mathrm{W,Z})\mathrm{V}]. \quad (15)$$

In Gray's decomposition we have the subsequent subspaces:

(i) The *trivial subspace* is characterized by $\nabla S = 0$.
(ii) The *subspace* $\mathcal{I}$ is characterized by $\tilde{R}(Z,V)W = 0$, that is,

$$(\nabla_Z S)(V,W) = \alpha(Z)g(V,W) + \omega(V)g(Z,W) + \omega(W)g(V,Z). \quad (16)$$

Such manifolds equipped with the condition (16) are called *Sinyukov manifolds* [18,19].

(iii) The *orthogonal complements* $\mathcal{I}'$ (also called as the subspace $\mathcal{A}$) is characterized by

$$(\nabla_Z S)(V,W) + (\nabla_V S)(Z,W) + (\nabla_W S)(Z,V) = 0, \tag{17}$$

which also infers that the scalar curvature r is constant. Also, according to the above equation the Ricci tensor is Killing [21].

(iv) In *the subspaces* $\mathcal{B}$ *and* $\mathcal{B}'$ the Ricci tensor is of Codazzi type, that is,

$$(\nabla_Z S)(V,W) = (\nabla_V S)(Z,W). \tag{18}$$

(v) In *the subspace* $\mathcal{I} \oplus \mathcal{A}$, the Ricci tensor obeys the following cyclic condition

$$\begin{aligned}&(\nabla_Z S)(V,W) + (\nabla_V S)(Z,W) + (\nabla_W S)(Z,V) \\ =&2\frac{dr(Z)}{n+2}g(V,W) + 2\frac{dr(V)}{n+2}g(Z,W) + 2\frac{dr(W)}{n+2}g(Z,V).\end{aligned} \tag{19}$$

(vi) In *the subspace* $\mathcal{I} \oplus \mathcal{B}$, the Ricci tensor fulfills the subsequent Codazzi condition

$$\nabla_Z \Big[S(V,W) - \frac{r}{2(n-1)}g(V,W)\Big] = \nabla_V \Big[S(Z,W) - \frac{r}{2(n-1)}g(Z,W)\Big]. \tag{20}$$

which entails that the Weyl conformal curvature tensor C is of divergence-free.

(vii) In *the subspace* $\mathcal{A} \oplus \mathcal{B}$, the scalar curvature is covariant constant.

Now we will examine each of these seven cases separately:
Case (i): The condition $\nabla S = 0$ implies the scalar curvature r is constant. Putting $V = \rho$ in the equation (11), we acquire

$$\begin{aligned}&\frac{2}{n}[nS(X,U) - rg(X,U)]A(\rho) + \frac{1}{n}[nS(U,\rho) - rg(U,\rho)]A(X) \\ &\frac{1}{n}[nS(X,\rho) - rg(X,\rho)]A(U) + \frac{n-1}{n}[R(X,\rho,\rho,U) + R(X,\rho,\rho,U)] \\ &-\frac{1}{n}[\frac{2r}{n}g(X,U)A(\rho) - \frac{r}{n}g(\rho,U)A(X) - S(X,\rho)A(U)] = 0.\end{aligned} \tag{21}$$

Using the equation (10), the foregoing equation yields

$$S(X,U) = \frac{r(n+1)}{n^2}g(X,U) - \frac{r}{n^2}A(U)A(X), \tag{22}$$

provided $A(R(\rho,X)U) = 0$. Thus, we have the following theorem:

Theorem 1. *If a pseudo projective symmetric manifold belongs to the trivial subspace, then the manifold becomes a quasi-Einstein manifold, provided* $A(R(\rho,X)U) = 0$.

Case (ii): In the subspace $\mathcal{I}$, the Ricci tensor fulfills the condition $\tilde{R}(Z,V)W = 0$, for all $Z,V,W \in \chi(M)$ and hence from the relation (15) we get $div\, C = 0$. Hence we have [9]

$$\begin{aligned}(\nabla_V S)(X,U) - (\nabla_X S)(V,U) = \frac{1}{6}[&g(V,X)dr(U) \\ &-g(V,U)dr(X)].\end{aligned} \tag{23}$$

Making use of the equation (10), the previous equation gives

$$(\nabla_V S)(X,U) = (\nabla_X S)(V,U). \tag{24}$$

Using the equation (11) and putting $V = \rho$ in the equation (23), we obtain

$$\begin{aligned}
&\frac{2}{n}[nS(X,U) - rg(X,U)]A(\rho) + \frac{1}{n}[nS(U,\rho) - rg(U,\rho)]A(X) \\
&\frac{1}{n}[nS(X,\rho) - rg(X,\rho)]A(U) + \frac{n-1}{n}[R(X,\rho,\rho,U) + R(X,\rho,\rho,U)] \\
&-\frac{1}{n}[\frac{2r}{n}g(X,U)A(\rho) - \frac{r}{n}g(\rho,U)A(X) - S(X,\rho)A(U)] \\
&= \frac{2}{n}[nS(\rho,U) - rg(\rho,U)]A(X) + \frac{1}{n}[nS(U,X) - rg(U,X)]A(\rho) \\
&\frac{1}{n}[nS(\rho,X) - rg(\rho,X)]A(U) + \frac{n-1}{n}[R(\rho,X,\rho,U) + R(\rho,\rho,X,U)] \\
&-\frac{1}{n}[\frac{2r}{n}g(\rho,U)A(X) - \frac{r}{n}g(X,U)A(\rho) - S(\rho,X)A(U)].
\end{aligned} \tag{25}$$

Utilizing the equation (10), the above equation infers

$$S(X,U) = \frac{2r}{n^2}g(X,U) - \frac{3r}{n^2}A(U)A(X), \tag{26}$$

provided $A(R(\rho,X)U) = 0$.

Thus, we conclude the following:

Theorem 2. *If a pseudo projective symmetric manifold belongs to the class $\mathcal{I}$, then the manifold becomes a quasi-Einstein manifold, provided $A(R(\rho,X)U) = 0$.*

Case (iii): Assume that the Ricci tensor belongs to the subspace $\mathcal{A}$. Then, the condition (17) holds which entails that

$$(\nabla_\rho S)(X,U) + (\nabla_X S)(U,\rho) + (\nabla_U S)(\rho,X) = 0. \tag{27}$$

Making use of the equation (11) in the equation (23), we obtain

$$\frac{2}{n}[nS(X,U)-rg(X,U)]A(\rho)+\frac{1}{n}[nS(U,\rho)-rg(U,\rho)]A(X)$$
$$\frac{1}{n}[nS(X,\rho)-rg(X,\rho)]A(U)+\frac{n-1}{n}[R(X,\rho,\rho,U)+R(X,\rho,\rho,U)]$$
$$-\frac{1}{n}[\frac{2r}{n}g(X,U)A(\rho)-\frac{r}{n}g(\rho,U)A(X)-S(X,\rho)A(U)]$$
$$+\frac{2}{n}[nS(\rho,U)-rg(\rho,U)]A(X)+\frac{1}{n}[nS(U,X)-rg(U,X)]A(\rho)$$
$$\frac{1}{n}[nS(\rho,X)-rg(\rho,X)]A(U)+\frac{n-1}{n}[R(\rho,X,\rho,U)+R(\rho,\rho,X,U)]$$
$$-\frac{1}{n}[\frac{2r}{n}g(\rho,U)A(X)-\frac{r}{n}g(X,U)A(\rho)-S(\rho,X)A(U)]$$
$$+\frac{2}{n}[nS(\rho,X)-rg(\rho,X)]A(U)+\frac{1}{n}[nS(U,X)-rg(U,X)]A(\rho)$$
$$\frac{1}{n}[nS(\rho,U)-rg(\rho,U)]A(X)+\frac{n-1}{n}[R(\rho,U,\rho,X)+R(\rho,\rho,U,X)]$$
$$-\frac{1}{n}[\frac{2r}{n}g(\rho,X)A(U)-\frac{r}{n}g(X,U)A(\rho)-S(\rho,U)A(X)]. \tag{28}$$

Using the equation (10), the foregoing equation yields

$$S(X,U)=\frac{3r}{4n-1}g(X,U), \tag{29}$$

provided $A(R(\rho,X)U)=0$.

Thus, we can state that:

Theorem 3. *If a pseudo projective symmetric manifold belongs to the class $\mathcal{A}$, then the manifold becomes an Einstein manifold, provided $A(R(\rho,X)U)=0$.*

Case (iv): If a perfect fluid spacetime belongs to $\mathcal{B}$ and $\mathcal{B}'$, then the Ricci tensor is of Codazzi-type and so the equation (25) holds.

Therefore, we can state the result as in Theorem 2 :

Theorem 4. *If a pseudo projective symmetric manifold belongs to classes $\mathcal{B}$ and $\mathcal{B}'$, then the manifold becomes a quasi-Einstein manifold, provided $A(R(\rho,X)U)=0$.*

Case (v): In this case, we have the relation (19), from which we infer the scalar curvature r is constant and hence $div\,C=0$. Hence we can state that:

Theorem 5. *If a pseudo projective symmetric manifold belongs to the subspace $\mathcal{I}\oplus\mathcal{A}$, then the manifold becomes a quasi-Einstein manifold, provided $A(R(\rho,X)U)=0$.*

Case (vi): Let the pseudo projective symmetric manifold belongs to $\mathcal{I}\oplus B$. In this case, we get $div\,C=0$ and state the same result as in Theorem 2.

Case (vii): In *the subspace* $\mathcal{A}\oplus\mathcal{B}$, the scalar curvature is covariant constant and hence $r=$ constant. Therefore, from [6] we conclude that $\frac{r}{n}$ is an eigen value of the Ricci tensor S and ρ is an eigen vector corresponding to the eigenvalue.

4. Conformally flat pseudo projective symmetric manifolds

In this section we characterize conformally flat pseudo projective symmetric manifolds. Conformally flat manifold reflects that the manifold is of divergence free conformal curvature tensor, that is, $div\, C = 0$. Hence, from case (ii) it follows that the manifold is quasi-Einstein of the form (26).

Now, $C = 0$ implies that

$$\begin{aligned} R(U,V,W,Z) = \frac{1}{n-2}[g(V,W)S(U,Z) - g(U,W)S(V,Z) \\ +S(V,W)g(U,Z) - S(U,W)g(V,Z)] \\ -\frac{r}{(n-1)(n-2)}[g(V,W)g(U,Z) - g(U,W)g(V,Z)]. \end{aligned} \tag{30}$$

Using (26) in the above expression yields

$$\begin{aligned} R(U,V,W,Z) = -\alpha[g(V,W)g(U,Z) - g(U,W)g(V,Z)] \\ +\beta[g(U,W)A(V)A(Z) + g(V,Z)A(U)A(W) \\ -g(V,W)A(U)A(Z) - g(U,Z)A(V)A(W)], \end{aligned} \tag{31}$$

where $\alpha = [\frac{4r}{n^2(n-2)} - \frac{r}{(n-1)(n-2)}]$ and $\beta = \frac{3r}{n^2(n-2)}$, which implies that a conformally flat pseudo projective symmetric manifold is a manifold of quasi constant curvature [7]. Thus, we conclude the following:

Theorem 6. *A conformally flat pseudo projective symmetric manifold is a manifold of quasi constant curvature.*

References

1. T. Adati and T. Miyazawa, *On a Riemannian space with recurrent conformal curvature*, Tensor(N.S.), **18** (1967), 348-354.
2. A. L. Besse, *Einstein manifolds,* Springer-Verlag, Berlin (2008).
3. Cartan, E., *Sur une classes remarquable d'espaces de Riemannian*, Bull. Soc. Math. France, **54**(1926), 214-264.
4. M.C. Chaki and B. Gupta, *On conformally symmetric spaces*, Indian J. Math., **5**(1963), 113-295.
5. M.C. Chaki, *On pseudo symmetric manifolds*, Ann. St. Univ. Al I Cuza Iasi, **33**(1987), 53-58.
6. M.C. Chaki and S.K. Saha, *On pseudo projective symmetric manifolds*, Bull. Ins. Math. academia Sinica, **17**(1999), 59-65.
7. B. Y. Chen and K. Yano, *Hypersurfaces of a conformally flat spaces*, Tensor(N.S.), **26**(1972), 318-322.
8. R. Deszcz , M. Hotlos and Z. Senturk , *On curvature properties of quasi-Einstein hypersurfaces in semi-Eucledian spaces*, Soochow J. Math. **27** (2001), 375-389.
9. L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1949.
10. A. Gray, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata, **7** (1978) 259-280.

11. D. Krupka, *The trace decomposition problem,* Beitrage Algebra Geom., **36**, (1995) 303-315.
12. C. A. Mantica, L. G. Molinari, Y. J. Suh, S. Shenawy, *Perfect-fluid, generalized Robertson-Walker spacetime, and Gray's decomposition,* J. Math. Phys., **60** (2019) 052506.
13. R.S. Mishra, *Structures on differentiable manifold and their applications*, Chandrama Prakasana, Allahabad, India, 1984.
14. B. O'Neill, *Semi-Riemannian Geometry with Applications to the Relativity*, Academic Press, New York-London, 1983.
15. W. Roter, *On conformally symmetric Ricci-recurrent spaces*, Colloquium Mathematicum, **31**(1974), 87-96.
16. R. N. Sen and M. C. Chaki, *On curvature restrictions of a certain kind of conformally flat Riemannian space of class one*, Proc. Nat. Inst. Sci. India Part A **33** (1967), 100-102.
17. S. Shenawy, U.C. De, N.B. Turki, S. Alsaeed, A.A. Syied, *Investigation of Pseudo-Ricci Symmetric Spacetimes in Gray's Subspaces,* Journal of Mathematics 2021, Article ID 6188097, 6 pages. https://doi.org/10.1155/2021/6188097
18. E. N. Sinyukov, *Geodesic mappings of L2 spaces,* Izv. Vyssh. Ucheb. Zaved. Mat., **3**, (1982) 57-61 (in Russian).
19. N. S. Sinyukov, *Geodesic mappings of Riemannian spaces,* Nauka, Moscow (in Russian) (1979).
20. A.G. Walker, *On Ruse's space of recurrent curvature*, Proc. London Math. Soc., **52**(1950), 36-54.
21. M. Walker and R. Penrose, *On quadratic first integrals of the geodesic equations for {22} spacetimes*, Commun. Math. Phys., **18** (1970), 265-274.
22. Yano, K. and Bochner, S., *Curvature and Betti numbers*, Annals of Math. Studies 32 (Princeton University Press)1953.