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An Unified Integral and Its Generating Functions through Lie-Group of Operators

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Abstract

In the present paper, we define an unified integral and then, obtain its generating functions on introducing Lie-group of operators. Finally, we obtain some generating relations involving H-function of two variables (cf. [2], [3], [4], [6]).

Key Words : An unified integral formula, differential recurrence relations, generating functions, Lie-groups, H-function of two variables.
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1. Introduction

The unified integrals are occurred in the solutions of various scientific problems of physical sciences, chemical sciences, biomechanics, integral equations and probability theory etcetera (see Srivastava, Gupta and Goyal [6]) and for computational work, they may be converted into numerical generating relations through Lie group of operators. The integrals involving generalized hypergeometric functions such as Fox's [1] H-function have a great importance and become very good generalized functions in an analytic continuation theory.

Here, we define an unification formula in the form

$$F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_1,A_1),\dots\dots\dots,(\alpha_n,A_n),(\alpha_{n+1},A_{n+1}),\dots\dots\dots,(\alpha_p,A_p):}_{b,y,z,N,M,(\beta_1,B_1),\dots\dots,(\beta_m,B_m),(\beta_{m+1},B_{m+1}),\dots\dots,(\beta_q,B_q):}\begin{bmatrix}h\\x\end{bmatrix}\right]$$

$$= F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j,A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j,B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right] = x^{\gamma}(1+bxz)^{-\gamma}\int_0^b t^{\alpha-\gamma-1}(b-t)^{\beta-1}$$

$$\left(z+\frac{t}{b}\right)^{-(\alpha-2\gamma)-\beta}\left\{1-\frac{t(1-bx)}{b(1+bxz)}\right\}^{-\gamma}$$

$$H_{p,q}^{m,n}\left[h\left\{1+\frac{(b-t)}{xb^2(z+t/b)}\right\}^{-\nu}\Big|_{(\beta_j:B_j)_{1,q}}^{(\alpha_j:A_j)_{1,p}}\right]S_N^M\left[y\left(\frac{t}{z+t/b}\right)^{\mu}\right]dt \tag{1.1}$$

where, $b>0, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0$ and $\mathrm{Re}(\gamma)>0, \nu$ is any positive real number, μ is any complex number such that $\mathrm{Re}(\mu)>\mathrm{Re}\left(\frac{\gamma-\alpha}{k}\right)$, $\forall k=1,2,.....,\left[\frac{N}{M}\right]$, M is any positive integer,

$N\in[\{N_0\}=\{0,1,2,......\}]$, $\left[\frac{N}{M}\right]$, is the greatest integral function not less than, (Also $\mathrm{Re}(\alpha)>\mathrm{Re}(\gamma)$ when $k=0$), $x\neq 0, h\neq 0$, $\left|\arg(z+\frac{t}{b})\right|\leq \pi-\in$, $0<t<\pi$ for $0\leq t\leq b$.

The Fox's [1] H-function is defined by (Also, see Srivastava and Manocha [7])

$$H_{p,q}^{m,n}\left[h\Big|_{(\beta_j:B_j)_{1,q}}^{(\alpha_j:A_j)_{1,p}}\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m}\Gamma(\beta_j-B_ju)\prod_{j=1}^{n}\Gamma(1-\alpha_j+A_ju)h^u}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j+B_ju)\prod_{j=n+1}^{p}\Gamma(\alpha_j-A_ju)}du \tag{1.2}$$

$i=\sqrt{(-1)}$, the integers m,n,p,q are such that $0\leq m\leq q$ and $0\leq n\leq p$, the coefficients $A_1,......,A_p$ and $B_1,.....,B_q$ are positive real numbers and the complex parameters $\alpha_1,.....,\alpha_p$ and $\beta_1,.....,\beta_q$ are so constrained that no poles of the integrand in (1.2) coincide, also, if

$$\Delta=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p}A_j+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q}B_j>0 \tag{1.3}$$

The integral in (1.2) is absolutely convergent and analytic in the sector $|\arg(h)|<\frac{1}{2}\Delta\pi$, the point $h=0$ being tacitly excluded.

The general class of polynomials is introduced by Srivastava [5] in the form

$$S_N^M[y]=\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\frac{y^k}{k!} \tag{1.4}$$

where, N and M are given in (1.1), the coefficients $\phi_{N,k}$ $(k\geq 0)$ are arbitrary constants, real or complex and are independent of y. $S_N^M[.]$ yields number of known and unknown polynomials as its particular cases. These include among other Laguerre polynomials, Hermite polynomials, Goulds and Hopper polynomials and several others.

Here, in our investigations, we first evaluate an analytic function due to the formula given in (1.1) and derive its differential recurrence relations and then, on defining a basis function, we obtain some of its generating functions through Lie-group theoretic techniques. Finally, we apply them to derive generating relations involving H-function of two variables (cf. [2], [3], [4], [6]).

2. An Analytic Function Due To Unified Integral

In this section, we evaluate an analytic function by the unified integral formula (1.1) for the given conditions in (1.1) - (1.4).

Theorm 1: For the given conditions in (1.1) - (1.4), due to integral formula (1.1) an analytic function exists and there holds the formula

$$F\left[\begin{smallmatrix}\alpha,\beta,\gamma,\nu,\mu,(\alpha_j,A_j)_{1,p}: \\ b,y,z,N,M,(\beta_j,B_j)_{1,q}:\end{smallmatrix}\left[\begin{smallmatrix}h\\x\end{smallmatrix}\right]\right] = \frac{b^{\alpha+\beta-2\gamma-1}}{(1+z)^{\alpha-\gamma}z^{\beta}}\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\Gamma(\alpha-\gamma+\mu k)$$

$$\frac{\left(y\left(\frac{b}{1+z}\right)^{\mu}\right)^k}{k!}H^{0,1:m,n:1,1}_{1,0:p,q+1:2,1}\times$$

$$\left[\begin{smallmatrix}[1-\gamma:\nu,1]:(\alpha_j:A_j)_{1,p} & : & (1-\beta:1),(\ldots\ldots\ldots\ldots):\\ [\ldots\ldots\ldots]:(\beta_j:B_j)_{1,q},(1-\gamma:\nu) & : & (0:1),(1+\gamma-\alpha-\beta-\mu k:1):\end{smallmatrix}h, b^{-1}x^{-1}z^{-1}\right] \tag{2.1}$$

provided that $|\arg(h)| < \frac{1}{2}\Delta\pi$, $\left|\arg(x^{-1}z^{-1})\right| < \pi$, Δ is given in (1.3) and it is in analytic continuation for $\sum\limits_{j=1}^{p} A_j - \sum\limits_{j=1}^{q} B_j \leq 0$.

Proof: The right hand side of (1.1) may be written in the form

$$b^{-\gamma}\int_0^b t^{\alpha-\gamma-1}(b-t)^{\beta-1}\left(z+\frac{t}{b}\right)^{-(\alpha-\gamma)-\beta}\left\{1+\frac{(b-t)}{xb^2(z+t/b)}\right\}^{-\gamma}$$

$$.H^{m,n}_{p,q}\left[h\left\{1+\frac{(b-t)}{xb^2(z+t/b)}\right\}^{-\nu}\Bigg|\begin{smallmatrix}(\alpha_j:A_j)_{1,p}\\(\beta_j:B_j)_{1,q}\end{smallmatrix}\right]S^M_N\left[y\left(\frac{t}{z+t/b}\right)^{\mu}\right]dt \tag{2.2}$$

Now, in (2.2), express $S^M_N[.]$ due to (1.4) and then change the order of summation and integration and thus write the Mellin-Barnes integral on defining

$H_{p,q}^{m,n}[.]$ given in (1.2), and then, change the order of integration we get,

$$b^{-\gamma}\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\frac{y^k}{k!}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m}\Gamma(\beta_j-B_ju)\prod_{j=1}^{n}\Gamma(1-\alpha_j+A_ju)h^u}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j+B_ju)\prod_{j=n+1}^{p}\Gamma(\alpha_j-A_ju)}$$

$$\int_0^b t^{\alpha-\gamma+\mu k-1}(b-t)^{\beta-1}\left(z+\frac{t}{b}\right)^{-\alpha+\gamma-\beta+\mu k}\left\{1+\frac{(b-t)}{xb^2(z+t/b)}\right\}^{-\gamma-\nu u}dt\,du \tag{2.3}$$

Next, use the Mellin-Barnes formula

$$(1+z)^{-\gamma}=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(-\upsilon)\Gamma(\gamma+\upsilon)z^{\upsilon}}{\Gamma(\gamma)}d\upsilon, |\arg(z)|<\pi, \tag{2.4}$$

in (2.3) and then, change the order of integration, we find that

$$b^{-\gamma}\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\frac{y^k}{k!}\frac{1}{(2\pi i)^2}$$

$$\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m}\Gamma(\beta_j-B_ju)\prod_{j=1}^{n}\Gamma(1-\alpha_j+A_ju)\Gamma(-\upsilon)h^u(x^{-1}b^{-2})^{\upsilon}\Gamma(\gamma+\nu u+\upsilon)}{\Gamma(\gamma+\upsilon u)\prod_{j=m+1}^{q}\Gamma(1-\beta_j+B_ju)\prod_{j=n+1}^{p}\Gamma(\alpha_j-A_ju)}$$

$$.\int_0^b t^{\alpha-\gamma+\mu k-1}(b-t)^{\beta+\upsilon-1}(z+t/b)^{-\alpha+\gamma-\beta-\mu k-\upsilon}dt\,du\,d\upsilon \tag{2.5}$$

Finally, making an appeal to the formula

$$\int_0^b t^{\alpha-1}(b-t)^{\beta-1}(z+t/b)^{-\alpha-\beta}dt=\frac{b^{\alpha+\beta-1}}{(1+z)^{\alpha}z^{\beta}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{2.6}$$

where, $b>0$, $|\arg(z+t/b)|\le\pi-\in$, $0<\in<\pi$, for, $0\le t\le b$, $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$, in the inner integral of (2.5), and then on defining H-function of two variables given by Mittal and Gupta [3] (Also, see Saxena and Nishimoto [4], Srivastva, Gupta and Goyal [6] and Kumar and Srivastava [2]), we approach at the right hand side of (2.1), which is absolutely convergent when $|\arg(h)|<\frac{1}{2}\Delta\pi$, Δ is given in (1.3) and $\left|\arg(x^{-1}z^{-1})\right|<\pi$. Also it is in analytic continuation when $\sum_{j=1}^{p}A_j-\sum_{j=1}^{q}B_j\le 0$.

3. Differential Recurrence Relations

$$\left\{h\frac{\partial}{\partial h}+\frac{\gamma-1}{\nu}\right\}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right]=\frac{1}{b\nu}\left(h\nu\frac{\partial}{\partial h}-x\frac{\partial}{\partial x}+\gamma-1\right)$$
$$.F\left[{}^{\alpha-1,\beta,\gamma-1,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right] \tag{3.1}$$

$$\left\{h\frac{\partial}{\partial h}+\frac{1-\alpha_j}{A_j}\right\}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right]$$
$$=\frac{1}{A_j}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j-1:A_j)_{1,n},(\alpha_j:A_j)_{n+1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right] \tag{3.2}$$

when, $1\le j\le n$.

$$\left\{h\frac{\partial}{\partial h}+\frac{1-\alpha_j}{A_j}\right\}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right]$$
$$=-\frac{1}{A_j}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,n},(\alpha_j-1:A_j)_{n+1,p}:}_{b,y,z,N,M,(\beta_j,B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right] \tag{3.3}$$

when, $n+1\le j\le p$.

$$\left\{h\frac{\partial}{\partial h}-\frac{\beta_j}{B_j}\right\}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right]$$
$$=-\frac{1}{B_j}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j+1,B_j)_{1,m},(\beta_j:B_j)_{m+1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right] \tag{3.4}$$

when, $1\le j\le m$.

$$\left\{h\frac{\partial}{\partial h}-\frac{\beta_j}{B_j}\right\}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right]$$
$$=\frac{1}{B_j}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j,B_j)_{1,m},(\beta_j+1:B_j)_{m+1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right] \tag{3.5}$$

when, $m+1\le j\le q$.

$$x^2\frac{\partial}{\partial x}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j,A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j,B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right]=xF\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right] \tag{3.6}$$

4. Basis Function and Actions of Lie-Group of Operaters

In this section, we define following basis function related to the analytic function (2.1) as introducing some new parameters $s;t_1,....,t_n,t_{n+1},....,t_p;u_1,....,u_m,u_{m+1},....,u_q$ in the form,

$$G\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{Bmatrix}h,s,t_1,....,t_p\\u_1,....,u_q,x\end{Bmatrix}\right]=F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{bmatrix}h\\x\end{bmatrix}\right]$$
$$.s^{\gamma}t_1^{\alpha_1}....t_n^{\alpha_n}t_{n+1}^{\alpha_{n+1}}....t_p^{\alpha_p}....u_1^{\beta_1}....u_m^{\beta_m}....u_{m+1}^{\beta_{m+1}}....u_q^{\beta_q} \tag{4.1}$$

Also, we define following Lie-group of operators

$$T_j \equiv \frac{t_j}{A_j}\frac{\partial}{\partial t_j}, \qquad 1 \le j \le n \le p,$$

$$U_j \equiv \frac{u_j}{B_j}\frac{\partial}{\partial u_j}, \qquad 1 \le j \le m \le q,$$

$$HT_j \equiv t_j^{-1}\left(h\frac{\partial}{\partial h} - \frac{t_j}{A_j}\frac{\partial}{\partial t_j} + \frac{1}{A_j}\right), \qquad 1 \le j \le n \le p,$$

$$HU_j \equiv u_j\left(h\frac{\partial}{\partial h} - \frac{u_j}{B_j}\frac{\partial}{\partial u_j}\right), \qquad 1 \le j \le m \le q,$$

$$HS \equiv s^{-1}\left(h\frac{\partial}{\partial h} + \frac{s}{\nu}\frac{\partial}{\partial s} - \frac{1}{\nu}\right),$$

$$X = x^2\frac{\partial}{\partial x} \tag{4.2}$$

Then, action of above operators given in (4.2) on the basis function (4.1) are given by

$$T_j G\left[\begin{matrix}\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}: \\ b,y,z,N,M,(\beta_j:B_j)_{1,q}:\end{matrix}\left\{\begin{matrix}h,s,t_1,...,t_p, \\ u_1,...,u_q,x\end{matrix}\right\}\right]$$
$$= \frac{\alpha_j}{A_j} G\left[\begin{matrix}\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}: \\ b,y,z,N,M,(\beta_j:B_j)_{1,q}:\end{matrix}\left\{\begin{matrix}h,s,t_1,....,t_p, \\ u_1,....,u_q,x\end{matrix}\right\}\right] \tag{4.3}$$

when, $1 \le j \le n \le p$.

$$U_j G\left[\begin{matrix}\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}: \\ b,y,z,N,M,(\beta_j:B_j)_{1,q}:\end{matrix}\left\{\begin{matrix}h,s,t_1,....,t_p, \\ u_1,....,u_q,x\end{matrix}\right\}\right]$$
$$= \frac{\beta_j}{B_j} G\left[\begin{matrix}\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}: \\ b,y,z,N,M,(\beta_j:B_j)_{1,q}:\end{matrix}\left\{\begin{matrix}h,s,t_1,....,t_p, \\ u_1,....,u_q,x\end{matrix}\right\}\right] \tag{4.4}$$

when, $1 \le j \le m \le q$.

$$HT_j G\left[\begin{matrix}\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}: \\ b,y,z,N,M,(\beta_j:B_j)_{1,q}:\end{matrix}\left\{\begin{matrix}h,s,t_1,....,t_p, \\ u_1,....,u_q,x\end{matrix}\right\}\right]$$
$$= \frac{1}{A_j} G\left[\begin{matrix}\alpha,\beta,\gamma,\nu,\mu,(\alpha_j-1:A_j)_{1,n},(\alpha_j:A_j)_{n+1,p}: \\ b,y,z,N,M,(\beta_j:B_j)_{1,q}:\end{matrix}\left\{\begin{matrix}h,s,t_1,....,t_p, \\ u_1,....,u_q,x\end{matrix}\right\}\right] \tag{4.5}$$

when, $1 \le j \le n$.

$$HT_j G\left[\begin{matrix}\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}: \\ b,y,z,N,M,(\beta_j:B_j)_{1,q}:\end{matrix}\left\{\begin{matrix}h,s,t_1,....,t_p, \\ u_1,....,u_q,x\end{matrix}\right\}\right]$$
$$= -\frac{1}{A_j} G\left[\begin{matrix}\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,n},(\alpha_j-1:A_j)_{n+1,p}: \\ b,y,z,N,M,(\beta_j:B_j)_{1,q}:\end{matrix}\left\{\begin{matrix}h,s,t_1,....,t_p, \\ u_1,....,u_q,x\end{matrix}\right\}\right] \tag{4.6}$$

when, $n+1 \le j \le q$.

$$HU_j G\left[\begin{matrix}\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}: \\ b,y,z,N,M,(\beta_j:B_j)_{1,q}:\end{matrix}\left\{\begin{matrix}h,s,t_1,....,t_p, \\ u_1,....,u_q,x\end{matrix}\right\}\right]$$

$$= -\frac{1}{B_j} G \left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j+1:B_j)_{1,m},(\beta_j:B_j)_{m+1,q}:} \left\{ {}^{h,s,t_1,\ldots,t_p,}_{u_1,\ldots,u_q,x} \right\} \right] \tag{4.7}$$

when, $1 \leq j \leq m$.

$$HU_j G \left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:} \left\{ {}^{h,s,t_1,\ldots,t_p,}_{u_1,\ldots,u_q,x} \right\} \right]$$

$$= \frac{1}{B_j} G \left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,m},(\beta_j+1:B_j)_{m+1,q}:} \left\{ {}^{h,s,t_1,\ldots,t_p,}_{u_1,\ldots,u_q,x} \right\} \right] \tag{4.8}$$

when, $m+1 \leq j \leq q$.

$$(HS)^n G \left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:} \left\{ {}^{h,s,t_1,\ldots,t_p,}_{u_1,\ldots,u_q,x} \right\} \right]$$

$$= \frac{\left(-\gamma - \nu h \frac{\partial}{\partial h} + 1\right)_n (-1)^n}{(\nu s)^n} G \left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:} \left\{ {}^{h,s,t_1,\ldots,t_p,}_{u_1,\ldots,u_q,x} \right\} \right] \tag{4.9}$$

$$XG \left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:} \left\{ {}^{h,s,t_1,\ldots,t_p,}_{u_1,\ldots,u_q,x} \right\} \right]$$

$$= xG \left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:} \left\{ {}^{h,s,t_1,\ldots,t_p,}_{u_1,\ldots,u_q,x} \right\} \right] \tag{4.10}$$

5(i) Generating Functions Resulting From the Lie-group Theoretic Techniques for One Parameter Subgroup exp$[\lambda HT_j + \lambda X]$ when $1 \leq j \leq n \leq p$.

From the Lie-group theoretic techniques due to Srivastava and Manocha [7, p.320], the transformations due to action of one parameter subgroup $\exp[\lambda HT_j + \lambda X]$, $1 \leq j \leq n \leq p$ on the basis function (4.1), are given by

$$\begin{bmatrix} h \\ t_j \\ u_j \\ x \\ s \end{bmatrix} \rightarrow \begin{bmatrix} h\,(1 - \lambda/A_j t_j)^{-A_j} \\ (t_j - \lambda/A_j) \\ u_j \\ x/(1 - \lambda x) \\ s \end{bmatrix}, \tag{5.1}$$

and the multiplier $w = (1 - \lambda/A_j T_j)^{-1}$ where $1 \leq j \leq n \leq p$.

Then, we obtain the generating functions for the analytic function (2.1) corresponding to the operators $\lambda(HT_j + X)$ when $1 \leq j \leq n \leq p$, by direct expansion on one hand and then substituting transformations for $\exp[\lambda HT_j + \lambda X] G \left[\left\{ {}^{h,s,t_j,}_{u_j,x} \right\} \right]$ on the other hand given in (5.1) such that

$$\left(1 - \frac{\lambda}{A_j t_j}\right)^{-1} F \left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:} \left\{ {}^{h(1-\lambda/A_j t_j)^{-A_j}}_{x/(1-\lambda x)} \right\} \right] (t_j - \lambda/A_j)^{\alpha_j}$$

$$=\sum_{r=0}^{\infty}\frac{(\lambda x)^r}{r!(A_j)^r}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j-r:A_j)_{1,n},(\alpha_j:A_j)_{n+1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{Bmatrix}h\\x\end{Bmatrix}\right]t_j^{\alpha_j-r}, \tag{5.2}$$

when, $1\le j\le n$, and

$$\left(1-\frac{\lambda}{A_jt_j}\right)^{-1}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{Bmatrix}h(1-\lambda/A_jt_j)^{-A_j}\\x/(1-\lambda x)\end{Bmatrix}\right](t_j-\lambda/A_j)^{\alpha_j}$$

$$=\sum_{r=0}^{\infty}\frac{(-\lambda x)^r}{r!(A_j)^r}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,n},(\alpha_j-r:s_j)_{n+1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{Bmatrix}h\\x\end{Bmatrix}\right]t_j^{\alpha_j-r}, \tag{5.3}$$

when, $n+1\le j\le p$.

(ii) Generating Functions Resulting From the Lie-group Theoretic Techniques for One Parameter Subgroup $\exp[\lambda HU_j+\lambda X]$ when $1\le j\le m\le q$.

In the similar fashion, the transformations due to action of one parameter, subgroup $\exp[\lambda HU_j+\lambda X]$, $1\le j\le m\le q$ on the basis function (4.1) are given by

$$\begin{bmatrix}h\\t_j\\u_j\\x\\s\end{bmatrix}\to\begin{bmatrix}h\,(1+\lambda u_j/B_j)^{B_j}\\t_j\\u_j\,(1+\lambda u_j/B_j)^{-1}\\x/(1-\lambda x)\\s\end{bmatrix}, \tag{5.4}$$

when, $1\le j\le m\le q$.

Now with the help of above transformation (5.4) and techniques applied above, we obtain the generating functions

$$F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{Bmatrix}h(1+\lambda u_j/B_j)^{B_j}\\x/(1-\lambda x)\end{Bmatrix}\right]u_j^{\beta_j}\left(1+\frac{\lambda u_j}{B_j}\right)^{-\beta_j}$$

$$=\sum_{r=0}^{\infty}\frac{(-\lambda x)^r}{r!(B_j)^r}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j+r:B_j)_{1,m},(\beta_j:B_j)_{m+1,q}:}\begin{Bmatrix}h\\x\end{Bmatrix}\right]u_j^{\beta_j+r}, \tag{5.5}$$

when, $1\le j\le m$.

$$F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:}\begin{Bmatrix}h(1+\lambda u_j/B_j)^{B_j}\\x/(1-\lambda x)\end{Bmatrix}\right]u_j^{\beta_j}\left(1+\frac{\lambda u_j}{B_j}\right)^{-\beta_j}$$

$$=\sum_{r=0}^{\infty}\frac{(\lambda x)^r}{r!(B_j)^r}F\left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,m},(\beta_j+r:B_j)_{m+1,q}:}\begin{Bmatrix}h\\x\end{Bmatrix}\right]u_j^{\beta_j+r}, \tag{5.6}$$

when, $m+1\le j\le q$.

(iii) Generating Functions Resulting From the Lie-group Theoretic Techniques for One Parameter Subgroup exp$[\boldsymbol{\lambda HS + \lambda X}]$.

The transformations due to action of one parameter, subgroup exp$[\lambda HS + \lambda X]$ on the basis function (4.1), are given by

$$\begin{bmatrix} h \\ t_j \\ u_j \\ x \\ s \end{bmatrix} \rightarrow \begin{bmatrix} h\,(1+\lambda/\nu s)^{\nu} \\ t_j \\ u_j \\ x/(1-\lambda x) \\ (s+\lambda/\nu) \end{bmatrix}, \tag{5.7}$$

with the multiplier $w = \left(1 + \frac{\lambda}{\nu s}\right)$.

Then, with the help of above transformations (5.7) and the techniques analysed in (5.2) and (5.3), we find the identity

$$\left(1+\frac{\lambda}{\nu s}\right) F \left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p}:}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:} \begin{Bmatrix} h(1+\lambda/\nu s)^{\nu} \\ x/(1-\lambda x) \end{Bmatrix} \right] \left(s + \frac{\lambda}{\nu}\right)^{\gamma}$$

$$= \left(1+\frac{\lambda x}{\nu s}\right)^{\gamma-1} F \left[{}^{\alpha,\beta,\gamma,\nu,\mu,(\alpha_j:A_j)_{1,p},f(\gamma,\nu;r):}_{b,y,z,N,M,(\beta_j:B_j)_{1,q}:} \begin{Bmatrix} h(1+\lambda x/\nu s)^{\nu} \\ x \end{Bmatrix} \right] s^{\gamma}, \tag{5.8}$$

provided that $\left|\frac{\lambda x}{\nu s}\right| < 1,\ \ f(\gamma,\nu;r) = (-1)^r(1-\gamma-\nu u)_r,\ -i\infty < u < i\infty,\ r = 0, 1, 2, ...$

6. Applications

In this section, making an appeal to (2.1) and the generating functions from the section 5, we evaluate following generating relations involving H-function of two variables [cf. 2, 3, 4, 6]

$$\sum_{k=0}^{[N/M]} (-N)_{Mk}\phi_{N,k}\Gamma(\alpha-\gamma+\mu k)\frac{(y(b/1+z)^{\mu})^k}{k!}\left(1-\frac{\lambda}{A_j t_j}\right)^{-1}$$

$$.H^{0,1:m,n:1,1}_{1,0:p,q+1:2,1}\left[\begin{matrix} [1-\gamma:\nu,1]:(\alpha_j:A_j)_{1,p},(......):(1-\beta:1): \\ [......]:(\beta_j,B_j)_{1,q},(1-\gamma:\nu):(0:1),(1+\gamma-\alpha-\beta-\mu k:1): \end{matrix}\right.$$

$$\left. h\left(1-\frac{\lambda}{A_j t_j}\right)^{-A_j}, b^{-1}\left(\frac{1-\lambda x}{x}\right)z^{-1}\right]\left(t_j - \frac{\lambda}{A_j}\right)^{\alpha_j}$$

$$= \sum_{r=0}^{\infty}\sum_{k=0}^{[N/M]} \frac{(\lambda x)^r}{r!(A_j)^r}(-N)_{Mk}\phi_{N,k}\Gamma(\alpha-\gamma+\mu k)\frac{(y(b/1+z)^{\mu})^k}{k!}$$

$$.H^{0,1:m,n:1,1}_{1,0:p,q+1:2,1}\left[\begin{array}{c}[1-r:\nu,1]:(\alpha_j-r:A_j)_{1,n},(\alpha_j:A_j)_{n+1,p}:(1-\beta:1),(.....):\\ [..........]:(\beta_j,B_j)_{1,q},(1-\gamma:\nu):(0:1),(1+\gamma-\alpha-\beta-\mu k:1):\end{array} h,b^{-1}x^{-1}z^{-1}\right] t_j^{\alpha_j-r} \tag{6.1}$$

when, $1\leq j\leq n$.

$$\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\Gamma(\alpha-\gamma+\mu k)\frac{(y(b/1+z)^{\mu})^k}{k!}\left(1-\frac{\lambda}{A_jt_j}\right)^{-1}\left(t_j-\frac{\lambda}{A_j}\right)^{-\alpha_j}$$

$$.H^{0,1:m,n:1,1}_{1,0:p,q+1:2,1}\left[\begin{array}{c}[1-\gamma:\nu,1]:(\alpha_j:A_j)_{1,p},(...........):(1-\beta:1):(...........):\\ [.....]:(\beta_j,B_j)_{1,q},(1-\gamma:\nu):(0:1),(1+\gamma-\alpha-\beta-\mu k:1):\end{array} h\left(1-\frac{\lambda}{A_jt_j}\right)^{-A_j},b^{-1}\left(x^{-1}-\lambda\right)z^{-1}\right]$$

$$=\sum_{r=0}^{\infty}\left(\frac{-\lambda x}{A_j}\right)^r\frac{1}{r!}\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\Gamma(\alpha-\gamma+\mu k)\frac{(y(b/1+z)^{\mu})^k}{k!}$$

$$.H^{0,1:m,n:1,1}_{1,0:p,q+1:2,1}\left[\begin{array}{c}[1-\gamma:\nu,1]:(\alpha_j:A_j)_{1,n},(\alpha_j-r:A_j)_{n+1,p}:(1-\beta:1),(.......):\\ [........]:(\beta_j,B_j)_{1,q},(1-\gamma:\nu):(0:1),(1+\gamma-\alpha-\beta-\mu k:1):\end{array} h,b^{-1}x^{-1}z^{-1}\right] t_j^{\alpha_j-r} \tag{6.2}$$

when, $n+1\leq j\leq p$.

$$\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\Gamma(\alpha-\gamma+\mu k)\frac{(y(b/1+z)^{\mu})^k}{k!}u_j^{\beta_j}\left(1+\frac{\lambda u_j}{B_j}\right)^{-\beta_j}$$

$$.H^{0,1:m,n:1,1}_{1,0:p,q+1:2,1}\left[\begin{array}{c}[1-\gamma:\nu,1]:(\alpha_j:A_j)_{1,p}:(1-\beta:1):(...........):\\ [.....]:(\beta_j,B_j)_{1,q},(1-\gamma:\nu):(0:1),(1+\gamma-\alpha-\beta-\mu k:1):\end{array} h\left(1+\frac{\lambda u_j}{B_j}\right)^{B_j},b^{-1}\left(x^{-1}-\lambda\right)z^{-1}\right]$$

$$=\sum_{r=0}^{\infty}\left(\frac{-\lambda x}{B_j}\right)^r\frac{1}{r!}\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\Gamma(\alpha-\gamma+\mu k)\frac{(y(b/1+z)^{\mu})^k}{k!}.H^{0,1:m,n:1,1}_{1,0:p,q+1:2,1}$$

$$\left[\begin{array}{c}[1-\gamma:\nu,1]:(\alpha_j:A_j)_{1,p},(.........):(1-\beta:1),(.........):\\ [.....]:(\beta_j+r,B_j)_{1,m},(\beta_j:B_j)_{m+1,q},(1-\gamma:\nu):(0:1),(1+\gamma-\alpha-\beta-\mu k:1):\end{array} h,b^{-1}x^{-1}z^{-1}\right] u_j^{\beta_j+r}, \tag{6.3}$$

when, $1\leq j\leq m$.

$$\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\Gamma(\alpha-\gamma+\mu k)\frac{(y(b/1+z)^{\mu})^k}{k!}u_j^{\beta_j}\left(1+\frac{\lambda u_j}{B_j}\right)^{-\beta_j}$$

$$.H^{0,1:m,n:1,1}_{1,0:p,q+1:2,1}\left[\begin{array}{c}[1-\gamma:\nu,1]:(\alpha_j:A_j)_{1,p}:(1-\beta:1):(..........):\\ [.....]:(\beta_j,B_j)_{1,q},(1-\gamma:\nu):(0:1),(1+\gamma-\alpha-\beta-\mu k:1):\\ h\left(1+\dfrac{\lambda u_j}{B_j}\right)^{B_j},b^{-1}\left(x^{-1}-\lambda\right)z^{-1}\end{array}\right]$$

$$=\sum_{r=0}^{\infty}\left(\frac{\lambda x}{B_j}\right)^r\frac{1}{r!}\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\Gamma(\alpha-\gamma+\mu k)\frac{(y(b/1+z)^{\mu})^k}{k!}.H^{0,1:m,n:1,1}_{1,0:p,q+1:2,1}$$

$$\left[\begin{array}{c}[1-\gamma:\nu,1]:(\alpha_j:A_j)_{1,p},(...........),(...........):(1-\beta:1),(...........):\\ [.....]:(\beta_j,B_j)_{1,m},(\beta_j+r,B_j)_{m+1,q},(1-\gamma:\nu):(0:1),(1-\gamma-\alpha-\beta-\mu k:1):\\ h,b^{-1}x^{-1}z^{-1}\end{array}\right]u_j^{\beta_j+r}, \tag{6.4}$$

when, $m+1\leq j\leq q$.

$$\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\Gamma(\alpha-\gamma+\mu k)\frac{(y(b/1+z)^{\mu})^k}{k!}\left(1+\frac{\lambda}{\nu s}\right)\left(s+\frac{\lambda}{\nu}\right)^{\gamma}$$

$$.H^{0,1:m,n:1,1}_{1,0:p,q+1:2,1}\left[\begin{array}{c}[1-\gamma:\nu,1]:(\alpha_j:A_j)_{1,p},(..........):(1-\beta:1),(..........):\\ [.....]:(\beta_j,B_j)_{1,q},(1-\gamma:\nu):(0:1),(1+\gamma-\alpha-\beta-\mu k:1):\\ h\left(1+\dfrac{\lambda}{\nu s}\right)^{\nu},b^{-1}\left(x^{-1}-\lambda\right)z^{-1}\end{array}\right]$$

$$=\sum_{r=0}^{\infty}\frac{(\lambda x/\nu)^r}{r!}\sum_{k=0}^{[N/M]}(-N)_{Mk}\phi_{N,k}\Gamma(\alpha-\gamma+\mu k)\frac{(y(b/1+z)^{\mu})^k}{k!}.H^{0,1:m,n:1,1}_{1,0:p,q+1:2,1}$$

$$\left[\begin{array}{c}[1-\gamma:\nu,1]:(\alpha_j:A_j)_{1,p},(...........):(1-\beta:1),(...........):\\ [.....]:(\beta_j,B_j)_{1,q},(1+r-\gamma:\nu):(0:1),(1-\gamma-\alpha-\beta-\mu k:1):\\ h,b^{-1}x^{-1}z^{-1}\end{array}\right]s^{\gamma-r}. \tag{6.5}$$

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