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ON THE BI-RECURRENT BOCHNER CURVATURE TENSOR

By

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ABSTRACT : In the present paper, we have studied bi-recurrent and bi-symmetric properties of Bochner Curvature tensor and several theorems have been established.

1. INTRODUCTION

The projective curvature tensor of an n-dimensional Riemannian space M^n is given by

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{(n-1)}(R_{ik}\delta^h_j - R_{jk}\delta^h_i) \tag{1.1}$$

which is invariant under any projective correspondence, where R^h_{ijk} and R_{ik} are the Riemannian curvature and Ricci tensor.

The conformal curvature tensor of M^n is given by

$$C^h_{ijk} = R^h_{ijk} + \frac{1}{(n-1)}(R^i_{jk}\delta^h_j - R_{jk}\delta^h_j + g_{ik}R^h_j - g_{jk}R^h_i)$$
$$-\frac{R}{(n-1)(n-2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i) \tag{1.2}$$

where g_{ij} is the Riemannian metric of M^n and

$$R^h_i = g^{ha}R_{ia}, \quad R = g^{ij}R_{ij}$$

Let K^n be an n(=2m) dimensional Kaehlerian space with the structure tensor g_{ij} and F_i^h. It is known that tensor

$$P_{ijk}^h = R_{ijk}^h + \frac{1}{(n+2)}(R_{ik}\,\delta_j^h - R_{jk}\,\delta_i^h + S_{ik}\,F_j^h - S_{jk}\,F_i^h + 2S_{ij}F_k^h) \tag{1.3}$$

Called the holomorphically Projective curvature tensor of K^n, is invariant under any holomorphically Projective correspondence (Tachibana and Ishihara [6]). P_{ijk}^h may be considered as the tensor corresponding to W_{ijk}^h.

On the other hand, (S. Bochner [4], Yano and Bochner [5]) has introduced a tensor in K^n given by

$$K_{hijk} = R_{hijk} - \frac{1}{(n+2)}(R_{ik}\,g_{hk} + R_{hi}\,g_{jk} + g_{ij}R_{hk} + g_{hi}R_{jk})$$

$$+\frac{R}{2(n+1)(n+2)}(g_{ij}\,g_{hk} + g_{hi}\,g_{jk})$$

with respect to complex local coordinate.

2. KAEHLERIAN SPACE

An n(=2m) dimensional Kaehlerian space K^n is a Riemannian space which admits a tensor field F_i^j satisfying the conditions

$$F_a^i F_j^a = -\delta_j^i \tag{2.1}$$

$$F_{ij} = -F_{ji} \quad (F_i^a = F_i^a\,g_{aj}) \tag{2.2}$$

and $$F_{i,k}^j = 0 \tag{2.3}$$

where the comma(,) followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ik} of the Riemannian space.

The Riemannian curvature tensor R_{ijk}^h is given by

$$R^h_{ijk} = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop ia} \right\} \left\{ {a \atop jk} \right\} - \left\{ {h \atop ja} \right\} \left\{ {a \atop ik} \right\} \tag{2.4}$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$ and $\{x^i\}$ denotes the real local coordinates.

The Ricci tensor and the scalar curvature tensor are respectively given by

$R_{jk} = R^a_{ajk}$ and $R = g^{ij} R_{ij}$

It is well known that these tensors satisfy the following identities

$$R^h_{ajk} F^a_i = -R^h_{iak} F^a_j \tag{2.5}$$

$$R^h_{ijk} F^a_k = R^a_{ijk} F^h_a \tag{2.6}$$

$$F^a_i R_{aj} = -R_{ia} F^a_j \tag{2.7}$$

$$F^a_i R^h_i = R^a_i F^h_a \tag{2.8}$$

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j} \tag{2.9}$$

and
$$R_{,i} = 2R^a_{i,a} \tag{2.10}$$

If we define a tensor S_{jk} by

$$S_{jk} = F^a_j R_{ak} \tag{2.11}$$

then we have

$$S_{jk} = -S_{kj} \tag{2.12}$$

$$F^a_i S_{ak} = -S_{ia} F^a_k \tag{2.13}$$

$$S_{jk} = -\left(\frac{1}{2}\right) F^{ab} R_{abjk} \tag{2.14}$$

$$2S^a_{i,a} = F^a_i R_{,a} \tag{2.15}$$

Now we shall consider a tensor K^h_{ijk} defined by

$$K_{ijk}^{h} = R_{ijk}^{h} + \frac{1}{(n+4)}(R_{ik}\delta_j^h - R_{jk}\delta_i^h + g_{ik}R_j^h - g_{jk}R_i^h + S_{ik}F_j^h$$

$$- S_{jk}F_i^h + F_{ik}S_i^h - F_{jk}S_i^h + 2S_{ij}F_k^h + 2F_{ij}S_k^h)$$

$$- \frac{R}{(n+2)(n+4)}(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h) \qquad (2.16)$$

which is constructed formally from C_{ijk}^{h} by taking account of the formal resemblence between W_{ijk}^{h} and P_{ijk}^{h}. Then we can prove that the tensor $K_{ijka} = g_{ha}K_{ijk}^{h}$ has components of tensor given by S. Bochner with respect to complex local coordinates.

Hence we call tensor the Bochner curvature tensor.

Now if we put

$$L_{ij} = R_{ij} - \frac{R}{2(n+2)}g_{ij} \quad \text{and} \qquad (2.17)a$$

$$M_{ij} = F_i^a L_{aj} = S_{ij} - \frac{R}{2(n+2)}F_{ij} \qquad (2.17)b$$

then (2.16) reduces to the form

$$K_{ijk}^{h} = R_{ijk}^{h} + \frac{1}{(n+4)}(L_{ik}\delta_j^h - L_{jk}\delta_i^h + g_{ik}L_j^h - g_{jk}L_i^h + M_{ik}F_j^h - M_{jk}F_i^h$$

$$+ F_{ik}M_j^h - F_{jk}M_i^h + 2M_{ij}F_k^h + 2F_{ij}M_k^h) \qquad (2.18)$$

The following identities are obtained by the straight forward computations.

$$K_{ijk}^{h} = -K_{jki}^{h}, \quad K_{ijka} = -K_{ijak}$$

$$K_{ijk}^{h} + K_{jki}^{h} + K_{kij}^{h} = 0$$

$$K_{ajk}^{a} = 0, \quad K_{ija}^{a} = 0$$

$$K_{ijk}^{a}F_a^h = K_{ija}^{h}F_k^a, \quad K_{ajk}^{h}F_i^a = -K_{iak}^{h}F_j^a,$$

$$K^b_{ija}F^a_b = 0, \quad K^b_{ajk}F^a_b = 0 \qquad (*1)$$

Now we shall use the following definitions.

DEFINITION 2.1 : A Kaehler space satisfying the relation

$$R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk} = 0 \tag{2.19}$$

for some non-zero tensor λ_{ab}, will be called a Kaehlerian bi-recurrent space and the space is called Ricci bi-recurrent (or semi bi-recurrent) if it satisfies

$$R_{ij,ab} - \lambda_{ab} R_{ij} = 0 \tag{2.20}$$

Multiplying the above equation by g^{ij}, we have

$$R_{,ab} - \lambda_{ab} R = 0 \tag{2.21}$$

DEFINITION 2.2 : A Kaehler space satisfying the relation

$$P^h_{ijk,ab} - \lambda_{ab} P^h_{ijk} = 0 \tag{2.22}$$

for some non-zero tensor λ_{ab}, will be called a Kaehlerian space with bi-recurrent H-projective curvature tensor or briefly K-P space.

DEFINITION 2.3 : A Kaehler space satisfying the relation

$$B^h_{ijk,ab} - \lambda_{ab} B^h_{ijk} = 0 \tag{2.23}$$

for some non-zero tensor λ_{ab}, will be called a Kaehlerian space with bi-recurrent Bochner curvature tensor or briefly K-B space.

THEOREM 1 : The necessary and sufficient condition that a Kaehlerian space is Kaehlerian Ricci bi-recurrent is that

$$K^h_{ijk,ab} - \lambda_{ab} K^h_{ijk} = R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk}$$

(*1) The holomorphically projective curvature tensor satisfies these all identities except the second one see, S. Tachibana and S. Ishihara [6].

PROOF : Let the space be Kaehlerian Ricci bi-recurrent, then the relation (2.20), is satisfied.

Differentiating (2.18), covariantly with respect to x^a and again differentiating the result thus obtained with respect to x^b, we get

$$K^h_{ijk,ab} = R^h_{ijk,ab} + \frac{1}{(n+4)}(L_{ik,ab}\delta^h_j - L_{jk,ab}\delta^h_i + g_{ik}L^h_{j,ab} - g_{jk}L^h_{i,ab} + M_{ik,ab}F^h_j$$
$$- M_{jk,ab}F^h_i + F_{ik,ab}M^h_j - F_{jk,ab}M^h_i + 2M_{ij,ab}F^h_k + 2F_{ij,ab}M^h_k) \qquad (2.24)$$

Transvecting (2.18) with λ_{ab}, and subtracting the result thus obtained from (2.24), we get

$$K^h_{ijk,ab} - \lambda_{ab}K^h_{ijk} = R^h_{ijk,ab} - \lambda_{ab}R^h_{ijk} + \frac{1}{(n+4)}[(L_{ik,ab} - \lambda_{ab}L_{ik})\delta^h_j$$
$$-(L_{jk,ab} - \lambda_{ab}L_{jk})\delta^h_i + g_{ik}(L^h_{j,ab} - \lambda_{ab}L^h_j) - g_{jk}(L^h_{i,ab} - \lambda_{ab}L^h_i)$$
$$+(M_{ik,ab} - \lambda_{ab}M_{ik})F^h_j - (M_{jk,ab} - \lambda_{ab}M_{jk})F^h_i + (F_{ik,ab} - \lambda_{ab}F_{ik})M^h_j$$
$$-(F_{jk,ab} - \lambda_{ab}F_{jk})M^h_i + 2F^h_k(M_{ij,ab} - \lambda_{ab}M_{ij}) + 2M^h_k(F_{ij,ab} - \lambda_{ab}F_{ij})] \quad (2.25)$$

Since the space is Ricci bi-recurrent then the equation (2.25), in view of (2.17a) and (2.17b), reduces to

$$K^h_{ijk,ab} - \lambda_{ab}K^h_{ijk} = R^h_{ijk,ab} - \lambda_{ab}R^h_{ijk} \qquad (2.26)$$

Conversely, if in a Kaehlerian space equation (2.26), is satisfied then we have from (2.25),

$$[(L_{ik,ab} - \lambda_{ab}L_{ik})\delta^h_j - (L_{jk,ab} - \lambda_{ab}L_{jk})\delta^h_i + g_{ik}(L^h_{j,ab} - \lambda_{ab}L^h_j)$$
$$- g_{jk}(L^h_{i,ab} - \lambda_{ab}L^h_i) + (M_{ik,ab} - \lambda_{ab}M_{ik})F^h_j - (M_{jk,ab} - \lambda_{ab}M_{jk})F^h_i$$
$$+(F_{ik,ab} - \lambda_{ab}F_{ik})M^h_j - (F_{jk,ab} - \lambda_{ab}F_{jk})M^h_i$$
$$+2F^h_k(M_{ij,ab} - \lambda_{ab}M_{ij}) + 2M^h_k(F_{ij,ab} - \lambda_{ab}F_{ij})] = 0 \qquad (2.27)$$

which yields with the help of (2.17a) and (2.17b),

$$R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk} = 0$$

i.e. the space is Kaehlerian Ricci bi-recurrent.

THEOREM 2 : Every Kaehlerian bi-recurrent space is a Kaehlerian space with bi-recurrent Bochner curvature tensor.

PROOF : If the space is Kaehlerian bi-recurrent, then equations (2.19) and (2.20) are satisfied and equation (2.25), in view of (2.19), (2.20), (2.17a) and (2.17b) reduces to

$$K^h_{ijk,ab} - \lambda_{ab} K^h_{ijk} = 0$$

which shows that the space will also be Kaehlerian space with bi-recurrent Bochner curvature tensor.

Next we consider a tensor K_{ijk}, given by

$$K_{ijk} = R_{jk,i} - R_{ik,j} + \frac{R_l}{2(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^l_j - F_{jk}F^l_i + 2F_{ij}F^l_k)$$

then we can get the following identity

$$K^a_{ijk,a} = \frac{n}{n+4} K_{ijk}$$

Now we cansider a tensor

$$U^h_{ijk} = R^h_{ijk} + \frac{R}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k)$$

then we can obtain the following.

THEOREM 3 : The Bochner curvature tensor coincides with U^h_{ijk} of a Kaehlerian space K^n is an Einstein space.

Remark : The term of a Riemannian space defined by

$$Z^h_{ijk} = R^h_{ijk} + \frac{R}{n(n-1)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i)$$

Is called the concircular curvature tensor and is invariant under any concircular correspondence (by Yano [2]) U^h_{ijk} correspondence to Z^h_{ijk}. A Kaehlerian space is called a space of constant holomorphically sectional curvature if its U^h_{ijk} vanishes identically.

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