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ON LPS - RIEMANNIAN MANIFOLD

By

H. B. Pandey, Manohar Singh & S. P. Pandey
Department of Mathematics
R. B. S. College, Agra,
F.E.T., Dr. Ram Manohar Lohia University, Faizabad

In this paper, Lorentzian Para Sasakian Riemannian manifold has been studied. The first section is introductory. Basic definitions and known results are defined. Second section deals with LPS - Riemannian manifold and the third section is devoted for KLPS - Riemannian manifold. Some interesting results have been investigated.

1. INTRODUCTION

DEFINITION 1.1 : An n - dimensional differentiable manifold Mn is called a Lorentzian Para Sasakian (LP - Sasakian) manifold if it admits (1,1) tensor field F, a vector field T, a 1 - form A and a Lorentzian metric 'g' satisfying,

$$A(T) = -1 \tag{1.1)a}$$

$$F^2(X) = X + A(X)T \tag{1.1)b}$$

$$g(FX,FY) = -g(X,Y) + A(X)A(Y) \tag{1.1)c}$$

$$g(X,T) = -A(X) \tag{1.1)d}$$

$$F(X) = D_x T \tag{1.1)e}$$

$$(D_x F)Y = g(X,Y) + A(X)A(Y)T + (X + A(X)T)\,A(Y) \tag{1.1)f}$$

where D denotes the covarient differentiation with respect to g.

In an LP - Sasakian manifold Mn with structure (F, T, A, g), we can easily show that

$$FT = 0 \tag{1.2)a}$$

$$A(FX) = 0 \tag{1.2)b$$

$$\text{rank}\,(E) = n - 1 \tag{1.2)c$$

More over, if we put

$$'F(X,Y) = g(\overline{X},Y) \tag{1.3}$$

where 'F is a skew symmetric

DEFINITION 1.2 : An LP - Sasakian metric manifold on which the fundamental 2 - form 'F is such that

$$2\,'F = d\,A \tag{1.4}$$

is satisfied is called LP - Sasakian manifold. [2]

2. SOME PROPERTIES OF LPS - RIEMANNIAN MANIFOLD

On LP Sasakian manifold we have,

$$\begin{aligned} 2\,'F(X,Y) &= dA(X,Y), \qquad (2.1)\\ &= XA(Y) - YA(X) - A([X,Y]),\\ &= (D_x A)(Y) - (D_y A)(X), \end{aligned}$$

where 'D' is Riemannian connexion.

Thus we have

THEOREM 2.1 : On LP Sasakian - Riemannian manifold, we have

$$'F(X,Y) = \frac{1}{2}[(D_x A)(Y) - (D_y A)(X)] \tag{2.2}$$

We have from (1.4)

$$(d\,'F) = d^2A = 0 \tag{2.3}$$

$$\begin{aligned} (d\,'F)(X,Y,Z) &= X\,'F(Y,Z) - Y\,'F(X,Z) + Z\,'F(X,Y) - 'F([X,Y],Z)\\ &\quad + 'F([X,Z],Y) - 'F([Y,Z],X)\\ &= D_x\,'F(Y,Z) + 'F(D_x Y,Z) + 'F(Y,D_x Z) - (D_y\,'F)(X,Z) - 'F(D_y X,Z)\\ &\quad - 'F(X,D_y Z) + (D_z\,'F)(X,Y) + 'F(D_z X,Y) + 'F(X,D_z Y)\\ &\quad - 'F((D_x Y - D_y X),Z) + 'F((D_x Z - D_z X),Y) - 'F((D_y Z - D_z Y),X) \end{aligned}$$

$$(d\,'F)(X,Y,Z) = (D_x\,'F)(Y,Z) + (D_y\,'F)(Z,X) + (D_z\,'F)(X,Y). \tag{2.4}$$

Thus we have

THEOREM 2.2 : On LPS - Riemannian manifold, we have

$$(d\,'F) = 0 \text{ (i.e. } 'F \text{ is closed)} \Leftrightarrow (D_x\,'F)(Y,Z) + (D_y\,'F)(Z,X) + (D_z\,'F)(X,Y) = 0 \quad (2.5)$$

DEFINITION 2.1 : An LP Sasakian manifold on which 'F is closed is called quasi - LP Sasakian - manifold or in short QLPS manifold.

DEFINITION 2.2 : An LP Sasakian manifold on which

$$(D_x A)(Y) + (D_y A)(X) = 0 \quad (2.6)$$

holds, is called K - LP Sasakian Riemannian manifold or KLPS - Riemannian manifold. [2]

3. THEOREMS ON KLPS - RIEMANNIAN MANIFOLD

From (2.1) and (2.6)

$$2\,'F(X,Y) + 0 = [(D_x A)(Y) - (D_y A)(X)] + [(D_x A)(Y) + (D_y A)(X)]$$

$$2\,'F(X,Y) = 2(D_x A)(Y)$$

$$'F(X,Y) = (D_x A)(Y) = -(D_y A)(X).$$

Thus we have

THEOREM 3.1 : On KLPS - Riemannian manifold, we have

$$'F(X,Y) = (D_x A)(Y) = -(D_y A)(X). \quad (3.1)$$

From (3.1)

$$'F(X,Y) = (D_X A)(Y)$$

$$(D_Z\,'F)(X,Y) + 'F(D_Z X,Y) + 'F(X,D_Z Y) = (D_Z D_X A)(Y) + (D_X A)(D_Z Y)$$

$$(D_Z\,'F)(X,Y) + (D_{D_Z X} A)(Y) + (D_X A)(D_Z Y) = (D_Z D_X A)(Y) + (D_X A)(D_Z Y)$$

$$(D_z\,'F)(X,Y) = (D_z D_x A)(Y) - (D_{D_z X} A)(Y) \quad (3.2)$$

$$(D_x\,'F)(Y,Z) = (D_x D_y A)(Z) - (D_{D_x Y} A)(Z) \quad (3.3)$$

$$(D_y\,'F)(Z,X) = (D_y D_z A)(X) - (D_{D_y Z} A)(X) \quad (3.4)$$

Replace Z by X

$$(D_y\ 'F)(X,Z) = (D_y D_x A)(Z) - (D_{D_y Z} A)(Z) \qquad (3.5)$$

Subtracting (3.5) from (3.3) we get

$$(D_x\ 'F)(Y,Z) - (D_y\ 'F)(X,Z) = (D_x D_y A)(Z) - (D_y D_x A)(Z)$$
$$- (D_{D_x Y} A - D_{D_y X} A)(Z) \qquad (3.6)$$

$$(D_x\ 'F)(Y,Z) + (D_y\ 'F)(Z,X) = (D_x D_y A)(Z) - (D_y D_x A)(Z) - (D_{[X,Y]} A)(Z) \qquad (3.7)$$

Using (2.5), we get

$$-(D_z\ 'F)(X,Y) = -A(K(X,Y,Z)),$$
$$(D_z\ 'F)(X,Y) = A(K(X,Y,Z))$$

Thus we have

THEOREM 3.2 : On KLPS - Riemannian manifold, we have

$$(D_z\ 'F)(X,Y) = A(K(X,Y,Z)) \qquad (3.8)$$

We have from (1.3)

$$'F(Y,T) = g(\overline{Y},T) = -A(\overline{Y}) = 0$$
$$'F(Y,T) = 0 \qquad (3.9)$$
$$(D_x\ 'F)(Y,T) + 'F(D_x Y,T) + 'F(Y,D_x T) = 0$$
$$(D_x\ 'F)(Y,T) + 0 - 'F(D_x T,Y) = 0$$

Using (1.1)(e), we get

$$(D_x\ 'F)(Y,T) = 'F(\overline{X},Y)$$
$$(D_x\ 'F)(Y,T) = 'F(\overline{X},Y)$$
$$(D_x\ 'F)(Y,T) = -'F(Y,\overline{X}) \qquad (3.10)$$

Using (1.3)

$$(D_x\ 'F)(Y,T) = -g(\overline{Y},\overline{X})$$
$$(D_x\ 'F)(Y,T) = -g(\overline{X},\overline{Y})$$

Thus we have

THEOREM 3.3 : On KLPS manifold, we have

$$(D_x\ 'F)(Y,T) = -g(\overline{X},\overline{Y}) \qquad (3.11)$$

We have from (1.3)

$$'F(X,Y) = g(\overline{X},Y)$$

$$'F(\overline{X},\overline{Y}) = g(\overline{\overline{X}}\,\overline{Y})$$

$$'F(\overline{X},\overline{Y}) = g(X + A(X)T, \overline{Y})$$

$$'F(\overline{X},\overline{Y}) = g(X, \overline{Y}) + A(X)\, g(T, \overline{Y})$$

$$'F(\overline{X},\overline{Y}) = g(X, \overline{Y}) - A(X)\, A(\overline{Y})$$

$$'F(\overline{X},\overline{Y}) = g(X, \overline{Y})$$

$$'F(\overline{X},\overline{Y}) = 'F(Y,X)$$

$$'F(\overline{X},\overline{Y}) = -'F(X,Y) \quad (3.12)$$

$$(D_z\,'F)(\overline{X},\overline{Y}) + 'F((D_z F)(X) + F(D_z X), \overline{Y}) + 'F(\overline{X}, (D_z F)(Y) + F(D_z Y)$$
$$= -(D_z\,'F)(X,Y) - 'F(D_z X, Y) - 'F(X, D_z Y) \quad (3.13)$$

$$(D_z\,'F)(\overline{X},\overline{Y}) + 'F((D_z F)(X), \overline{Y}) + 'F(F(D_z X), \overline{Y}) + 'F(\overline{X}, (D_z F)(Y))$$
$$+ 'F(\overline{X}, F(D_z \overline{Y})) = -(D_z\,'F)(X,Y) - 'F(D_z X, Y) - 'F(X, D_z Y)$$

Using (3.12), we get

$$(D_z\,'F)(\overline{X},\overline{Y}) + 'F((D_z F)(X), \overline{Y}) + 'F(\overline{X}, (D_z F)(Y)) = -(D_z\,'F)(X,Y) \quad (3.14)$$

Using (1.3), we get

$$(D_z\,'F)(\overline{X},\overline{Y}) + g(\overline{((D_Z F)(X))}, \overline{Y}) - g(\overline{((D_Z F)(Y))}, \overline{X}) = -(D_z\,'F)(X,Y) \quad (3.15)$$

Using (1.1)c, we get

$$(D_z\,'F)(\overline{X},\overline{Y}) - g((D_z F)(X), Y) + A((D_z F)(X))\, A(Y) + g((D_z F)(Y), X)$$
$$- A((D_z F)(Y))\, A(X) = -(D_z\,'F)(X,Y) \quad (3.16)$$

$$(D_z\,'F)(\overline{X},\overline{Y}) + (D_z\,'F)(Y,X) + A((D_z F)(X))\, A(Y) - A((D_z F)(Y))\, A(X) = 0 \quad (3.17)$$

$$(D_z\,'F)(\overline{X},\overline{Y}) + (D_z\,'F)(Y,X) + g(\overline{Z},\overline{X})A(Y) - g(\overline{Z},\overline{Y})A(X) = 0 \quad (3.18)$$

Using (1.1)(c), we get

$$(D_z\,'F)(\overline{X},\overline{Y}) - (D_z\,'F)(X,Y) - g(Z, X)A(Y) + g(Z, Y)A(X) = 0.$$

Thus we have

THEOREM 3.4 : On KLPS - Riemannian manifold, we have

$$(D_z{}'F)(\overline{X},\overline{Y}) - (D_z{}'F)(X,Y) - g(Z,X)A(Y) + g(Z,Y)A(X) = 0 \qquad (3.19)$$

We know (Mishra - 84) [2]

$$(D_z{}'F)(X,Y) = A(X)g(Y,Z) - A(Y)g(X,Z) \qquad (3.20)$$

Barring X and Y

$$(D_z{}'F)(\overline{X},\overline{Y}) = A(\overline{X})g(\overline{Y},Z) - A(\overline{Y})g(\overline{X},Z) \qquad (3.21)$$

$$(D_z{}'F)(\overline{X},\overline{Y}) = 0 \qquad (3.22)$$

Using (3.5), we get

$$(D_z{}'F)(\overline{X},\overline{Y}) = A(K(\overline{X},\overline{Y},Z)) = 0 \qquad (3.23)$$

From (3.19)

$$(D_z{}'F)(X,Y) + A(Y)g(Z,X) - A(X)g(Z,Y) = 0 \qquad (3.24)$$

$$(D_z{}'F)(X,Y) = A(X)g(Z,Y) - A(Y)g(Z,X)$$

Using (1.2)(b), we get

$$(D_z{}'F)(X,Y) = -A(X)g(\overline{Z},\overline{Y}) + A(Y)g(\overline{Z},\overline{X}) \qquad (3.25)$$

Using (3.8), we get

$$(D_z{}'F)(X,Y) = A(Y)(D_z{}'F)(X,T) - A(X)(D_z{}'F)(Y,T). \qquad (3.26)$$

Thus we have

THEOREM 3.5 : On KLPS - Riemannian manifold, we have

$$(D_z{}'F)(X,Y) = A(Y)(D_z{}'F)(X,T) - A(X)(D_z{}'F)(Y,T).$$

DEFINITION 3.1 : On KLPS - Riemannian manifold structure {F, T, A} is said to be normal if

$$\underset{O}{N}(X,Y) = 0 \qquad (3.27)$$

where $\underset{O}{N}(X,Y)\ \underset{F}{N}(X,Y) + dA(X,Y)T = 0.$ [2]

$$\underset{O}{N}(X,Y) = [\overline{X},\overline{Y}] + \overline{\overline{[X,Y]}} - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]} + \{X\,A(Y) - Y\,A(X) - A(X,Y)\}T$$

$$= D_{\overline{X}}\overline{Y} - D_{\overline{Y}}\overline{X} + [X,Y] + A([X,Y])T - \overline{D_{\overline{X}}Y} + \overline{D_Y\overline{X}} - \overline{D_X\overline{Y}} + \overline{D_{\overline{Y}}X}$$
$$+\{X\,A(Y) - Y\,A(X) - A(X,Y)\}T$$

$$= (D_{\bar{X}} F)(Y) + F(D_{\bar{X}} Y) - (D_{\bar{Y}} F)(X) - F(D_{\bar{Y}} X) + D_X Y - D_Y X$$

$$+ A(D_X Y)T - A(D_Y X)T - \overline{D_{\bar{X}} Y} + \overline{(D_Y F)(X)} + \overline{\overline{D_Y X}}$$

$$- \overline{(D_X F)(Y)} - \overline{\overline{D_X Y}} + \overline{D_{\bar{Y}} X} + (D_X A)(Y)T + A(D_X Y)T$$

$$- (D_Y A)(X)T - A(D_Y X)T - A(D_X Y)T + A(D_Y X)T,$$

$$= (D_{\bar{X}} F)(Y) + \overline{D_{\bar{X}} Y} - (D_{\bar{Y}} F)(X) - (\overline{D_{\bar{Y}} X}) + D_X Y - D_Y X + A(D_X Y)T$$

$$- A(D_Y X)T - \overline{D_{\bar{X}} Y} + \overline{(D_Y F)(X)} + D_Y X + A(D_Y X)T - \overline{(D_X F)(Y)}$$

$$- D_X Y - A(D_X Y)T + \overline{D_{\bar{Y}} X} + (D_X A)(Y)T + A(D_X Y)T$$

$$- (D_Y A)(X)T - A(D_Y X)T - A(D_X Y)T + A(D_Y X)T$$

so we have

$$\underset{O}{N}(X,Y) = (D_{\bar{X}} F)(Y) - (D_{\bar{Y}} F)(X) + A(D_X Y)T - A(D_Y X)T + \overline{(D_Y F)(X)}$$

$$- \overline{(D_X F)(Y)} - A(D_X Y)T + (D_X A)(Y)T - (D_Y A)(X)T + A(D_Y X)T \quad (3.28)$$

Differentiating covariantly the equation

$$\overline{\overline{Y}} = F\overline{Y}$$

and using (1.1)e and (1.1)b, we get

$$\overline{(D_X F)(Y)} = -(D_X F)\overline{Y} + (D_X A)(Y)T + A(Y)(\overline{X}). \quad (3.29)$$

Using (3.28) and (3.29), we see that

$$\underset{O}{N}(X,Y) = 0 \Leftrightarrow (D_{\bar{X}} F)(Y) - (D_{\bar{Y}} F)(X) + A(D_X Y)T - A(D_Y X)T - (D_Y F)\overline{X}$$

$$+ (D_Y A)(X)T + A(X)\overline{Y} + (D_X F)\overline{Y} - (D_X A)(Y)T - A(Y)\overline{X}$$

$$- A(D_X Y)T + (D_X A)(Y)T - (D_Y A)(X)T + A(D_Y X)T = 0$$

$$\Leftrightarrow (D'_{\bar{X}} F)(Y) - (D_{\bar{Y}} F)(X) - (D_Y F)\overline{X} + A(X)\overline{Y} + (D_X F)\overline{Y} - A(Y)\overline{X} = 0$$

$$\Leftrightarrow (D_{\bar{X}} F)(Y) - (D_{\bar{Y}} F)X + (D_X F)\overline{Y} - (D_Y F)\overline{X} - A(Y)\overline{X} + A(X)\overline{Y} = 0$$

So we get $\underset{O}{N}(X, Y) = 0$ if and only if

$$(D_{\bar{X}}F)(Y)-(D_{\bar{Y}}F)(X)+(D_XF)\bar{Y}-(D_YF)\bar{X}-A(Y)(\bar{X})+A(X)(\bar{Y})=0 \qquad (3.30)$$

which is equivalent to

$$g((D_{\bar{X}}F)Y,Z)-g((D_{\bar{Y}}F)X,Z)+g((D_XF)\bar{Y},Z)-g((D_YF)\bar{X},Z)$$
$$-A(Y)g(\bar{X},Z)+A(X)g(\bar{Y},Z)=0$$

or

$$(D_{\bar{X}}{'F})(Y,Z)+(D_{\bar{Y}}{'F})(Z,X)+(D_X{'F})(\bar{Y},Z)-(D_Y{'F})(\bar{X},Z)$$
$$-A(Y)g(\bar{X},Z)+A(X)g(\bar{Y},Z)=0 \qquad (3.31)$$

Using (2.4), (3.31) becomes

$$(d'F)(\bar{X},Y,Z)-(D_Y{'F})(Z,\bar{X})-(D_Z{'F})(\bar{X},Y)+(d'F)(X,\bar{Y},Z)$$
$$-(D_X{'F})(\bar{Y},Z)-(D_Z{'F})(X,\bar{Y})+(D_X{'F})(\bar{Y},Z)-(D_Y{'F})(\bar{X},Z)$$
$$-A(Y)g(\bar{X},Z)+A(X)g(\bar{Y},Z)=0$$

or

$$(d'F)((\bar{X},Y,Z)+(d'F)(X,\bar{Y},Z)-(D_Z{'F})(\bar{X},Y)-(D_Z{'F})(X,\bar{Y}))$$
$$-A(Y)g(\bar{X},Z)+A(X)g(\bar{Y},Z)=0 \qquad (3.32)$$

Since on a LP - Sasakian Riemannian manifold, we have (d 'F) = 0, the above equation is equivalent to

$$(D_Z{'F})(\bar{X},Y)+(D_Z{'F})(X,\bar{Y})=A(X)g(\bar{Y},Z)-A(Y)g(\bar{X},Z) \qquad (3.33)$$

thus we have

THEOREM 3.6 : KLPS - Riemannian structure is normal if (3.33) holds.

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