Journal of the Tensor Society
Volume 1, 2007
https://doi.org/10.56424/jts.v1i01.9947

ON H-PROJECTIVELY FLAT KH-STRUCTURE SUBMANIFOLD

By

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In this paper we have obtained the condition for a submanifold of a KH-structure manifold to be KH-structure. A few relations are found in the second fundamental tensors H and K. Using these results we have also find out the flatness of H-projective curvature tensor in KH-structure submanifold.

1. INTRODUCTION

Let us consider a 2n-dimensional differentiable manifold M^{2n} of class C^{∞}. Let there exist in M^{2n} a vector valued linear function F such that

$$F^2X = a^2X , \tag{1.1}$$

for an arbitrary vector field X in M^{2n}, where a is any complex number, not equal to zero. Then F gives to M^{2n} a GF-structure and the manifold is called a GF-structure manifold [1].

If the GF-structure is endowed with metric tensor G, such that

$$G(FX,FY) = -a^2 G(X,Y), \tag{1.2}$$

then M^{2n} is called an H-structure manifold.

(1.1) AGREEMENT

In this paper the equations containing X, Y, Z, U will hold for arbitrary vector field X, Y, Z, U in M^{2n-2}.

Let us put

$$'F(X,Y) \stackrel{def}{=} G(FX,Y), \tag{1.3}$$

then 'F is skew symmetric.

Let M^{2n} be given with H-structure, such that

$$(E_X F)Y = 0, \tag{1.4}$$

is satisfied. Where E_X is the Riemannian Connexion in M^{2n}, then M^{2n} is said to have a KH-structure and the manifold is called KH-structure manifold [3].

Let M^{2n-2} be a submanifold of M^{2n} with b at p

$$b : M^{2n-2} \to M^{2n}.$$

be the inclusion map such that $p \in M^{2n-2} \Rightarrow bp \in M^{2n}$ is called the Jacobian map, then

$$G(BX, BY)ob = g(X, Y). \tag{1.5}$$

Since G is real valued positive, definite, bilinear and symmetric function in M^{2n}, g is also real valued positive, definite, bilinear and symmetric function in M^{2n-2}, then g is called the first fundamental magnitudes in M^{2n-2}. If B is the induced Riemannian connection in M^{2n-2}, we have the following equation

$$E_{BX} BY = BD_X Y + H(X,Y)M + K(X,Y)N, \tag{1.6}$$

where M and N be the unit normal vectors to the submanifold M^{2n-2} and H and K are Geometric bilinear functions in M^{2n-2}.

Weingarten equations in the submanifold M^{2n-2} are given by [1].

$$E_{BX} M = -B'H(X) + 'L(X)N, \tag{1.7)a$$

$$E_{BX} N = -B'K(X) + 'L(X)M, \tag{1.7)b$$

where

$$g('H(X), Y) \overset{\text{def}}{=} H(X, Y), \quad (1.8)a$$

$$g('K(X), Y) \overset{\text{def}}{=} K(X, Y), \quad (1.8)b$$

and 'L(K) is the third fundamental tensor.

Let '*K and *K are the curvature tensor in the manifold M^{2n} and submanifold M^{2n-2}. Ric and Ric are corresponding Ricci tensor.

Then from the Gauss and Mainardi-Codazzi equation [2]

$$\begin{aligned} '{*K}(BX, BY, BZ, BU)ob = {*K}(X, Y, Z, U) - H(X, U)\,H(Y, Z) + H(Y, U)\,H(X, Z) \\ - K(X, U)\,K(Y, Z) + K(Y, U)\,K(X, Z), \end{aligned} \quad (1.9)$$

we have

$$\begin{aligned} 'Ric\,(BY, BZ) = Ric\,(Y, Z) - (C_1^1\,'H)\,H(Y, Z) + H('H(Y), Z) \\ - (C_1^1\,'K)\,K(Y, Z) + K('K(Y), Z), \end{aligned} \quad (1.10)$$

$$'r\,(BY) = Br\,(Y) - (C_1^1\,'H)\,B'H(Z) + 'H('H(Y)) - (C_1^1\,'K)\,'BK(Y) + 'K('K(Y)). \quad (1.11)$$

where C_1^1 is the contraction operator.

$$'Ric\,(BY, BZ) \overset{\text{def}}{=} G('r\,(BY), BZ), \quad (1.12)a$$

$$Ric\,(Y, Z) \overset{\text{def}}{=} g(r\,(Y), Z). \quad (1.12)b$$

The nacessary and sufficient condition that M^{2n-2} be an H-structure submanifold with the structure (f, g) in the H-structure manifold M^{2n}, we have [4].

$$F(BX) = B\overline{X} \quad (1.13)a$$

where

$$\overline{X} = \acute{F}(X) \quad (1.13)b$$

Also when M^{2n-2} is an H-structure submanifold in the H-structure manifold M^{2n} we have [1].

$$F(M) = N, \tag{1.14a}$$

$$F(N) = M, \tag{1.14b}$$

An H-structure submanifold M^{2n-2} in a KH-structure manifold M^{2n} is KH-structure manifold. H and K of KH-structure submanifold M^{2n-2} immersed in KH-structure manifold M^{2n} are related by [3].

$$H(X,\overline{Y}) = H(\overline{X},Y) \tag{1.15a}$$

$$K(X,\overline{Y}) = K(\overline{X},Y) \tag{1.15b}$$

2. SUBMANIFOLD

THEOREM 2.1 : Let M^{2n-2} be a KH-structure submanifold in a KH-structure manifold M^{2n} of constant holomorphic sectional curvature 'K, then in M^{2n-2} we have

$$\begin{aligned} {}^*K(X, Y, Z, U) &= H(X, U)\,H(Y, Z) - H(Y, U)\,H(X, Z) + K(X, U)\,K(Y, Z) \\ &- K(Y, U)\,K(X, Z) + \frac{1}{4}K\,[g(X, U)\,g(Y, Z) - g(Y, U)\,g(X, Z) \\ &+ {}'f(X, U)\,{}'f(Y, Z) - {}'f(Y, U)\,{}'f(X, Z) - 2\,{}'f(Z, U)\,{}'f(X, Y)], \end{aligned} \tag{2.1a}$$

where ${}'f(X, Y) = h(fX, Y) = -{}'f(Y, X)$.

$$Ric\,(Y, Z) = \frac{m}{2}\,{}'Kg(Y, Z) - 2H({}'H(Y), Z'), \tag{2.1b}$$

$$R = m(m-1)\,{}'K - 2C_1^1({}'H({}'H)), \tag{2.1c}$$

where R is the scalar curvature.

PROOF : Let us consider that the enveloping manifold M^{2n} is of constant holomorphic sectional curvature 'K, then the curvature tensor of M^{2n} is given by

$$\begin{aligned} {}'^*K(BX, BY, BZ, BU)ob &= \frac{1}{4}\,{}'K[(G(BX, BU)ob)(G(BY, BZ)ob) \\ &- (G(BX, BZ)ob)(G(BY, BU)ob) + (G(F(BX), BU)ob)(G(F(BY), BZ)ob) \\ &- (G(F(BX), BZ)ob)(G(F(BY), BU)ob) - 2(G(F(BX), BY)ob)(G(F(BZ), BU)ob)]. \end{aligned}$$

By using (1.5), (1.9) and (1.13), we have (2.1)a on contracting (2.1)a, using $H('H(\overline{Y}), Z) = K('K(Y), Z)$ and $(C_1^1 'H) = (C_1^1 'K) = 0$, we get (2.1)b, (2.1)c follows from (2.)b.

THEOREM 2.2 : Let M^{2n-2} be a KH-structure submanifold in a KH-structure manifold M^{2n} of constant holomorphic sectional curvature 'K, then for H - projectively flat submanifold M^{2n-2}, we have

$$H('H(X),Y) = \frac{C_1^1('H('H))}{2(m-1)} g(X,Y). \tag{2.2a}$$

PROOF : The H - projectively curvature tensor 'P in M^{2n-2} as given by [4]

$$'P(X,Y,Z,U) = {}^*K(X,Y,Z,U) - \frac{1}{2m}[g(X,U)Ric(Y,Z) - g(Y,U)Ric(X,Z)$$

$$+'f(X,U)Ric(\overline{Y},Z) - 'f(Y,U)Ric(\overline{X},Z) - 2'f(Z,U)Ric(\overline{X},Y)] \tag{2.2b}$$

Hence making use of the equation (2.1) and (2.2)b, the H - projectively curvature tensor 'P in submanifold M^{2n-2} becomes

$$'P(X, Y, Z, U) = H(X, U)\, H(Y, Z) - H(Y, U)\, H(X, Z) + K(X, U)\, K(Y, Z)$$

$$- K(Y, U)\, K(X, Z) + \frac{1}{m}[g(X, U)\, H('H(Y), Z) - g(Y, U)\, H('H(X), Z)$$

$$+ 'f(X, U)\, K('H(Y, Z)) - 'f(Y, U)\, K('H(X), Z) - 2'f(Z, U)\, K('H(X), Y)]. \tag{2.2c}$$

Let $'P(X, Y, Z, U) = 0,$

then from (2.2)c, we have

$$'H(X, U)\, 'H(Y) - 'H(X)\, H(Y, U) + K(X, U)\, 'K(Y) - K(Y, U)\, 'K(X)$$

$$+ \frac{1}{2}[g(X, U)\, 'H('H(Y)) - g(Y, U)\, 'H('H(X)) + 'f(X, U)\, 'K('H(Y))$$

$$- 'f(Y, U)\, 'K('H(X)) + 2\overline{u}\, K('H(X), Y)] = 0, \tag{2.2d}$$

contracting (2.2)d, we get

$$H('H(Y), U) - C_1^1('H)\, H(Y, U) + K('K(U), U) - C_1^1('K)\, K(Y, U)$$

$$+ \frac{1}{m}[g('H('H(Y)), U) - g(Y, U)\, C_1^1{'H}('H) + {'f}('K('H(Y)), U))$$

$$- {'f}(Y, U)\, C_1^1('K('H)) + 2K('H(\overline{U}), Y)] = 0 . \qquad (2.2)e$$

Making (2.2)e, we have

$$2(m - 1)\, H('H(Y), U) = C_1^1('H('H))\, g(Y, U) \qquad (2.2)f$$

from (2.2)f, we get

$$H('H(Y),U) = \frac{C_1^1('H('H))}{2(m-1)} g(Y,U) .$$

THEOREM 2.3 : Let M^{2n-2} be a KH-structure submanifold in a KH-structure manifold M^{2n} of constant holomorphic sectional curvature 'K then for a totally geodesic manifold M^{2n-2}, we have

It is H - projectively flat. (2.3)a

It is an Einstein manifold. (2.3)b

Hence the manifold M^{2n-2} is of canstant holomorphic sectional curvature,

$$'K = \frac{R}{m(m-1)}$$

PROOF : Let M^{2n-2} be totally geodesic then

$$H(X, Y) = 0 \text{ or } 'H(X) = 0.$$

Using (2.3)a in (2.2)c, we have

$$'P(X, Y, Z, U) = 0.$$

Using (2.3)a and the theorem (2.1)b, we get

$$\text{Ric}\,(Y, Z) = \frac{m}{2}\, 'Kg(Y, Z) ,$$

and in the theorem (2.1)c, we get

$$'K = \frac{R}{m(m-1)}$$

Hence the proof follows.

THEOREM 2.4 : Let M^{2n-2} be a KH-structure submanifold of H - projectively flat KH-structure manifold M^{2n} then for H - projectively flat manifold M^{2n-2}, we have

$$H('H(X),Y) = \frac{C_1^1('H('H))}{2(m-1)} g(X,Y) .$$

PROOF : The H - projectively curvature tensor '*P in M^{2n} is given by [4].

$$*P(BX, BY, BZ, BU) \, ob = '*K(BX, BY, BZ, BU) \, ob$$
$$+ \frac{1}{2(m+1)} [G(BX, BU) \, ob \, 'Ric(BY, BZ) - G(BY, BU) \, ob \, Ric(BX, BZ)$$
$$+ 'F(BZ, BU) \, 'Ric(F(BY), BZ) - 'F(BY, BU) \, 'Ric(F(BX), BZ)$$
$$- 2'F(BZ, BU) \, 'Ric('F(BX), BY)] , \tag{2.4)a}$$

Let $'*P(BX, BY, BZ, BU) \, ob = 0$,

then the manifold

$$Ric(BY,BZ) = \frac{R'}{2m} G(BY,BZ) ob \tag{2.4)b}$$

By using (1.10) and (1.11) in (2.4)b, we get

$$Ric (Y, Z) + 2H('H(Y), Z) = \frac{1}{2m} [R + 2C_1^1('H('H))] g(Y, Z) . \tag{2.4)c}$$

contracting (2.4)c, we get

$$R + 2C_1^1('H('H)) = 0 \tag{2.4)d}$$

hence, we have

$$Ric (Y, Z) + 2H('H(Y), Z) = 0 , \tag{2.4)e}$$

from (2.4)e and making use of (1.10), we get

$$'Ric(BY, BZ) = 0, \tag{2.4f}$$

$$*K(X, Y, Z, U) = H(X, U)\, H(Y, U) - H(Y, U)\, H(X, Z) + K(X, U)\, K(Y, Z) - K(Y, U)\, K(X, Z) \tag{2.4g}$$

Now from (2.4)e, (2.4)g and (2.2)b the H - projective curvature tensor 'P in H - projectively flat KH-structure manifold M^{2n} becomes (2.2)c. By the same method as in them (2.2)f, we get the result.

THEOREM 2.5 : Let M^{2n-2} be a KH-structure submanifold in a H - projectively flat KH-structure manifold M^{2n} then for the totally geodesic submanifold M^{2n-2}, we have

(a) It is H - projectively flat

(b) It is flat manifold M^{2n}.

THEOREM 2.6 : Let M^{2n-2} be a KH-structure submanifold in a H - projectively flat KH-structure manifold M^{2n} then for H- projectively flat submanifold M^{2n-2}, it is an Einstein manifold.

PROOF : From (2.4)e, we have

(a) $\quad Ric(Y, Z) = -2H('H(Y), Z)$

By using (2.2)a and putting X = Y & Y = Z,

(b) $\quad Ric(Y, Z) \dfrac{-C_1^1(H('H))}{(m-1)} g(Y, Z)$,

which proves the statement.

THEOREM 2.7 : Let M^{2n-2} be a KH-structure submanifold in a KH-structure manifold M^{2n} of constant holomorphic sectional curvature 'K, then for H- projectively flat KH-structure submanifold M^{2n-2} is an Einstein manifold.

PROOF : Let us consider (2.1)b, we have

$$\text{Ric}(Y, Z) = \frac{m}{2}\,{}'Kg(Y, Z) - 2H('H(Y), Z)$$

By using (2.2)a in the above equation, we get

$$\text{Ric}(Y, Z)\frac{R}{2m(m-1)}g(Y, Z)$$

THEOREM 2.8 : Let M^{2n-2} be a KH-structure submanifold in a KH-structure manifold M^{2n} of constant holomorphic sectional curvature 'K, then for H- projectively flat submanifold M^{2n-2} is of constant holomorphic sectional curvature 'K and it is an totally geodesic.

PROOF : The proof follows from the above them.

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