Journal of the Tensor Society
Volume 1, 2007
https://doi.org/10.56424/jts.v1i01.9948

ON P3-LIKE FINSLER SPACE

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ABSTRACT : There are three kinds of torsion tensor in Cartan's theory of Finsler spaces. Two of them are (h) hv-torsion tensor C_{ijk} and (v) hv-torsion tensor P_{ijk}, which are symmetric in all their indices and both are indicatory tensors. Mathematicians have studied various interesting special forms of these torsion tensors. For example C-reducible [2], semi C-reducible [6], C2-like [6] and C3-like [9] Finsler spaces are based on the special forms of C_{ijk} where as P-reducible [1] [5] and P-symmetric [5] Finsler spaces are based on the special forms of P_{ijk}. In the present paper we shall discuss another special form of P_{ijk}.

1. INTRODUCTION

DEFINITION : A Finsler space F^n $(n > 3)$ is called P3-like, if there exist four scalars A, B, C, D and two linearly independent vector fields X_i, Y_i which are positively homogeneous of degree 0 in "y^i" (briefly, (0)p-homogeneous) and satisfy

$$P_{ijk} = A X_i X_j X_k + B Y_i Y_j Y_k + C \pi_{(ijk)} \{X_i X_j Y_k\} + D \pi_{(ijk)} \{X_i Y_j Y_k\} \tag{1.1}$$

where $\pi_{(ijk)}\{\dots\}$ denotes the cyclic interchange of i, j, k and summation. For instance

$$\pi_{(ijk)}\{A_i B_j C_k\} = A_i B_j C_k + A_j B_k C_i + A_k B_i C_j$$

and there exist no scalar γ and vector field Z_i satisfying $P_{ijk} = \gamma Z_i Z_j Z_k$.

Since P_{ijk} of every three dimensional Finsler space may be written in the form of (1.1) [see theorem (2.1)], that is why we say F^n ($n > 3$) as P3-like Finsler space whose P_{ijk} is of the form (1.1).

2. P3-LIKE FINSLER SPACE

We are concerned with P3-like Finsler spaces. First we prove

PROPOSITION 2.1 : The matrix $\begin{bmatrix} A & B \\ B & C \\ C & D \end{bmatrix}$ is of rank two.

PROOF : If the rank is one then there exist scalar λ such that $B = \lambda A$, $C = \lambda B$, $D = \lambda C$. By putting $U_i = X_i + \lambda Y_i$, (1.1) reduces to $P_{ijk} = A U_i U_j U_k$ which is contrary to a condition for the P3-like property.

We again consider the equation (1.1). Indicatrization of (1.1) leads us to

$$P_{ijk} h^i_a h^j_b h^k_c = A X_a X_b X_c + B Y_a Y_b Y_c + C \pi_{(abc)} \{X_a X_b Y_c\}$$
$$+ D \pi_{(abc)} \{X_a Y_b Y_c\} \tag{1.1)'$$

where $h^i_a = g^{ib} h_{ab}$, $h_{ab} = g_{ab} - l_a l_b$ and we put $X_a = X_i h^i_a$, $Y_a = Y_i h^i_a$. Then X_a and Y_a are linearly independent vectors such that $X_0 = Y_0 = 0$, where and in the following the subscript 0 indicates contraction by y^i. In fact if $Y_a = \mu X_a$ for some scalar μ then (1.1)′ is written in the form $P_{ijk} = (A + B\mu^3 + 3\mu C + 3D\mu^3) X_a X_b X_c$ which contradicts the definition of P3-like space. We thus may assume that X_i and Y_i are orthogonal to the supporting element y^i.

It is obvious that X_i and Y_i may be taken as unit vectors, by suitable modification of scalars A, B, C, D. Further it may be supposed that X_i is orhtogonal to Y_i. If Y_i is not arthogonal to X_i, then we take the vector

$$W_i = Y_i - (X,Y)X_i, \quad (X,Y) = g^{ij} X_i Y_j \neq 0$$

It is clear that this is orhtogonal to X_i and P_{ijk} is written in a similar form (1.1) in terms of X_i and W_i. Consequently this leads to

PROPOSITION 2.2 : In the equation (1.1) it may be assumed that

$$X_i y^i = Y_i y^i = 0, \quad g_{ij} X^i X^i = g_{ij} Y^i Y^i = 1, \quad g_{ij} X^i Y^i = 0.$$

Throughout the following we shall suppose that in the definition (1.1) of P3-like Finsler space X_i and Y_i satisfy the assumptions in proposition (2.2).

From $X_i y^i = Y_i y^i = 0$, taking v- covariant derivative of this equation with respect to $C\Gamma$ gives

$$X_i\big|_j y^i + X_j = 0, \quad Y_i\big|_j y^i + Y_j = 0. \tag{2.1}$$

If the vector fields X_i and Y_i are v-covariantly constant then from (2.1) it follows that $X_i = Y_i = 0$, which is a contradiction. Hence we have the following

PROPOSITION 2.3 : The vector field X_i and Y_i in P3-like Finsler space is not v-covariantly constant.

The concurrent vector field has been defined by Matsumoto [4] and Tachibana [12]. The semi parallel vector field, the semi parallel h-vector field and the concircular vector field have been defined by Singh and Prasad [10], Singh and Kumari [11] and Prasad et all [8] respectively. In all these works the vector filds are v- covariantly constant. Hence we may state that

PROPOSITION 2.4 : The vector fields X_i and Y_i occuring in P3-like Finsler space is neither of any of the following:

(a) a concurrent vector field, (b) a semi parallel vector field,

(c) a semi parallel h-vector field, (d) a concircular vector field.

Next we consider the three-dimensional Finsler space F^3 in which the Moor's frame $(l_i, m_i n_i)$ play an important role. With respect to this frame, the metric tensor and (h) hv-torsion tensor are respectively given by $g_{ij} = l_i l_j + m_i m_j + n_i n_j$ and

$$LC_{ijk} = H m_i m_j m_k - J\pi_{(ijk)}\{m_i m_j n_k\} + I\pi_{(ijk)}\{m_i n_j n_k\} + J n_i n_j n_k. \qquad (2.2)$$

where H, I, J are main scalars of three-dimensional Finsler space F^3 such that H + I = L C [3]. Taking h-covariant derivative of (2.2) with respect to x^h and using $m_{i|j} = n_i h_j$, $n_{i|j} = -m_i h_j$ [3] then contracting the resulting equation with y^h, we have

$$P_{ijk} = C_{ijk|0} = (H_{,1} + 3Jh_1) m_i m_j m_k + (J_{,1} + 3Ih_1) n_i n_j n_k$$
$$+ \{-J_{,1} + (H - 2I)h_1\}\pi_{(ijk)}\{m_i m_j n_k\} + (I_{,1} - 3Jh_1)\pi_{(ijk)}\{m_i n_j n_k\}, \qquad (2.3)$$

where $H_{,1} = L^{-1} H_{|0}$ and $h_1 = h_i l^i$. Equation (2.3) is of the same form as (1.1) with $A = H_{,1} + 3Jh_1, B = J_{,1} + 3Ih_1, C = -J_{,1} + (H - 2I)h_1, D = I_{,1} - 3Jh_1$ and $X_i = m_i, Y_i = n_i$.

THEOREM 2.1 : Every three dimensional Finsler space is P3-like.

3. FOUR-DIMENSIONAL P3-LIKE FINSLER SPACE

Consider the four-dimensional Finsler space F^4 in which the Moor's frame (l_i, m_i, n_i, p_i) plays the important role. For this frame, the metric tensor and (h) hv-torsion tensor are respectively given by [7] $g_{ij} = l_i l_j + m_i m_j + n_i n_j + p_i p_j$ and

$$LC_{ijk} = Hm_i m_j m_k + Jn_i n_j n_k + H'p_i p_j p_k + I\pi_{(ijk)}\{m_i n_j n_k\} + K\pi_{(ijk)}\{m_i p_j p_k\}$$
$$-(J+J')\pi_{(ijk)}\{m_i m_j n_k\} + I'\pi_{(ijk)}\{n_i n_j p_k\} + J'\pi_{(ijk)}\{n_i p_j p_k\}$$
$$-(H'+I')\pi_{(ijk)}\{m_i m_j p_k\} + K'\pi_{(ijk)}\{m_i(n_j p_k + n_k p_j)\}, \quad (3.1)$$

where H, I, J, K, H', I', J', K' are main scalars of four-dimensional Finsler space F^4 such that H + I + K = L C. Differentiating (3.1) h-covariantly with respect to x^h, using [7] $m_{i|j} = n_i h_j + p_i j_j$, $n_{i|j} = -m_i h_j + p_i k_j$, $p_{i|j} = -m_i j_j - n_i k_j$, then contracting with y^h and putting $h_0 = h_i y^i = Lh_1$, $H_{,0} = H_{,i} y^i = LH_{,1}$ etc, we have

$$P_{ijk} = C_{ijk|0} = [H_{,1} + 3(J+J')h_1 + 3(H'+I')j_1]m_i m_j m_k$$
$$+[J_{,1} + 3Ih_1 - 3I'k_1]n_i n_j n_k + [H'_{,1} + 3Kj_1 + 3J'k_1]p_i p_j p_k$$
$$+[-(J+J')_{,1} + (H-2I)h_1 - 2K'j_1 + (H'+I')k_1]\pi_{(ijk)}\{m_i m_j n_k\}$$
$$+[I_{,1} - (3J+2J')h_1 - I'j_1 - 2K'k_1]\pi_{(ijk)}\{m_i n_j n_k\}$$
$$+[K_{,1} - J'h_1 - (3H'+2I')j_1 + 2K'k_1]\pi_{(ijk)}\{m_i p_j p_k\}$$
$$+[J'_{,1} + Kh_1 + 2K'j_1 - (H'-2I')k_1]\pi_{(ijk)}\{n_i p_j p_k\}$$
$$+[I'_{,1} + 2K'h_1 + Ij_1 + (J-2J')k_1]\pi_{(ijk)}\{n_i n_j p_k\}$$
$$+[-(H'+I')_{,1} - 2K'h_1 + (H-2K)j_1 - (J+J')k_1]\pi_{(ijk)}\{m_i m_j p_k\}$$
$$+[K'_{,1} - (H'+2I')h_1 - (J+2J')j_1$$
$$+(I-K)k_1]\pi_{(ijk)}\{m_i n_j p_k + m_i p_j n_k\} \quad (3.2)$$

Let X_α and Y_α $(\alpha = 1, 2, 3, 4)$ are scalar components of X_i and Y_i respectively with respect to Moor's frame (l_i, m_i, n_i, p_i) i.e.

$$X_i = X_1 l_i + X_2 m_i + X_3 n_i + X_4 p_i, \quad Y_i = Y_1 l_i + Y_2 m_i + Y_3 n_i + Y_4 p_i. \quad (3.3)$$

Then $X_i y^i = Y_i y^i = 0$ implies that $X_1 = Y_1 = 0$ and $g^{ij} X_i X_j = g^{ij} Y_i Y_j = 1$, $g^{ij} X_i Y_j = 0$ implies that

$$X_2^2 + X_3^2 + X_4^2 = 1, \quad Y_2^2 + Y_3^2 + Y_4^2 = 1,$$
$$X_2 Y_2 + X_3 Y_3 + X_4 Y_4 = 0. \tag{3.4}$$

If F^4 is P3-like Finsler space, then from (1.1) and (3.3), we get

$$\begin{aligned}
P_{ijk} = & [AX_2^3 + BY_2^3 + 3CX_2^2 Y_2 + 3DX_2 Y_2^2] m_i m_j m_k \\
& + [AX_3^3 + BY_3^3 + 3CX_3^2 Y_3 + 3DX_3 Y_3^2] n_i n_j n_k \\
& + [AX_4^3 + BY_4^3 + 3CX_4^2 Y_4 + 3DX_4 Y_4^2] p_i p_j p_k \\
& + [AX_2^2 X_3 + BY_2^2 Y_3 + C(X_2^2 Y_3 + 2X_2 X_3 Y_2) \\
& \qquad + D(Y_2^2 X_3 + 2X_2 Y_2 Y_3)] \pi_{(ijk)} \{m_i m_j n_k\} \\
& + [AX_2 X_3^2 + BY_2 Y_3^2 + C(Y_2 X_3^2 + 2X_2 X_3 Y_2) \\
& \qquad + D(X_2 Y_3^2 + 2Y_2 X_3 Y_3)] \pi_{(ijk)} \{m_i n_j n_k\} \\
& + [AX_2 X_4^2 + BY_2 Y_4^2 + C(Y_2 X_4^2 + 2X_2 X_4 Y_4) \\
& \qquad + D(X_2 Y_4^2 + 2Y_2 X_4 Y_4)] \pi_{(ijk)} \{m_i p_j p_k\} \\
& + [AX_3 X_4^2 + BY_3 Y_4^2 + C(Y_3 X_4^2 + 2X_3 X_4 Y_4) \\
& \qquad + D(X_3 Y_4^2 + 2Y_3 X_4 Y_4)] \pi_{(ijk)} \{n_i p_j p_k\} \\
& + [AX_3^2 X_4 + BY_3^2 Y_4 + C(X_3^2 Y_4 + 2Y_3 X_3 X_4) \\
& \qquad + D(Y_3^2 X_4 + 2Y_3 X_3 Y_4)] \pi_{(ijk)} \{n_i n_j p_k\} \\
& + [AX_2^2 X_4 + BY_2^2 Y_4 + C(X_2^2 Y_4 + 2Y_2 X_2 Y_4) \\
& \qquad + D(Y_2^2 X_4 + 2Y_2 X_2 Y_4)] \pi_{(ijk)} \{m_i m_j p_k\} \\
& + [AX_2 X_3 X_4 + BY_2 Y_3 Y_4 + C(X_2 X_3 Y_4 + Y_2 X_3 X_4 + X_2 Y_2 X_4) \\
& \qquad + D(X_2 Y_3 Y_4 + Y_2 X_3 Y_4 + Y_2 Y_3 X_4)] \pi_{(ijk)} \{m_i m_j n_k\}
\end{aligned} \tag{3.5}$$

Comparing (3.2) and (3.5), we have

$$AX_2^3 + BY_2^3 + 3CX_2^2 Y_2 + 3DX_2 Y_2^2 = H_{,1} + 3(J+J')h_1 + 3(H'+I')j_1. \tag{3.6)a$$

$$AX_3^3 + BY_3^3 + 3CX_3^2 Y_3 + 3DX_3 Y_3^2 = J_{,1} + 3Ih_1 - 3I'k_1. \tag{3.6)b$$

$$AX_4^3 + BY_4^3 + 3CX_4^2 Y_4 + 3DX_4 Y_4^2 = H'_{,1} + 3Kj_1 + 3J'k_1. \tag{3.6)c$$

$$AX_2^2 X_3 + BY_2^2 Y_3 + C(X_2^2 Y_3 + 2X_2 X_3 Y_2) + D(Y_2^2 X_3 + 2X_2 Y_2 Y_3) = -(J+J')_{,1} + (H-2I)h_1 - 2K'j_1 + (H'+I')k_1 \tag{3.6)d$$

$$AX_2 X_3^2 + BY_2 Y_3^2 + C(Y_2 X_3^2 + 2X_2 X_3 Y_3) + D(X_2 Y_3^2 + 2Y_2 X_3 Y_3) = I_{,1} - (3J+2J')h_1 + I'j_1 - 2K'k_1 \tag{3.6)e$$

$$AX_2 X_4^2 + BY_2 Y_4^2 + C(Y_2 X_4^2 + 2X_2 X_4 Y_4) + D(X_2 Y_4^2 + 2Y_2 X_4 Y_4) = K_{,1} - J'h_1 - (3H'+2I')j_1 + 2K'k_1 \tag{3.6)f$$

$$AX_3 X_4^2 + BY_3 Y_4^2 + C(Y_3 X_4^2 + 2X_3 X_4 Y_4) + D(X_3 Y_4^2 + 2Y_3 X_4 Y_4) = J'_{,1} + Kh_1 + 2K'j_1 - (H'-2I')k_1 \tag{3.6)g$$

$$AX_3^2 X_4 + BY_3^2 Y_4 + C(X_3^2 Y_4 + 2Y_3 X_3 X_4) + D(Y_3^2 X_4 + 2Y_3 X_3 Y_4) = I'_{,1} + 2K'h_1 + Ij_1 + (J-2J')k_1 \tag{3.6)h$$

$$AX_2^2 X_4 + BY_2^2 Y_4 + C(X_2^2 Y_4 + 2Y_2 X_2 Y_4) + D(Y_2^2 X_4 + 2Y_2 X_2 Y_4) = -(H'+I')_{,1} - 2K'h_1 + (H-2K)j_1 - (J+J')k_1 \tag{3.6)i$$

$$AX_2 X_3 X_4 + BY_2 Y_3 Y_4 + C(X_2 X_3 Y_4 + Y_2 X_3 X_4 + X_2 Y_2 X_4) + D(X_2 Y_3 Y_4 + Y_2 X_3 Y_4 + Y_2 Y_3 X_4) = K'_{,1} - (H'+2I')h_1 - (J+2J')j_1 + (I-K)k_1 \tag{3.6)j$$

Adding (a), (e) and (f) of equation (3.6), we have $(A+D)X_2 + (B+C)Y_2 = (LC)_{,1}$

Adding (b), (d) and (g) of equation (3.6), we have $(A+D)X_3 + (B+C)Y_3 = (LC)h_1$

Adding (c), (h) and (i) of equation (3.6), we have $(A+D)X_4+(B+C)Y_4=(LC)j_1$

Solving above, we have the following:

THEOREM 3.1 : In a four-dimensional P3-like Finsler space the scalars A+B and B+C are given in terms of scalar components of X_i and Y_i as

$$A+D=\frac{Y_3(LC)_{,1}-Y_2h_1LC}{X_2Y_3-X_3Y_2},\qquad B+C=\frac{X_2h_{,1}LC-Y_3(LC)_{,1}}{X_2Y_3-X_3Y_2}.$$

The equation (3.6) is more complicated to discuss any other geometrical results, so we consider the particular cases of four-dimensional P3-like Finsler space.

Let X_i is along m_i and Y_i is along n_i then $X_2=1, X_3=0, X_4=0, Y_2=0,$ $Y_3=1, Y_4=0$. Therefore (3.6) reduces to

$$A=H_{,1}+3(J+J')h_1,\qquad B=J_{,1}+3Ih_1-3I'k_1,$$
$$C=-(J+J')_{,1}'+(H-2I)h_1-+(H'+I')k_1,\quad D=I_{,1}-(3J+2J')h_1-2K'k_1, \tag{3.7}$$

and $j_1=0$. Therefore we have

THEOREM 3.2 : If in a four-dimensional P3-like Finsler space the vector fields X_i and Y_i are along m_i and n_i respectively of the Moor's frame (l_i,m_i,n_i,p_i), then $j_i=0$ and the scalars A, B, C and D are given by (3.7).

If X_i and Y_i are along m_i and p_i respectively, the $X_2=Y_4=1, X_3=X_4=$ $Y_2=Y_3=0$. Therefore we have

THEOREM 3.3 : If in a four-dimensional P3-like Finsler space the vector fields X_i and Y_i are along m_i and p_i respectively of the Moor's frame (l_i,m_i,n_i,p_i), then $h_i=0$ and

$$A=H_{,1}+3(J+J')h_1+3(H'+I')j_1,\qquad B=H'_{,1}+3Kj_1+3J'k_1,$$
$$C=-(H'+I')_{,1}+(H-2K)j_1-(J+J')k_1,\qquad D=K_{,1}-(3H'+2I')j_1+2K'k_1.$$

Let X_i and Y_i are along n_i and p_i respectively, the $X_3 = Y_4 = 1, X_2 = X_4 = Y_2 = Y_3 = 0$. Therefore we have

THEOREM 3.4 : If in a four-dimensional P3-like Finsler space the vector fields X_i and Y_i are along n_i and p_i respectively of the Moor's frame (l_i, m_i, n_i, p_i), then $(LC)_{,1} = 0$ and

$$A = J_{,1} + 3Ih_1 - 3I'k_1, \qquad B = H'_{,1} + 3Kj_1 + 3J'k_1,$$

$$C = I'_{,1} + 2K'h_1 + Ij_1 + (J - 2J)k_1, \qquad D = J'_{,1} + Kh_1 + 2K'j_1 - (H' - 2I')k_1.$$

4. P3-LIKE N-DIMENSIONAL SPECIAL FINSLER SPACES

A Finsler space of dimension n $(n \geq 3)$ is called C-reducible, [2] if C_{ijk} is written in the form $C_{ijk} = \frac{1}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)$ where $C_i = g^{jk}C_{ijk}$. Therefore

$$P_{ijk} = C_{ijk|0} = \frac{1}{n+1}(h_{ij}C_{k|0} + h_{jk}C_{i|0} + h_{ki}C_{j|0}). \tag{4.1}$$

Comparing (4.1) with (1.1) and then contracting with g^{jk}, we get

$$A + D = C_{i|0}X^i, \qquad B + C = C_{i|0}Y^i. \tag{4.2}$$

A Finsler space of dimension more than two is called semi C-reducible space of constant scalar [6], if C_{ijk} is written in the form

$$C_{ijk} = \frac{p}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j) + \frac{q}{C^2}C_iC_jC_k$$

where p and q are constant such that p + q = 1 and $C^2 = g^{jk}C_iC_j$. Therefore

$$P_{ijk} = \frac{p}{n+1}(h_{ij}C_{k|0} + h_{jk}C_{i|0} + h_{ki}C_{j|0}) + \frac{q}{C^2}(C_{i|0}C_jC_k + C_iC_{j|0}C_k$$
$$+ C_iC_jC_{k|0}) - \frac{2q}{C^3}C_{|0}C_iC_jC_k. \tag{4.3}$$

Comparing (4.3) with (1.1) and then contracting with g^{jk}, we get (4.2).

A Finsler space of dimension n (n > 2) is called C2-like [6], if C_{ijk} is written in the form $C_{ijk} = \frac{1}{C^2} C_i C_j C_k$. Therefore

$$P_{ijk} = \frac{1}{C^2}(C_{i|0} C_j C_k + C_i C_{j|0} C_k + C_i C_j C_{k|0}) - \frac{2}{C^3} C_{|0} C_i C_j C_k. \qquad (4.4)$$

Comparing (4.4) with (1.1) and then contracting with g^{jk}, we get (4.2).

THEOREM 4.1 : If P3-like Finsler space is a C-reducible or P-reducible or Semi C-reducible with constant coefficients or C2-like the scalars A + D and B + C are given by $A + D = C_{i|0} X^i$, $B + C = C_{i|0} Y^i$.

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