Journal of the Tensor Society
Volume 1, 2007
https://doi.org/10.56424/jts.v1i01.9951

AFFINE CURVATURE INHERITANCE IN AN NP-F_n

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ABSTRACT : The concept of on the curvature inheritance in Finsler space II has studies by Singh [4]. In the present paper, we study affine curvature inheritance in an NP-F_n. Some special cases are also considered at the end.

Key words : NP-F_n, RNP-F_n, Affine motion, Contra and concurrent field.

1. INTRODUCTION

K.Yano [2] defined a set of parameters

$$\Pi^i_{kh} = G^i_{kh} - \frac{\dot{x}^i}{n+1} G^r_{khr}, \tag{1.1}$$

which forms a connection called the normal projective connection. The functions Π^i_{kh}, G^i_{kh} and G^i_{jkh} are symmeric in their lower indices and are positively homogeneous of degree zero, zero and-1 respectively in their directional arguments. The functions G^i_{jk} are the Berwald's connection parameters. The derivatives

$$\Pi^i_{jkh} = \dot{\partial}_j \Pi^i_{kh} = G^i_{jkh} - \frac{1}{n+1}(\delta^i_j G^r_{khr} + \dot{x}^i G^r_{jkhr}), \tag{1.2}$$

is symmetric in k and h only and is positively homogeneous of degree -1 in directional arguments. Therefore, the following relations which will be used in our discussion follow from (1.1) and (1.2):

a) $\Pi^i_{kh}\dot{x}^k = \Pi^i_{hk}\dot{x}^k = G^i_h$, b) $\Pi^i_{ki} = G^i_{ki}$,

c) $\dot{x}^j\Pi^i_{jkh} = 0$ d) $\Pi^i_{ikh} = \frac{2}{n+1}G^i_{ikh}$,

e) $\Pi^i_{jki} \doteq G^i_{jki} = \Pi^i_{jik}$. (1.3)

The normal projective connection parameters define the projective convariant derivative :

$$\nabla_k X^i = \partial_k X^i - (\dot{\partial}_j X^i)\Pi^j_{kh}\dot{x}^h + X^j\Pi^i_{jk},$$

$$(\partial_k = \partial/\partial x^k, \dot{\partial}_k = \partial/\partial\dot{x}^k), \quad (1.4)$$

and preserve the vector character of X^i. In particular, this derivative vanishes for $\dot{x}^i$. The corresponding curvature tensor $N^i_{jkh}(x,\dot{x})$ tensor so-called, by K. Yano, the normal projective curvature tensor, is given by

$$N^i_{jkh} = 2\{\partial_{[j}\Pi^i_{k]h} + \Pi^i_{lh[j}\Pi^l_{k]m}\dot{x}^m + \Pi^i_{l[j}\Pi^l_{k]h}\}. \quad (1.5)$$

DEFINITION : The manifold F_n with normal projective connection parameters Π^i_{kh} and the normal projective curvature tensor N^i_{jkh} is called the normal projective Finsler manifold which is denoted by NP-F_n.

The normal projective curvature tensor is skew-symmetric in j, k indices and is a homogeneous function of degree 0 in directional arguments, so by definition we have

a) $N^i_{jkh} = -N^i_{kjh}$, b) $\dot{\partial}_l N^i_{jkh}\dot{x}^l = 0$. (1.6)

The contraction of N^i_{jkh} with respect to i, j ; i, k and i, h give

a) $N^i_{ikh} = N_{kh}$, b) $N^i_{jih} = -N^i_{ijh} = -N_{jh}$,

and c) $N^i_{jki} = 2N_{[kj]}$ (1.7)

repectively, where [k j] represents the skew-symmetric part.

The covariant derivative gives rise to the commutation formula

$$2\nabla_{[j}\nabla_{k]}X^i = N^i_{jkh}X^h - (\dot{\partial}_l X^i)N^l_{jkh}\dot{x}^h \tag{1.8}$$

together with the normal projective curvature tensor N^i_{jkh}. In terms of the contracted tensor $N_{kh} = N^i_{ikh}$ there is defined a tensor

$$M_{kh} = -\frac{1}{n^2-1}(nN_{kh} + N_{hk}) \tag{1.9}$$

and the Weyl's projective curvature tensor W^i_{jkh} is given by K. Yano [2]

$$W^i_{jkh} = N^i_{jkh} + 2\{\delta^i_{[j}M_{k]h} - M_{[jk]}\delta^i_h\}. \tag{1.10}$$

The commutation formulae for any general tensor, involving the curvature tensor, are given as follows K. Yano [2]

$$2\nabla_{[k}\nabla_{h]}T^i_j = N^i_{khl}T^l_j - N^l_{khj}T^i_l - (\dot{\partial}_l T^i_j)N^l_{khm}\dot{x}^m, \tag{1.11}$$

$$(\dot{\partial}_j\nabla_k - \nabla_k\dot{\partial}_j)T^i_h = \Pi^i_{jkl}T^l_h - \Pi^l_{jkh}T^i_l - \Pi^l_{jkm}\dot{x}^m(\dot{\partial}_l T^i_h). \tag{1.12}$$

The Lie-derivative of a tensor T^i_j and the connection coefficients Π^i_{jk} defined by an infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon v^i(x) \tag{1.13}$$

are characterized by K. Yano [2]

$$\pounds T^i_j = v^h(\nabla_h T^i_j) - T^h_j(\nabla_h v^i) + T^i_h(\nabla_j v^h) + (\dot{\partial}_h T^i_j)(\nabla_s v^h)\dot{x}^s \tag{1.14}$$

and

$$\pounds \Pi^i_{jk} \doteq \nabla_j\nabla_k v^i - N^i_{hjk}v^h + \Pi^i_{hjk}(\nabla_l v^h)\dot{x}^l \tag{1.15}$$

respectively.

The commutation formulae with respect to Lie-derivative and other for any tensor T^i_{jk} are given by

$$\pounds(\nabla_l T^i_{jk}) - \nabla_l(\pounds T^i_{jk}) = (\pounds\Pi^i_{lh})T^h_{jk} - (\pounds\Pi^r_{jl})T^i_{rk}$$
$$- (\pounds\Pi^r_{kl})T^i_{jr} - (\pounds\Pi^r_{lm})\dot{x}^m(\dot{\partial}_r T^i_{jk}) \tag{1.16}$$

and

$$\dot{\partial}_l(\pounds T^i_{jk}) - \pounds(\dot{\partial}_l T^i_{jk}) = 0. \tag{1.17}$$

The Lie-derivative of the normal projective curvature tensor N^i_{kjh} is expressed in the form

$$\nabla_k(\pounds\Pi^i_{jh}) - \nabla_j(\pounds\Pi^i_{kh}) = \pounds N^i_{kjh} + (\pounds\Pi^r_{km})\dot{x}^m\Pi^i_{rjh}) - (\pounds\Pi^r_{jm})\dot{x}^m\Pi^i_{rkh}. \tag{1.18}$$

The infinitesimal transformation (1.13) defines an affine motion, it satisfies the condition by Misra and Meher [5]

$$\pounds\Pi^i_{jk} = 0 \tag{1.19}$$

2. AFFINE N-CURVATURE INHERITANCE

Singh [4] defined the R*-curvature inheritance as an infinitesimal transformation with respect to which the Lie-derivative of Berwald's curvature tensor R^{*i}_{jkh} is the relation of the form

$$\pounds R^{*i}_{jkh} = \alpha^*(x) R^{*i}_{jkh} \tag{2.1}$$

where $\alpha^*(x)$ is non-zero scalar function.

In the present paper, we consider the infinitesimal transformation (1.13) which admits an affine motion in a NP-F_n. Now we define and study the infinitesimal transformation (1.13) which is an affine motion in a NP-F_n.

DEFINITION 2.1' : In an NP-F_n, if the normal projective curvature tensor field N^i_{jkh} satisfies the relation

$$\pounds N^i_{jkh} = \alpha(x) N^i_{jkh} \tag{2.2}$$

where $\alpha(x)$ is non-zero scalar function and $\pounds$ denotes Lie-derivative defined by the infinitesimal transformation (1.13), then the transformation (1.13) is called an affine N-curvature inheritance.

Contracting with respect to the indices i, j in (2.2), we get

$$\pounds N_{kh} = \alpha N_{kh}. \tag{2.3}$$

is called an affine Ricci-like N-curvature inheritance.

Applying Lie operator to both sides of (1.9) and using (2.3) and (1.9), we obtain

$$\pounds M_{kh} = \alpha(x) M_{kh} \tag{2.4}$$

Applying Lie operator to both sides of (1.10), using (2.2), (2.4) and (1.10), we get

$$\pounds W^i_{jkh} = \alpha(x) W^i_{jkh}. \tag{2.5}$$

By virture of (2.2) and (1.19), the equation (1.8) reduces to

$$\alpha(x) N^i_{jkh} = 0 \tag{2.6}$$

by using (1.6)a.

Since α (x) is a non-zero scalar function, we have

$$N^i_{jkh} = 0 \tag{2.7}$$

Accordingly, we have

THEOREM 2.1 : In an NP-F_n, which admits an affine N- curvature inheritance, if the space is flat.

Contracting with respect to i, j in (2.6) and using (1.7)a), we obtain

$$\alpha(x) N_{kh} = 0. \tag{2.8}$$

Since α (x) is a non-zero scalor function, we get

$$N_{kh} = 0. \tag{2.9}$$

Thus, we state:

COROLLARY 2.1 : In an NP-F_n, which admits an affine Ricci-like N-curvature inheritance, if the space is flat.

Employing Lie operator to both sides of (1.10) and using (2.4), (2.5),

Theorem (2.1) and corollary (2.1), we obtain

$$\alpha(x)W^i_{jkh} = 0. \tag{2.6}$$

Since α (x) is a non-zero scalar function, we get

$$W^i_{jkh} = 0. \tag{2.10}$$

Thus, we state:

THEOREM 2.2 : In an NP -F_n, which admits an affine N - curvature inheritance, if the space is projectively flat.

Applying the commutation formula (1.16) for N^i_{jkh}, using the equations (2.2) and (1.19), we get

$$£(\nabla_l N^i_{jkh}) - \nabla_l\{\alpha(x)N^i_{jkh}\} = 0. \tag{2.11}$$

The equation (2.11) assumes the form

$$£(\nabla_l N^i_{jkh}) = \alpha(x)\nabla_l N^i_{jkh}. \tag{2.12}$$

If the gradient vector field $\nabla_l \alpha = \alpha_l$ is zero.

Hence, we state :

LEMMA 2.1 : When an affine N-curvature inheritance admitted in NP-F_n, the derivative of the normal projective curvature tensor N^i_{jkh} satisfies the inheritance property (2.12) provided the gradient vector field α_l is zero.

3. SPECIAL CASES

In this section, we discuss the following important cases :

(a) CONTRA FIELD : In an NP-F_n, if the vector field $v^i(x)$ satisfies the relation by Mishra and Yadav [6]

$$\nabla_j v^i = 0, \tag{3.1}$$

then the vector field $v^i(x)$ spans a contra field.

Here we consider a special infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon v^i(x), \quad \nabla_j v^i = 0, \tag{3.2}$$

which admits an affine motion in NP-F_n. It is assumed that the relation (2.2) is also satisfied in NP-F_n, then the transformation (3.2) defines an affine N-curvature inheritance.

Employing the equations (1.19) in (1.15, we obtain

$$0 = \nabla_j \nabla_k v^i - N^i_{hjk} v^h + \Pi^i_{hjk} (\nabla_l v^h) \dot{x}^l.$$

If $v^i(x)$ spans a contra field, it reduces to

$$N^i_{hjk} v^h = 0. \tag{3.3}$$

In addition the integrability condition of affine motion gives

$$£N^i_{jkh} v^h \equiv \nabla_l N^i_{jkh} v^h = 0. \tag{3.4}$$

In view of (2.2), the equation (3.2) implies that the space is flat.
Hence, we state :

THEOREM 3.1 : In an NP-F_n, which admits an affine N-curvature inheritance of the following conditions:

(i) the NP - F_n is flat

(ii) $N^i_{hjk} v^h = 0$

(iii) $\nabla_l N^i_{hjk} v^h = 0$

holds good.

(b) CONCURRENT FIELD : In an NP-F_n, if the vector field $v^i(x)$ satisfies the relation

$$\nabla_j v^i = c\delta^i_j, \tag{3.5}$$

where c is a non-zero constant then the vector field $v^i(x)$ determines a concurrent field.

In this section, we consider the infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon v^i(x), \quad \nabla_j v^i = c\delta^i_j, \tag{3.6}$$

which admits an affine motion and defines an affine N-curvature inheritance in NP - F_n.

The covariant derivative of (3.3) with respect to x^l and using (3.5), we get

$$\nabla_l N^i_{hjk} v^h + c N^i_{ljk} = 0$$

Applying Lie operator in above equation, using (2.12) and (2.2), we obtain

$$\alpha(x) \nabla_l N^i_{hjk} v^h + \alpha(x) c N^i_{ljk} = 0 \tag{3.7}$$

Since $\alpha(x)$ is a non-zero scalar function, we have

$$\nabla_l N^i_{hjk} v^h = c N^i_{jlk} \tag{3.8}$$

in view of the equation (1.6)a.

Hence, we state:

THEOREM 3.2 : In an NP - F_n, which admits an affine N-curvature inheritance, if the vector field $v^i(x)$ determines a coccurrent field, the relation (3.8) necessarily hold.

Let us assume that the space under consideration is a RNP-F_n and the transformation (3.6) defines an affine N-curvature inheritance in it. In this case the relation (3.8) assumes the form

$$\lambda_l N^i_{hjk} v^h = c N^i_{jlk}. \tag{3.9}$$

Accordingly, we state :

THEOREM 3.3 : In an RNP-F_n, which admits an affine N-curvature inheritance, if the vector field $v^i(x)$ determines a concurrent field, the relation (3.9) necessarily hold.

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