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GENERALISED KROPINA CHANGE OF FINSLER METRIC AND FINSLERIAN HYPERSURFACES

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1. INTRODUCTION

Hypersurfaces and sub spaces of Finsler space have been studied by M. Haimovice in (1939), O. Varga in (1942), E. T. Davies in (1947) etc. Later the theory of Finslerian hypersurfaces have been established by M. Matsumoto in 1985 ([9]). He has defined three types of hypersurfaces. Later these hypersurfaces known as I, II and III kinds of the hyperplanes by Rapesak ([1]), Kikuchi ([10]) ans Haimovici ([7]).

The purpose of the present paper is to study of the Finslarian hypersurfaces given by generalised Kropina change by using the field of linear frame ([2], [7], [9]). I have established some relations between original Finslerian hypersurfaces and generalised Kropina change make three types of hypersurfaces invariant under certain condition.

Let (M^n, L) be an n-dimensional Finsler space on differentiable manifolds M^n, equipped with the fundamental function $L(x, y)$. C. Shibata ([6]) introduced the transformation of Finsler metric in 1984.

$$L^*(x,y) = f(L,\beta) \tag{1.1}$$

where $\beta = b_i(x)y^i$, $b_i(x)$ are components of a covariant vector field in (M^n,L) and f is positively homogenous function of degree one in L and β. This change of metric is known as a β-change.

Finsler metric equipped with β-change is known as Kropina Change given by ([11]) which as follows:

$$L^*(x, y) = L^2(x, y) \backslash \beta(x, y)$$

A Finsler metric $L(x, y) = \alpha^p \beta^q$, where p+q=1 is a generalization of Kropina type i.e. $L(x,y) = \dfrac{\alpha^{m+1}}{\beta^m}$ where $m \neq 0, 1$, $\alpha = q_{ij}(x) y^i y^j$ and $\beta = b_i(x) y^i$. In view of (1.1) generalization of Kropina change metric

$L^*(x, y) = [L^2(x, y) \backslash \beta(x, y)] + \beta(x, y)$ is defined as :

$$L^*(x, y) = \mu^2(x, y) \backslash \beta(x, y), \ \beta(x, y) \text{ does not vanish,} \tag{1.2}$$

where $\mu^2(x, y) = (L^2 + \beta^2)(x, y)$

If $L(x,y)$ reduces to the metric function of Riemannian space then $L^*(x, y)$ reduces to the metric function of a Kropina space [3] as well as generalized Kropina space. Due to this reason the transformation (1.2) has been called the generalized Kropina Change of Finsler metric.

2. PRELIMINARIES

Let M^n be an n-dimensional smooth manifold and $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with the fundamental function $L(x, y)$ on M^n. The metric tensor $g_{ij}(x, y)$ and Cartan's C-tensor $C_{ijk}(x, y)$ are defined as :

$$g_{ij}(x,y) = \tfrac{1}{2}\partial^2 L \backslash \partial y^i \partial y^j \quad , \quad C_{ijk} = \tfrac{1}{2}\partial g_{ij} \backslash \partial y^k$$

The Cartan's Connection can be introduced in F^n which is as follows :

$$C\Gamma = (F^i_{jk}, N^i_j, C^i_{jk}).$$

A hypersurfaces M^{n-1} of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where u^α are Gaussian coordinates on M^{n-1} and greek indices vary from 1 to n - 1, where we shall assume that the metric consisting of the projection factors $B^i_\alpha = \partial x^i \backslash \partial u^\alpha$ is of rank n - 1. The following notations are also employed : $B^i_{\alpha\beta} = \partial^2 x^i \backslash \partial u^\alpha \partial u^\beta$, $B^i_{0\beta} = v^\alpha B^i_{\alpha\beta}$. If the supporting element y^i at a point (u^α) of M^{n-1} is assumed to be tangential to M^{n-1}, it may be written as $y^i = B^i_\alpha(u)v^\alpha$ i. e. v^α is thought of as the supporting element of M^{n-1} at the point (u^α). Hence, $\underline{L}(u,v) = L(x(u), y(v))$ is follows as Finsler metric of M^{n-1}, which gives (n-1) dimensional Finsler space $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$. At each point (u^α) of F^{n-1}, the unit normal vector $N^i(u,v)$ is defined by :

$$g_{ij} B^i_\alpha N^j = 0 \quad , \quad g_{ij} N^i N^j = 1 \tag{2.1}$$

If (B^α_i, N_i) is the inverse matrix of (B^i_α, N^i), we have

$$B^i_\alpha B^\alpha_i = \delta^\beta_\alpha, \; B^i_\alpha N_i = 0, \; N^i N^i = 1 \text{ and } B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j .$$

In view of inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \; N_i = g_{ij} N^j . \tag{2.2}$$

For the induced Cartan's connection $C\Gamma = (F^\alpha_{\beta\nu}, N^\beta_\alpha, C^\alpha_{\beta\nu}) F^{n-1}$, the second fundamental h-tensor $H_{\alpha\beta}$ and the normal Curvature vector H_α are respectively given by ([8]).

$$H_{\alpha\beta} = N_i (B^i_{\alpha\beta} + F^i_{jk} B^j_\alpha B^k_\beta) + M_\alpha H_\beta , \tag{2.3}$$

$$H_\alpha = N_i (B^i_{0\alpha} + N^i_j B^j_\alpha) ,$$

where

$$M_\alpha = C_{ijk} B^i_\alpha N^j N^k . \tag{2.4}$$

Contraction of $H_{\alpha\beta}$ by v^α, gives $H_{\alpha\beta} v^\alpha = H_\beta$. Further more the second fundamental v-tensor $M_{\alpha\beta}$ is given by ([10]) is defined as :

$$M_{\alpha\beta} = C_{ijk} B^i_\alpha B^j_\beta N^k . \tag{2.5}$$

3. GENERALISED KROPINA CHANGED FINSLER SPACE

Let $F^n = (M^n, L)$ be a given Finsler Space and let $b_i(x)dx^i$ be a 1-form on M^n. We shall define on M^n a function $L^*(x,y) > 0$ by the equation (1.2) where we put $\beta(x,y) = b_i(x)y^i$. The following results are obtained to find metric tensor g^*_{ij}, the angular metric tensor h^*_{ij} and the Cartan's C-tensor C^*_{ijk} of $F^{*n} = (M^n, L^*)$, as follows:

$$\partial\beta \backslash \partial y^i = b_i,\ \partial L \backslash \partial y^i = l_i,\ \partial l_j \backslash \partial y^i = L^{-1} h_{ij}, \tag{3.1}$$

where h_{ij} are components of angular metric tensor of F^n defined as :

$$h_{ij} = g_{ij} - l_i l_j = L\partial^2 L \backslash \partial y^i \partial y^j .$$

The successive differentiation of (1.2) with respect to y^i and y^j gives the following results :

$$l^*_i = (\alpha \backslash \beta)(Ll_i + \beta^2 b_i) - (\mu^2 \backslash \beta^2) b_i,\ \text{where } \mu^2 = L^2 + \beta^2 , \tag{3.2}$$

$$h^*_{ij} = Q_0 (h_{ij} + l_i l_j) - (LQ_0 \backslash \beta)(l_i b_j + l_j b_i) + (Q_0 \backslash \beta^2)(L^2 + \beta^2) b_i b_j \tag{3.3}$$

$$= Q_0 [(h_{ij} + l_i l_j) - (L \backslash \beta)(l_i b_j + l_j b_i) + (1 \backslash \beta^2)(L^2 + \beta^3) b_i b_j],$$

where $Q_0 = 2\mu^2 \backslash \beta^2$.

In view of (3.2) and (3.3), the metric tensor of F^{*n} is given as follows:

$$g^{*}_{ij} = Q_0 g_{ij} + (4L^2 \backslash \beta^2) l_i l_j + Q_1 b_i b_j + Q_2 (l_i b_j + l_j b_i), \tag{3.4}$$

where $Q_0 = 2\mu^2 \backslash \beta^2$

$Q_1 = 4 - 2\mu^2 \backslash \beta + \mu^4 \backslash \beta^4 + 2\mu^2 L \backslash \beta^4$

and $Q_2 = 4 - 2\mu^2 L \backslash \beta^3 - 2\mu^2 \backslash \beta^3$.

Differentiating (3.4) with respect to y^k and using (3.1), we get the following relation between the Cartan's C-tensor of F^n and F^{*n}.

$$C^{*}_{ijk} = Q_0 C_{ijk} + P_0 l_i l_j l_k + P_{01} l_k + P_{02} (h_{jk} l_i + h_{ik} l_j) + m_i P_{1(jk)} + m_j P_{2(ik)} + m_k P_{3(ijk)}, \tag{3.5}$$

where $Q_0 = 2\mu^2 \backslash \beta^2$,

$P_0 = 8(L+\beta) \backslash \beta + (\mu^2 \backslash \beta^2)(L-1)$,

$P_{01} = 2(L-\beta) \backslash \beta$,

$P_{02} = 2L \backslash \beta^2$,

$P_{1(jk)} = \frac{1}{2}\{Q_2 L^{-1} h_{jk} - 10(L \backslash \beta^3) l_j l_k - (l_k \backslash \beta^4)(L^{-1}\beta^2 - \mu^2)\}$,

$P_{2(ik)} = \frac{1}{2}\{Q_2 L^{-1} h_{ik} + (2 \backslash \beta^5) l_i l_k (L^{-1}\beta^4 + \mu^2 - 10L)\}$,

$P_{3(ijk)} = (2L^2 \backslash \beta^3) l_k + 2 \backslash \beta^3 (\mu^2 - 2L^2) l_i l_j$,

$m_i = (b_i - \beta \backslash 2) l_i$, $b_i = m_i + (\beta \backslash L) l_i$.

It is to be notted that

$$m_i l^i = 0, \; m_i b^i = b^2 - \beta^2 \backslash L^2, \; h_{ij} l_j = 0, \; h_{ij} m^j = h_{ij} b^j = m_i, \tag{3.6}$$

where $m^i = g^{ij} m_j = b^i - (\beta \backslash L) l^i$.

4. HYPERSURFACES GIVEN BY A GENERALISED KROPINA CHANGE

Consider a Finsler hypersurface $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$ of the F^n and another Finsler hypersurfaces $F^{*n-1} = (M^{n-1}, \underline{L}^*(u,v))$ of the F^{*n} given by the generalized Kropina change. Let N^i be the unit normal vector at each point of F^{n-1} and (B_i^α, N_i) be the inverse matric of (B^i_α, N^i). The function B^i_α may be considered as a components of n - 1 linearly independent tangent vectors of F^{n-1} and they are invariant under genralized Kropina change. Thus I shall show that a unit normal vector N^{*i} of F^{*n-1} is uniquely determined by:

$$g^*_{ij} B^i_\alpha N^{*j} = 0, \quad g^*_{ij} N^{*i} N^{*j} = 1. \tag{4.1}$$

Contracting (3.4) by $N^i N^j$ and in view of (2.1) and $l_i N^i = 0$ I have :

$$g^*_{ij} N^i N^j = (\mu^2 \backslash \beta^4)[2\beta^2 + (b_i N^i)^2 (4\beta^4 \mu^{-2} - 2\beta^3 + \mu^2 + L^2)].$$

Therefore we obtain

$$g^*_{ij} [\pm N^i \beta^2 \backslash \mu \sqrt{\{2\beta^2 + (b_i N^i)^2 (4\beta^4 \mu^{-2} - 2\beta^3 + \mu^2 + L^2)\}}$$
$$[\pm N^j \beta^2 \backslash \mu \sqrt{\{2\beta^2 + (b_i N^i)^3 (4\beta^4 \mu^{-2} - 2\beta^3 + \mu^2 + L^2)\}}] = 1.$$

Hence N^{*i} takes the following form :

$$N^{*i} = \beta^2 N^i \backslash \mu \sqrt{\{2\beta^2 + (b_i N^i)^2 (4\beta^4 \mu^{-2} - 2\beta^3 + \mu^2 + L^2)\}}, \tag{4.2}$$

where positive sign has choosen in order to fix orientation. In view of (3.1), (3.4) and (4.2) the first condition of (4.1) takes the following form :

$$(Q_1 b_i B^i_\alpha + Q_2 l_i B^i_\alpha) b_j \beta^2 N^j \backslash \mu \sqrt{\{2\beta^2 + (b_i N^i)^2 (4\beta^4 \mu^{-2} - 2\beta^3 + \mu^2 + L^2)\}}. \tag{4.3}$$

If $(Q_1 b_i B^i_\alpha + Q_2 l_i B^i_\alpha) = 0$, contracting it by v^α and in view of $y^i = B^i_\alpha v^\alpha$

and $L > 0$, we get $(Q_1 b_i B^i_\alpha + Q_2 l_i B^i_\alpha) \neq 0$. Hence $b_i N^i = 0$. Therefore (4.2) is rewritten as :

$$N^{*i} = (\beta \backslash \mu \sqrt{2}) N^i. \tag{4.4}$$

Summarising the above, I have :-

PROPOSITION 4.1 : For a field of linear frame $(B^i_1, B^i_2, \ldots B^i_{n-1}, N^i)$ of F^n there exists a field of linear frame $(B^i_1, B^i_2, \ldots B^i_{n-1}, N^{*i} = (\beta \backslash \mu \sqrt{2}) N^i)$ of F^{*n} such that (4.1) is satisfied along F^{*n-1} and then b_i is tangential to both the hypersurfaces F^{n-1} and F^{*n-1}.

The quantities $B^{*\alpha}_i$ are uniquely determined along F^{n-1} by

$$B^{*\alpha}_i = g^{*\alpha\beta} g^*_{ij} B^j_\beta,$$

where $(g^{*\alpha\beta})$ is the inverse matrix of $(g^*_{\alpha\beta})$. Let $(B^{*\alpha}_i, N^*_i)$ be the inverse of (B^i_α, N^i), then $B^i_\alpha B^{*\beta}_i = \delta^\beta_\alpha$, $N^{*i} N^*_i = 1$ and furthermore $B^i_\alpha B^{*\alpha}_j + N^{*i} N^*_j = \delta^i_j$, $N^*_i = g^*_{ij} N^{*j}$ which in view of (3.4), (3.2) and (4.4) gives

$$N^*_i = (\beta \backslash \mu \sqrt{2}) N_i), \tag{4.5}$$

The Cartan's connection of F^n and F^{*n} are denoted by $(F^i_{jk}, N^i_j, C^i_{jk})$ and $(F^{*i}_{jk}, N^{*i}_j, C^{*i}_{jk})$ and the difference tensor is defined as : $D^i_{jk} = F^{*i}_{jk} - F^i_{jk}$

The vector field b_i in F^n can be taken as :

$$D^i_{jk} = A_{jk} b^i - B_{jk} l^i, \tag{4.6}$$

where A_{jk} and B_{jk} are components of a symmetric covariant tensor of second order. In view of $N_i b^i = 0$ and $N_i l^i = 0$ and (4.6), I get $N_i D^i_{jk} = 0$ and $N_i D^i_{0k} = 0$, on account of (2.3) and (4.5), I get

$$H^*_\alpha = (\mu \sqrt{2} \backslash \beta) H_\alpha, \quad \text{where} \quad \mu^2 = L^2 + \beta^2 \tag{4.7}$$

If each path of the hypersurfaces F^{n-1} with respect to the induced connection is also a path of enveloping space F^n, then F^{n-1} is called a hyperplane of the first kind ([8]). A hyperplane of the first kind is characterised by $H_\alpha = 0$. Hence from (4.7), I have

THEOREM 4.1 : If $b_i(x)$ be a vector field in F^n satisfying (4.6), then a hypersurface F^{n-1} is a hyperplane of the first kind if and only if the hypersurface F^{*n-1} is a hyperplane of first kind.

Next contracting (3.5) by $B_i^\alpha N^{*j} N^{*k}$ and on account of $m_i N^i = 0$, $h_{jk} N^j N^k = 1$ and $h_{ij} B_\alpha^i N^j = 0$. I get

$$M_\alpha^* = M_\alpha + B_\alpha^i \{(L l_i \backslash \mu^2) + m_i (L^{-1} \backslash 2)(L - 1 \backslash \beta)\}. \qquad (4.8)$$

From (2.3), (4.5), (4.6), (4.7) and (4.8), I have

$$H_{\alpha\beta}^* = (\mu \sqrt{2} \backslash \beta)[H_{\alpha\beta} + H_\alpha B_\alpha^i \{(L l^i \backslash \mu^2) + (m_i L^{-1} \backslash 2)(L - 1 \backslash \beta). \qquad (4.9)$$

If each h-path of a hypersurface F^{n-1} with respect to the induced connection is also h-path of the enveloping space F^n, then F^{n-1} is called hyperplane of second kind.

A hyperplane of the first kind is characterised by $H_{\alpha\beta} = 0$. Since $H_{\alpha\beta} = 0$ implies that $H_\alpha = 0$. From (4.7) and (4.9) I have the following theorem :

THEOREM 4.2 : If $b_i(x)$ be a vector field in F^n satisfying (4.6), then a hypersurface F^{n-1} is a hyperplane of the second kind if and only if the hypersurface F^{*n-1} is a hyperplane of second kind.

Finally contracting (3.5) by $B_\alpha^i B_\beta^j N^{*k}$ and on account of (4.4), I have

$$M_{\alpha\beta}^* = (\mu \sqrt{2} \backslash \beta) M_{\alpha\beta} \qquad (4.10)$$

If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1}, the F^{n-1} is called a hyperplane of third kind ([8]). A hyperplane of the third kind is characterised by $H_{\alpha\beta} = 0$, $M_{\alpha\beta} = 0$. From (4.7), (4.9) and (4.10) I have :

THEOREM 4.3 : If $b_i(x)$ be a vector field in F^n satisfying (4.6), then a hypersurface F^{n-1} is a hyperplane of the third kind if and only if the hypersurface F^{*n-1} is a hyperplane.

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