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DOUBLE SAMPLING GENERALIZED ESTIMATORS OF RATIO AND PRODUCT OF PARAMETERS

By

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ABSTRACT : Double sampling generalized estimators representing a wider classes of estimators than some previous ones in the literature, are proposed for the estimation of ratio and product parameters. Bias and mean square error of the proposed wider classes of estimators are found and their properties are studied. Subsets of optimum estimators in the sense of having minimum mean square error are investigated and subsets of estimators depending upon estimated optimum values and attaining the same minimum mean square error of the optimum are also obtained.

Key words : Optimum estimators, Bias and mean square error, Estimated optimum and Efficiency.

1. INTRODUCTION

Let a first phase large simple random sample of size n' be drawn from a population size N, let the auxiliary character x_2 be observed to find an estimate of population mean $\overline{X}_2$ of x_2, and further, let the characters y, x_1 be observed on the second phase simple random sample of size n from the first phase sample of size n'. Let $(\overline{Y},\overline{X}_1)$ be the population means of the characters (y,x_1) respectively, $\overline{x}'_2$ be the sample mean of n' first phase sample values on x_2 and $(\overline{y},\overline{x}_1)$ be the sample means of n second phase sample values on (y,x_1) respectively. Also let ρ_{01}, ρ_{02}

and ρ_{12} be the correlation coefficients between (y,x_1), (y,x_2) and (x_1,x_2) respectively, and

$$C_0 = \frac{S_y}{\overline{Y}}, \qquad C_1 = \frac{S_{x_1}}{\overline{X}_1}, \qquad C_2 = \frac{S_{x_2}}{\overline{X}_2}$$

$$S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_i - \overline{Y})^2,$$

$$S_{x_1}^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(X_{1i} - \overline{X}_1)^2,$$

$$S_{x_2}^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(X_{2i} - \overline{X}_1)^2,$$

where (Y_i, X_{1i}, X_{2i}) are the values on characters (y, x_1, x_2) respectively for the i^{th} $(i = 1,2,\ldots, N)$ unit of the population.

The double sampling estimators of Singh (1965)for the ratio $R = \overline{Y}/\overline{X}_1$ and the product $P = \overline{Y}\,\overline{X}_1$ are respectively

$$\hat{R}_{1d} = \frac{\overline{y}}{\overline{x}_1}\cdot\frac{\overline{x}_2}{\overline{x}_2'} = \hat{R}\left(\frac{\overline{x}_2}{\overline{x}_2'}\right),$$

$$\hat{R}_{2d} = \frac{\overline{y}}{\overline{x}_1}\cdot\frac{\overline{x}_2'}{\overline{x}_2} = \hat{R}\left(\frac{\overline{x}_2'}{\overline{x}_2}\right),$$

and

$$\hat{P}_{1d} = \frac{\overline{y}}{\overline{x}_1}\cdot\frac{\overline{x}_2}{\overline{x}_2'} = \hat{P}\left(\frac{\overline{x}_2}{\overline{x}_2'}\right),$$

$$\hat{P}_{2d} = (\overline{y}\,\overline{x}_1)\frac{\overline{x}_2'}{\overline{x}_2} = \hat{P}\left(\frac{\overline{x}_2'}{\overline{x}_2}\right),$$

where $\hat{R} = \dfrac{\overline{y}}{\overline{x}_1}$ and $\hat{P} = \overline{y}\,\overline{x}_1$.

For estimating R and P, the proposed ganeralized double sampling estimators are respectively

$$\hat{R}_{gd} = g(\hat{R}, \bar{x}_2, \bar{x}_2') \tag{1.1}$$

and $$\hat{P}_{gd} = h(\hat{P}, \bar{x}_2, \bar{x}_2') \tag{1.2}$$

where $g(\hat{R}, \bar{x}_2, \bar{x}_2')$ and $h(\hat{P}, \bar{x}_2, \bar{x}_2')$ satisfying the validity conditions of Taylor's series expansion are bounded functions of $(\hat{R}, \bar{x}_2, \bar{x}_2')$ and $(\hat{P}, \bar{x}_2, \bar{x}_2')$ respectively such that at the points $Q = (R, \bar{X}_2, \bar{X}_2)$ and $T = (P, \bar{X}_2, \bar{X}_2)$

(i) $$g(R, \bar{X}_2, \bar{X}_2) = R \tag{1.3}$$

and $$h(P, \bar{X}_2, \bar{X}_2) = P \tag{1.4}$$

(ii) first order pertial derivative of $g(\hat{R}, \bar{x}_2, \bar{x}_2')$ with respect to $\hat{R}$ and first order partial derivative of $h(\hat{P}, \bar{x}_2, \bar{x}_2')$ with respect to $\hat{P}$ are unity at the points Q and T respectively, that is

$$g_0 = \left.\frac{\partial g(\hat{R}, \bar{x}_2, \bar{x}_2')}{\partial \hat{R}}\right]_{Q=(R, \bar{X}_2, \bar{X}_2)} = 1 \tag{1.5}$$

and

$$h_0 = \left.\frac{\partial h(\hat{P}, \bar{x}_2, \bar{x}_2')}{\partial \hat{P}}\right]_{T=(P, \bar{X}_2, \bar{X}_2)} = 1 \tag{1.6}$$

(iii) $$g_1 = -g_2 \tag{1.7}$$

and $$h_1 = -h_2 \tag{1.8}$$

for first order partial derivatives

$$g_1 = \left.\frac{\partial g(\hat{R}, \bar{x}_2, \bar{x}_2')}{\partial \bar{x}_2}\right]_Q, \qquad g_2 = \left.\frac{\partial g(\hat{R}, \bar{x}_2, \bar{x}_2')}{\partial \bar{x}_2'}\right]_Q$$

and $$h_1 = \left.\frac{\partial h(\hat{P}, \bar{x}_2, \bar{x}_2')}{\partial \bar{x}_2}\right]_T, \qquad h_2 = \left.\frac{\partial h(\hat{P}, \bar{x}_2, \bar{x}_2')}{\partial \bar{x}_2'}\right]_T,$$

(iv) second order partial derivatives

$$g_{00} = \left.\frac{\partial^2 g(\hat{R}, \bar{x}_2, \bar{x}_2')}{\partial \hat{R}^2}\right]_Q = 0 \tag{1.9}$$

and $$h_{00} = \left.\frac{\partial^2 h(\hat{P}, \bar{x}_2, \bar{x}_2')}{\partial \hat{P}^2}\right]_T = 0 \tag{1.10}$$

and (v) $$g_{01} = -g_{02} \tag{1.11}$$

and $$h_{01} = -h_{02} \tag{1.12}$$

for $$g_{01} = \left.\frac{\partial^2 g(\hat{R}, \bar{x}_2, \bar{x}_2')}{\partial \hat{R} \partial \bar{x}_2}\right]_Q, \quad g_{02} = \left.\frac{\partial^2 g(\hat{R}, \bar{x}_2, \bar{x}_2')}{\partial \hat{R} \partial \bar{x}_2'}\right]_Q$$

and $$h_{01} = \left.\frac{\partial^2 h(\hat{P}, \bar{x}_2, \bar{x}_2')}{\partial \hat{P} \partial \bar{x}_2}\right]_T, \quad h_{02} = \left.\frac{\partial^2 h(\hat{P}, \bar{x}_2, \bar{x}_2')}{\partial \hat{P} \partial \bar{x}_2'}\right]_T$$

To mention some particular members belonging the generalized double sampling estimator $\hat{R}_{gd}$, we have

(i) $$\hat{R}_{1d} = \hat{R}\frac{\bar{x}_2}{\bar{x}_2'}$$

(ii) $$\hat{R}_{2d} = \hat{R}\frac{\bar{x}_2'}{\bar{x}_2}$$

(iii) $$\hat{R}_{3d} = \hat{R} + k(\bar{x}_2 - \bar{x}_2')$$

(iv) $$\hat{R}_{4d} = \hat{R}\left(\frac{\bar{x}_2}{\bar{x}_2'}\right)^{k_1} + k_2(\bar{x}_2 - \bar{x}_2')$$

and $$\hat{R}_{5d} = \hat{R}\left(\frac{\bar{x}_2}{\bar{x}_2'}\right)^{k_1} + k_2\left(\bar{x}_2^{k_3} - \bar{x}_2'^{k_3}\right)$$

where k, k_1, k_2 and k_3 are the characterizing scalars to be chosen suitably.

The corresponding perticular members $\hat{P}_{i\,d}$ (i = 1,2,..., 5) of the generalized double sampling estimator $\hat{P}_{g\,d}$ may be obtained by just replacing $\hat{R}$ by $\hat{P}$ in the estimators $\hat{R}_{i\,d}$ (i = 1,2,...,5) from (i) to (v).

It may be easily verified that conditions (1.3), (1.5), (1.7), (1.9) and (1.11) are satisfied for all the estimators $\hat{R}_{i\,d}$, i = 1,2,.... 5 and the conditions (1.4), (1.6), (1.8), (1.10) and (1.12) are satisfied for the corresponding estimators $\hat{P}_{i\,d}$ (i = 1,2,..., 5) for P.

For example, considering the estimator $\hat{R}_{5\,d}$, we have

$$\left.\left\{\hat{R}\left(\frac{\bar{x}_2}{\bar{x}_2'}\right)^{k_1}+k_2\left(\bar{x}_2^{k_3}-\bar{x}_2'^{k_3}\right)\right\}\right]_Q = R \text{ satisfying (1.3),}$$

$$g_0=\left.\frac{\partial\hat{R}_{5\,d}}{\partial\hat{R}}\right]_Q$$

$$=\left.\left(\frac{\bar{x}_2}{\bar{x}_2'}\right)^{k_1}\right]_{Q=(R,\bar{X}_2,\bar{X}_2)}$$

$$= 1 \text{ satisfying the condition (1.5).}$$

$$g_1=\left.\frac{\partial\hat{R}_{5\,d}}{\partial\bar{x}_2}\right]_Q$$

$$=\left.\left\{\hat{R}k_1\left(\frac{\bar{x}_2}{\bar{x}_2'}\right)^{k_1-1}\frac{1}{\bar{x}_2'}+k_2k_3\,\bar{x}_2^{k_3-1}\right\}\right]_{Q=(R,\bar{X}_2,\bar{X}_2)}$$

$$=k_1\frac{R}{\bar{X}_2}+k_2k_3\bar{X}_2^{k_3-1}.$$

$$g_2 = \frac{\partial \hat{R}_{5d}}{\partial \bar{x}_2'}\Bigg]_Q$$

$$= \left\{ \hat{R} k_1 \left(\frac{\bar{x}_2}{\bar{x}_2'} \right)^{k_1 - 1} \left(-\frac{\bar{x}_2}{\bar{x}_2'^2} \right) - k_2 k_3 (\bar{x}_2')^{k_3 - 1} \right\} \Bigg]_Q$$

$$= -k_1 \frac{R}{\bar{x}_2} - k_2 k_3 (\bar{x}_2)^{k_3 - 1}$$

$$= -g_1 \text{ satisfying (1.7)},$$

$$g_{00} = \frac{\partial^2 \hat{R}_{5d}}{\partial \hat{R}^2}\Bigg]_Q$$

$$= 0 \text{ satisfying (1.9)}.$$

$$g_{01} = \frac{\partial^2 \hat{R}_{5d}}{\partial \hat{R} \partial \bar{x}_2}\Bigg]_Q$$

$$= k_1 \left(\frac{\bar{x}_2}{\bar{x}_2'} \right)^{k_1 - 1} \cdot \frac{1}{\bar{x}_2'}\Bigg]_Q$$

$$= \frac{k_1}{\bar{X}_2}$$

and

$$g_{02} = \frac{\partial \hat{R}_{5d}}{\partial \hat{R} \partial \bar{x}_2'}\Bigg]_Q$$

$$= k_1 \left(\frac{\bar{x}_2}{\bar{x}_2'} \right)^{k_1 - 1} \left(-\frac{\bar{x}_2}{\bar{x}_2'^2} \right)\Bigg]_Q$$

$$= -\frac{k_1}{\overline{X}_2}$$

$$= -g_{01}$$

satisfying the condition (1.11) for the estimator $\widehat{R}_{5d}$. Similarly, we may easily verify the regularity conditions to hold for the estimators $\widehat{R}_{1d}$, $\widehat{R}_{2d}$, $\widehat{R}_{3d}$ and $\widehat{R}_{4d}$ also, and some other double sampling estimators available in the literature (see for further details - Murthy, 1967, Cochran, 1977, Sukhatme et al. 1984).

2. BIAS AND MEAN SQUARE ERROR

Let

$$e_0 = \frac{\overline{y} - \overline{Y}}{\overline{Y}}, \qquad e_1 = \frac{\overline{x}_1 - \overline{X}_1}{\overline{X}_1}, \qquad e_1' = \frac{\overline{x}_1' - \overline{X}_1}{\overline{X}_1},$$

$$e_2 = \frac{\overline{x}_2 - \overline{X}_2}{\overline{X}_2}, \qquad e_2' = \frac{\overline{x}_2' - \overline{X}_2}{\overline{X}_2}$$

so

$$E(e_0) = E(e_1) = E(e_1') = E(e_2) = E(e_2') = 0$$

$$E(e_0^2) = \frac{f_1}{n} C_0^2, \qquad E(e_1^2) = \frac{f_1}{n} C_1^2,$$

$$E(e_1'^2) = \frac{f_1'}{n'} C_1^2, \qquad E(e_0 e_1) = \frac{f_1}{n} \rho_{01} C_0 C_1,$$

$$E(e_0 e_1') = \frac{f_1'}{n'} \rho_{01} C_0 C_1, \qquad E(e_1 e_1^1) = \frac{f_1'}{n'} C_1^2,$$

$$E(e_2^2) = \frac{f_1}{n} C_2^2, \qquad E(e_2'^2) = \frac{f_1'}{n'} C_2^2,$$

$$E(e_0 e_2) = \frac{f_1}{n} \rho_{02} C_0 C_2, \qquad E(e_0 e_2') = \frac{f_1'}{n'} \rho_{02} C_0 C_2,$$

$$E(e_1 e_2) = \frac{f_1}{n}\rho_{12} C_1 C_2, \qquad E(e_1 e_2') = \frac{f_1'}{n'}\rho_{12} C_1 C_2,$$

$$E(e_1' e_2) = \frac{f_1'}{n'}\rho_{12} C_1 C_2, \qquad E(e_1' e_2') = \frac{f_1'}{n'}\rho_{12} C_1 C_2,$$

and $E(e_2 e_2') = \frac{f_1'}{n'} C_2^2$

where $f_1 = \frac{N-n}{N}$ and $f_1' = \frac{N-n'}{N}$

Further, it is assumed that the sample is large enough to ignore terms involving e_0; e_1, e_1', e_2, e_2' of degree greater than two, to justify the first degree approximation (see Murthy, 1967).

Expanding $g(\hat{R}, \bar{x}_2, \bar{x}_2')$ about the point $Q = (R, \bar{X}_2, \bar{X}_2)$, in Taylor's series we have

$$\hat{R}_{gd} = g(R, \bar{X}_2, \bar{X}_2) + (\hat{R} - R)g_0 + (\bar{x}_2 - \bar{X}_2)g_1 + (\bar{x}_2' - \bar{X}_2)g_2$$

$$+ \frac{1}{2!}\Big\{(\hat{R} - R)^2 g_{00} + (\bar{x}_2 - \bar{X}_2)^2 g_{11} + (\bar{x}_2' - \bar{X}_2)^2 g_{22} + 2(\hat{R} - R)$$

$$(\bar{x}_2 - \bar{X}_2)g_{01} + 2(\hat{R} - R)(\bar{x}_2' - \bar{X}_2)g_{02} + 2(\bar{x}_2 - \bar{X}_2)(\bar{x}_2' - \bar{X}_2)g_{12}$$

$$+ \frac{1}{3!}\left\{(\hat{R} - R)\frac{\partial}{\partial \hat{R}} + (\bar{x}_2 - \bar{X}_2)\frac{\partial}{\partial \bar{x}_2} + (\bar{x}_2' - \bar{X}_2)\frac{\partial}{\partial \bar{x}_2'}\right\}^3 g(\hat{R}*, \bar{x}_2 * \bar{x}_2' *) \quad (2.1)$$

where g_0, g_1, g_2, g_{00}, g_{01} and g_{02} are already defined, second order partial derivatives g_{11}, g_{22} and g_{12} are given by

$$g_{11} = \left.\frac{\partial^2 g(\hat{R}, \bar{x}_2, \bar{x}_2')}{\partial \bar{x}_2^2}\right]_Q$$

$$g_{22} = \left.\frac{\partial^2 g(\hat{R}, \bar{x}_2, \bar{x}_2')}{\partial \bar{x}_2'^2}\right]_Q.$$

$$g_{12} = \left.\frac{\partial^2 g(\hat{R}, \bar{x}_2, \bar{x}_2')}{\partial \bar{x}_2 \partial \bar{x}_2^1}\right]_Q$$

and $R* = R + \theta(\hat{R} - R), \quad \bar{x}_2* = \bar{X}_2 + \theta(\bar{x}_2 - \bar{X}_2),$

$$\bar{x}_2' = \bar{X}_2 + \theta(\bar{x}_2' - \bar{X}_2), \quad \text{for } 0 < \theta < 1$$

From regularity conditions, employing $g(R, \bar{x}_2, \bar{x}_2) = R$, $g_0 = 1$, $g_{00} = 0$, $g_1 = -g_2$ and $g_{01} = -g_{02}$ in (2.1), we have

$$\hat{R}_{gd} - R = (\hat{R} - R) + (\bar{x}_2 - \bar{X}_2)g_1 - (\bar{x}_2' - \bar{X}_2)g_1$$

$$+ \frac{1}{2}\Big\{(\bar{x}_2 - \bar{X}_2)^2 g_{11} + (\bar{x}_2' - \bar{X}_2)^2 g_{22} + 2(\hat{R} - R)(\bar{x}_2 - \bar{X}_2)g_{01}$$

$$- 2(\hat{R} - R)(\bar{x}_2' - \bar{X}_2)g_{01} + 2(\bar{x}_2 - \bar{X}_2)(\bar{x}_2' - \bar{X}_2)g_{12}$$

$$+ \frac{1}{3!}\left\{(\hat{R} - R)\frac{\partial}{\partial \hat{R}} + (\bar{x}_2 - \bar{X}_2)\frac{\partial}{\partial \bar{x}_2} + (\bar{x}_2' - \bar{X}_2)\frac{\partial}{\partial \bar{x}_2'}\right\}^3$$

$$g(\hat{R}*, \bar{x}_2 * \bar{x}_2' *).$$

$$= \Big\{R(1+e_0)(1+e_1)^{-1} - R\Big\} + \Big(\bar{X}_2 e_2 - \bar{X}_2 e_2'\Big)g_1 + \frac{1}{2}\Big[\bar{X}_2^2 e_2^2 g_{11}$$

$$+ \bar{X}_2^2 e_2'^2 g_{22} + 2\Big\{R(1+e_0)(1+e_1)^{-1} - R\Big\}\bar{X}_2 e_2 g_{01}$$

$$- 2\Big\{R(1+e_0)(1+e_1)^{-1} - R\Big\}\bar{X}_2 e_2' g_{01} + 2\bar{X}_2^2 e_2 e_2' g_{12}\Big]$$

$$+ \frac{1}{3!}\left[\Big\{R(1+e_0)(1+e_1)^{-1} - R\Big\}\frac{\partial}{\partial \hat{R}} + \bar{X}_2 e_2 \frac{\partial}{\partial \bar{x}_2} + \bar{X}_2 e_2' \frac{\partial}{\partial \bar{x}_2'}\right]^3$$

$$g(\hat{R}*, \bar{x}_2 * \bar{x}_2' *) \tag{2.2}$$

Taking expectation on both sides of (2.2), to the first degree of approximation

$$E(\hat{R}_{gd})-R=E\Big[R(e_0-e_1+e_1^2-e_0e_1)+\overline{X}_2(e_2-e_2')g_1$$
$$+\frac{1}{2}\Big\{\overline{X}_2^2e_2^2g_{11}+\overline{X}_2^2e_2'^2g_{22}+2R(e_0-e_1+e_1^2-e_0e_1)\overline{X}_2(e_2-e_2')g_{01}$$
$$+2\overline{X}_2^2e_2e_2'g_{12}\Big\}\Big]$$

or

$$\text{Bias}\quad(\hat{R}_{gd})=(f_1/n)R(C_1^2-\rho_{01}C_0C_1)+\frac{1}{2}\Big[(f_1/n)\overline{X}_2^2C_2^2g_{11}+(f_1'/n')\overline{X}_2^2C_2^2$$
$$(g_{22}+2g_{12})+2\left(\frac{1}{n}-\frac{1}{n'}\right)R\overline{X}_2(\rho_{02}C_0C_2-\rho_{12}C_1C_2)g_{01}\Big] \qquad (2.3)$$

Squaring both sides of (2.2) and taking expectation, $MSE(\hat{R}_{gd})=E(\hat{R}_{gd}-R)^2$ to the first degree of approximation, is

$$MSE(\hat{R}_{gd})=E\Big[R^2(e_0-e_1)^2+\overline{X}_2^2(e_2-e_2')^2g_1^2+2R\overline{X}_2(e_0-e_1)(e_2-e_2')g_1\Big]$$
$$\doteq(f_1/n)R^2(C_0^2-2\rho_{01}C_0C_1+C_1^2)+\left(\frac{1}{n}-\frac{1}{n'}\right)\overline{X}_2^2C_2^2g_1^2$$
$$+2\left(\frac{1}{n}-\frac{1}{n'}\right)R\overline{X}_2(\rho_{02}C_0C_2-\rho_{12}C_1C_2)g_1$$
$$=MSE(\hat{R})+\left(\frac{1}{n}-\frac{1}{n'}\right)\overline{X}_2^2C_2^2\left[g_1^2+2\left(\frac{R}{\overline{X}_2}\right)\left\{\rho_{02}\left(\frac{C_0}{C_2}\right)-\rho_{12}\left(\frac{C_1}{C_2}\right)\right\}g_1\right]$$
$$=MSE(\hat{R})+\left(\frac{1}{n}-\frac{1}{n'}\right)\overline{X}_2^2C_2^2\left[g_1^2+2\left(\frac{R}{\overline{X}_2}\right)Cg_1\right] \qquad (2.4)$$

where $C=\rho_{02}\left(\frac{C_0}{C_2}\right)-\rho_{12}\left(\frac{C_1}{C_2}\right)$.

Proceeding on the same lines as for $\hat{R}_{gd}$, the bias and mean square error of $\hat{P}_{gd}$ to the first degree of approximation, are

$$\text{Bias}\ (\hat{P}_{gd}) = \left(\frac{f_1}{n}\right)P(C_1^2 + \rho_{01}C_0C_1) + \frac{1}{2}\left[\left(\frac{f_1}{n}\right)\bar{X}_2^2C_2^2h_{11} + \left(\frac{f_1'}{n'}\right)\bar{X}_2^2C_2^2\right.$$

$$\left.(h_{22} + 2h_{12}) + 2\left(\frac{1}{n} - \frac{1}{n'}\right)P\bar{X}_2(\rho_{02}C_0C_2 + \rho_{12}C_1C_2)h_{01}\right] \qquad (2.5)$$

and $$\text{MSE}(\hat{P}_{gd}) = \text{MSE}(\hat{P}) + \left(\frac{1}{n} - \frac{1}{n'}\right)\bar{X}_2^2C_2^2\left[h_1^2 + 2\left(\frac{P}{\bar{X}_2}\right)C^*h_1\right] \qquad (2.6)$$

where $C^* = \rho_{02}\left(\frac{C_0}{C_2}\right) + \rho_{12}\left(\frac{C_1}{C_2}\right)$.

3. OPTIMUM AND ESTIMATED OPTIMUM VALUE

From (2.4), the optimum value of g_1 minimising $\text{MSE}(\hat{R}_{gd})$ is

$$g_1^* = -\left(\frac{R}{\bar{X}_2}\right)C \qquad (3.1)$$

and the minimum mean square error is given by

$$\text{MSE}(\hat{R}_{gd})_{min} = \text{MSE}(\hat{R}) - \left(\frac{1}{n} - \frac{1}{n'}\right)R^2C_2^2C^2 \qquad (3.2)$$

The optimum value $g_1^* = -\left(\frac{R}{\bar{X}_2}\right)C$ in (3.1) may not be known always in practice, hence the alternative is to replace the parameters involved in the optimum value by their unbiased or consistent estimators and thus get the estimated optimum value depending upon sample observations. We can write

$$-\left(\frac{R}{\bar{X}_2}\right)C = -\left(\frac{R}{\bar{X}_2}\right)\left\{\rho_{02}\left(\frac{C_0}{C_2}\right) - \rho_{12}\left(\frac{C_1}{C_2}\right)\right\}$$

$$= -\frac{R}{\overline{X}_2 C_2^2}(\rho_{02} C_0 C_2 - \rho_{12} C_1 C_2)$$

$$= -\frac{R}{S_{x_2}^2}\left(\frac{S_{yx_2}}{\overline{Y}} - \frac{S_{x_1 x_2}}{\overline{X}_1}\right). \tag{3.3}$$

For (y_i, x_{1i}, x_{2i}) being the i^{th} $(i = 1,2,\ldots, n)$ second phase sample value on (y, x_1, x_2) respectively, replacing $R, \overline{Y}, \overline{X}_1, S_{x_2}^2$,

$$S_{yx_2} = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_i - \overline{Y})(X_{2i} - \overline{X}_2) \quad \text{and}$$

$$S_{x_1 x_2} = \frac{1}{(N-1)}\sum_{i=1}^{N}(X_{1i} - \overline{X}_1)(X_{2i} - \overline{X}_2)$$

in (3.3) by their estimators $\hat{R}, \overline{y}, \overline{x}_1$, $s_{x_2}^2 = \frac{1}{n-1}\sum_{i=1}^{N}(x_{2i} - \overline{x}_2)^2$,

$$S_{yx_2} = \frac{1}{(n-1)}\sum_{i=1}^{n}(y_i - \overline{y})(x_{2i} - \overline{x}_2) \quad \text{and}$$

$$S_{x_1 x_2} = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_{1i} - \overline{x}_1)(x_{2i} - \overline{x}_2)$$

respectively, we get the estimated optimum value of $g_1{*} = -\frac{R}{\overline{X}_2}C$ as

$$\hat{g}_1 = -\frac{\hat{R}}{s_{x_2}^2}\left(\frac{s_{yx_2}}{\overline{y}} - \frac{s_{x_1 x_2}}{\overline{x}_1}\right)$$

$$= -\left(\frac{\hat{R}}{\overline{x}_2^1}\right)\hat{C} \tag{3.4}$$

We now investigate a class of double smpling estimators depending upon estimated optimum value $\hat{g}_1$ and attaining the minimum mean square error given by (3.2).

To attain (3.2), we need an estimator or equivalently the function $g(\hat{R}, \bar{x}_2, \bar{x}_2')$ satisfying (1.3), (1.5), (1.7), (1.9), (1.11) and

$$\left.\frac{\partial g(\hat{R}, \bar{x}_2, \bar{x}_2')}{\partial \bar{x}_2}\right]_Q = -\left(\frac{R}{\bar{X}_2}\right)C$$

from (3.1), making the requirement for the function g to involve not only $(\hat{R}, \bar{x}_2, \bar{x}_2')$ but R, $\bar{X}_2$ and C as well, and thus we want a function $g^*(\hat{R}, \bar{x}_2, \bar{x}_2', \bar{X}_2, R, C)$ such that

$$g^*(\hat{R}, \bar{x}_2, \bar{x}_2', \bar{X}_2, R, C)\Big]_{(R,\bar{X}_2,\bar{X}_2)} = R.$$

$$\left.\frac{\partial g^*}{\partial \hat{R}}\right]_{(R,\bar{X}_2,\bar{X}_2)} = 1,$$

$$g_1^* = \left.\frac{\partial g^*}{\partial \bar{x}_2}\right]_{(R,\bar{X}_2,\bar{X}_2)} = -\left.\frac{\partial g^*}{\partial \bar{x}_2'}\right]_{(R,\bar{X}_2,\bar{X}_2)} = -g_2^*,$$

$$\left.\frac{\partial^2 g^*}{\partial \hat{R}^2}\right]_{(R,\bar{X}_2,\bar{X}_2)} = 0,$$

$$g_{01}^* = \left.\frac{\partial^2 g^*}{\partial \hat{R}\partial \bar{x}_2}\right]_{(R,\bar{X}_2,\bar{X}_2)} = -\left.\frac{\partial^2 g^*}{\partial \hat{R}\partial \bar{x}_2'}\right]_{(R,\bar{X}_2,\bar{X}_2)} = -g_{02}^*,$$

$$\left.\frac{\partial g^*}{\partial \bar{x}}\right]_{(R,\bar{X}_2,\bar{X}_2)} = -\frac{R}{\bar{X}_2}C.$$

Since the function $g^*(\hat{R}, \bar{x}_2, \bar{x}_2', \bar{X}_2, R, C)$ so found, $\bar{X}_2$, R and C are unknown, we may take $g^{**}(\hat{R}, \bar{x}_2, \bar{x}_2', \hat{C}) = g^*(\hat{R}, \bar{x}_2, \bar{x}_2', \bar{x}_2', \hat{R}, \hat{C})$ as our estimator of R such that

$$g^{**}(\hat{R},\bar{x}_2,\bar{x}_2',\hat{C})\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=R, \qquad \frac{\partial g^{**}}{\partial \hat{R}}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=1,$$

$$g_1^{**}=\frac{\partial g^{**}}{\partial \bar{x}_2}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=-\frac{\partial g^{**}}{\partial \bar{x}_2'}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=-g_2^{**},$$

$$\frac{\partial^2 g^{**}}{\partial \hat{R}^2}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=0,$$

$$g_{01}^{**}=-\frac{\partial^2 g^{**}}{\partial \hat{R}\partial \bar{x}_2}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=-\frac{\partial^2 g^{**}}{\partial \hat{R}\partial \bar{x}_2'}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=-g_{02}^{**},$$

$$\frac{\partial g^{**}}{\partial \bar{x}}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=-\frac{R}{\bar{X}_2}C,$$

Expanding $g^{**}(\hat{R},\bar{x}_2,\bar{x}_2',\hat{C})$ about $W=(R,\bar{X}_2,\bar{X}_2,C)$ in Taylor's series, we have

$$g^{**}(\hat{R},\bar{x}_2,\bar{x}_2',\hat{C})=g^{**}(R,\bar{X}_2,\bar{X}_2,C)+(\hat{R}-R)\frac{\partial g^{**}}{\partial \hat{R}}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}$$

$$+(\bar{x}_2-\bar{X}_2)g_1^{**}+(\bar{x}_2-\bar{X}_2)g_2^{**}+(\hat{C}-C)\frac{\partial g^{**}}{\partial \hat{C}}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}+\dots$$

$$=R+(\hat{R}-R)+(\bar{x}_2-\bar{X}_2)g_1^{**}+(\bar{x}_2'-\bar{X}_2)-g_2^{**}+(\hat{C}-C)\frac{\partial g^{**}}{\partial \hat{C}}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}+\dots$$

or
$$g^{**}(\hat{R},\bar{x}_2,\bar{x}_2,\hat{C})-R=(\hat{R}-R)+(\bar{x}_2-\bar{X}_2)g_1^{**}+(\bar{x}_2'-\bar{X}_2)g_2^{**}$$
$$+(\hat{C}-C)g_3^{**}+\dots \qquad (3.5)$$

where $g_3^{**}=\frac{\partial g^{**}}{\partial \hat{C}}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}$.

Squaring both sides of (3.5) and taking expectation, we see that the mean square error $E\left[g^{**}(\hat{R},\bar{x}_2,\bar{x}_2',\hat{C})-R\right]^2$ to the first degree of approximation becomes equal to $MSE(\hat{R}_{gd})_{min}$ given by (3.2) if $g_3^{**}=0$, thus the estimator taken as a function $R_{d(est)}^{*}=g^{**}(\hat{R},\bar{x}_2,\bar{x}_2',\hat{C})$ depending upon estimated optimum value such that

$$\left.\begin{aligned}
&g^{**}(\hat{R},\bar{x}_2,\bar{x}_2',\hat{C})\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=R,\quad \frac{\partial g^{**}}{\partial \hat{R}}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=1,\\
&g_1^{**}=\frac{\partial g^{**}}{\partial \bar{x}_2}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=-\frac{\partial g^{**}}{\partial \bar{x}_2'}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=-g_2^{**}\\
&\frac{\partial^2 g^{**}}{\partial \hat{R}^2}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=0,\\
&g_{01}^{**}=-\frac{\partial^2 g^{**}}{\partial \hat{R}\,\partial \bar{x}_2}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=-\frac{\partial^2 g^{**}}{\partial \hat{R}\,\bar{x}_2'}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=-g_{02}^{**}\\
&\frac{\partial g^{**}}{\partial \bar{x}}\Big]_{(R,\bar{X}_2,\bar{X}_2,C)}=-\left(\frac{R}{\bar{X}_2}\right)C\\
&\text{and}\quad g_3^{**}=0
\end{aligned}\right\} \tag{3.6}$$

attains the minimum mean square erro given by (3.2).

Proceedings on same lines, we find the optimum value of h_1 minimizing $MSE(\hat{P}_{gd})$ as

$$h_1^{*}=-(P/\bar{X}_2)\,C^{*} \tag{3.7}$$

for which the minimum mean square error to the first degree of approximation, is

$$MSE(\hat{P}_{gd})_{min}=MSE'(\hat{P})-\left(\frac{1}{n}-\frac{1}{n'}\right)P^2C_2^2C^{*2} \tag{3.8}$$

and replacing the parameters involved in $h_1^* = -\frac{P}{\bar{X}_2} C^*$ by their unbiased or consistent estimators, we get the estimated optimum value

$$\hat{h}_1 = -(\hat{P} / \bar{x}_2')\hat{C}^* \tag{3.9}$$

We can now obtain the estimator $\hat{P}_{d(est)}^*$ depending on estimated optimum value on the same lines as for $\hat{R}_{d(est)}^*$ and can check that, to that first degree of approximation MSE ($\hat{P}_{d(est)}^*$) equals the minimum mean square error given by (3.8).

Some particular estimators depending on estimated optimum value, satisfying the conditions in (3.6) and attaining the minimum mean square error given by (3.2), are given in the following section 4.

4. CONCLUDING REMARKS

(a) Results of various estimators in the literature may be easily seen to the special cases of this study. For example, mean square error, to the first degree of approximation, for the double sampling estimators $\hat{R}_{1d}$ and $\hat{R}_{2d}$ of Singh (1965) are

$$\text{MSE}(\hat{R}_{1d}) = \text{MSE}(\hat{R}) + \left(\frac{1}{n} - \frac{1}{n'}\right) R^2 C_2^2 (1 + 2C) \tag{4.1}$$

and

$$\text{MSE}(\hat{R}_{2d}) = \text{MSE}(\hat{R}) + \left(\frac{1}{n} - \frac{1}{n'}\right) R^2 C_2^2 (1 - 2C) \tag{4.2}$$

respectively. We see from $\hat{R}_{1d} = \hat{R}\left(\frac{\bar{x}_2}{\bar{x}_2'}\right)$ and $\hat{R}_{2d} = \hat{R}\left(\frac{\bar{x}_2'}{\bar{x}_2}\right)$ that the value of g_1 for $\hat{R}_{1d}$ and $\hat{R}_{2d}$ are $\left(\frac{R}{\bar{X}_2}\right)$ and $-\left(\frac{R}{\bar{X}_2}\right)$ respectively so that by sustituting $g_1 = \frac{R}{\bar{X}_2}$ and

$g_1 = -\frac{R}{\overline{X}_2}$ in (2.4)dealing with the general expressions of mean square error of the generalized double sampling estimator $\hat{R}_{gd}$, we get the same experssions as in (4.1) and (4.2) of $MSE(\hat{R}_{1d})$ and $MSE(\hat{R}_{2d})$ respectively. Similarly, results of other estimators satisfying the regularity conditions of $\hat{R}_{gd}$ or $\hat{P}_{gd}$, may be easily shown to be special cases of those of $\hat{R}_{gd}$ or $\hat{P}_{gd}$.

(b) From (3.1) and (3.2), the optimum choice of the function $g(\hat{R}, \overline{x}_2, \overline{x}_2')$ in $\hat{R}_{gd}$ for which the mean square error of $\hat{R}_{gd}$ to the first degree approximation is minimized, is the function $g(\hat{R}, \overline{x}_2, \overline{x}_2')$ such that $g_1 = -\frac{R}{\overline{X}_2}C$ giving the minimum mean square error

$$MSE(\hat{R}_{gd})_{min} = MSE(\hat{R}) - \left(\frac{1}{n} - \frac{1}{n'}\right) R^2 C_2^2 C^2 \tag{4.3}$$

Thus, the estimators

$$\hat{R}\left(\frac{\overline{x}_2}{\overline{x}_2'}\right)^k, \quad \hat{R} + k(\overline{x}_2' - \overline{x}_2) \text{ and } \hat{R}\left\{1 + \frac{k(\overline{x}_2' - \overline{x}_2)}{x_2'}\right\}$$

belonging to the class $\hat{R}_{gd}$ of estimators and having the values of g_1 equal to

$$k(R/\overline{X}_2), \ -k \text{ and } -k(R/\overline{X}_2) \tag{4.4}$$

respectively, will attain the minimum mean square error given by (4.3) for the optimum values of k equal to -C, $(R/\overline{X}_2)C$ and C obtained respectively by equating each of (4.4) to the optimum value - $(R/\overline{X}_2)C$ of g_1, that is, the mean square error of the estimators

$$\hat{R}\left(\frac{\overline{x}_2}{\overline{x}_2'}\right)^{-C}, \quad \hat{R} + \left(\frac{R}{\overline{x}_2}\right) C(\overline{x}_2' - \overline{x}_2) \text{ and } \hat{R}\left\{1 + C\frac{(\overline{x}_2' - \overline{x}_2)}{\overline{x}_2'}\right\}$$

to the first degree of approximation will be equal to that of (4.3). But C or $(R/\overline{X}_2)C$ may be rarely known, hence replacing C or $(R/\overline{X}_2)C$ by consistent estimates from sample values, we get the estimators depending upon estimated optimum values to be

(i) $\hat{R}\left(\frac{\overline{x}_2}{\overline{x}_2'}\right)^{-\hat{C}}$ (ii) $\hat{R}+\frac{\hat{R}\hat{C}}{\overline{x}_2'}(\overline{x}_2'-\overline{x}_2)$ and

(iii) $\hat{R}\left\{1+\hat{C}\frac{(\overline{x}_2'-\overline{x}_2)}{\overline{x}_2'}\right\}$

which belong to the class $\hat{R}_{d(est)}$ * and satisfy the conditions in (3.6), and also attain the minimum mean square error given by (3.2) to the first degree of approximation. The general result regarding $\hat{R}_{d(est)}$ * which attains the minimum mean square error (to the first degree of approximation) given in (3.2) is already given to section 3.

(c) Similar remarks follow for $\hat{P}_{gd}$ and $\hat{P}_{d(est)}$ * also.

(d) Single sampling results may be easily found as the special cases of this study for n' = N

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