J. T. S.
Vol. 5 (2011), pp.61-66
https://doi.org/10.56424/jts.v5i01.10441

Peristaltic Transport in Elliptic Tubes

Vijai Shanker Verma

Department of Mathematics & Statistics
DDU Gorakhpur University, Gorakhpur, India
E-mail : vsverma01@yahoo.co.in
(Received: 12 January, 2010)

Abstract

In this paper, the peristaltic transport of a Newtonian bio-fluid through a tube of an elliptic cross-section is studied by considering the longitudinal motion of the wall. It is shown that the transport of the bio-fluid increases as the eccentricity of the tube decreases or the amplitude of the wave increases.

Keywords and Phrases : Peristaltic transport, Newtonian bio-fluid, elliptic tube, longitudinal motion.
2000 AMS Subject Classification : 76Z05.

1. Introduction

The peristaltic transport of bio-fluids in different geometries has many applications in Mathematics, Biology and Engineering. Peristaltic action i.e. peristalsis is the mechanism by which a bio-fluid is transported through a distensible tube when contraction or expansion wave propagate along its length. Peristalsis appears to be the mechanism for fluid transport in many physiological situations such as : transport of urine through ureter, swallowing of food through oesophagus, chyme movement through small intestine, the colonic transport in the large intestine, etc. The study of peristaltic transport of bio-fluid is based on the principles of fluid mechanics involving interaction of fluid motion in tubes with flexible boundaries. In such investigations, an approximate model of the physiological system is maid by keeping in view the nature of the physiological fluid (i.e. its Newtonian or non-Newtonian character, its viscosity, its behaviour as a two-phase mixture, etc.), the nature of the tube and other processes involved. The initial mathematical models of peristalsis obtained by a train of sinusoidal waves in an infinitely long symmetric channel or tube have been investigated by Shapiro et al. [1] and Fung and Yih [2]. After these studies, many investigations were done to understand the peristaltic action for Newtonian and non-Newtonian fluids in different situations. The importance of the study of peristaltic transport in an asymmetric channel has been brought out by Eytan and Elad [3] with an application in intra-uterine-fluid flow in a non-pregnant uterus. After this study, some investigations were done to understand the mechanism of peristalsis in asymmetric channels. Mishra and Rao [4] have investigated the flow in an asymmetric channel generated by peristaltic wave propagating on the walls with different amplitudes and phases. Rao and Mishra [5] discussed the non-linearity and curvature effects on the peristaltic flow of a viscous fluid in asymmetric channel when the ratio of the channel width to the wavelength is small. Pioneering work in this area has been also done by several investigators including Shukla and Gupta [6], Misra and Pandey [7], Subba Reddy et al. [8], Lykoudis and Roos [9], Barton and Raynor [10].

In real situations, when sinusoidal waves are taken into account, the lumen of the tube, in general, does not remain circular but it becomes elliptic due to heterogeneous structure of its wall. (Lykoudis and Roos [9]).

Therefore, in this paper, the peristaltic transport of a Newtonian fluid in an elliptic tube has been investigated by taking into account the longitudinal motion of the wall which may be related to electrophysiological activity of the organ. The expression for volumetric flow rate i.e. flux of the fluid is calculated and studied.

2. Mathematical Formulation

Consider the axi-symmetric flow of a Newtonian fluid in an elliptic tube with a sinusoidal wave propagating along its wall as shown in Fig. 1.

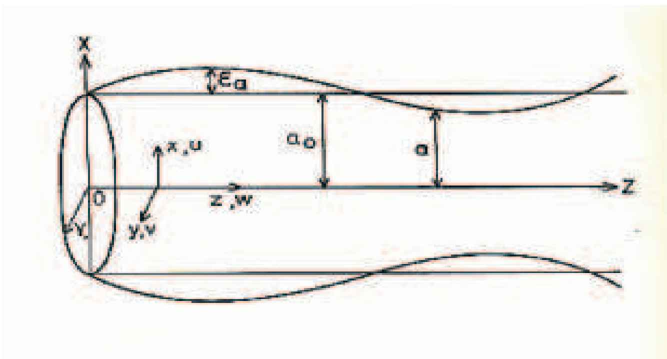

Fig. 1. Peristaltic Transport in Elliptic Tubes

The equations of momentum and continuity governing the motion of fluid under long wavelength approximation and neglecting inertia terms in stationary frame of reference are written as:

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = \frac{1}{\mu}\frac{\partial P}{\partial Z} \tag{1}$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} = 0 \tag{2}$$

The boundary conditions are taken as:

$$\frac{\partial W}{\partial X} = 0, \qquad \frac{\partial W}{\partial Y} = 0 \qquad \text{at } X = 0, Y = 0, \tag{3}$$

$$W = W_0 \sin\frac{2\pi}{\lambda}(Z - Ct), \quad \text{at } \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1, \tag{4}$$

where $W_0 \sin\frac{2\pi}{\lambda}(Z - Ct)$ is the longitudinal wave velocity.

3. Method of Solution

To study the problem in what follows, we transform the stationary co-ordinates (X, Y, Z) to moving co-ordinates (x, y, z) with U, V, W and u, v, w as respective fluid velocity components in these co-ordinates which move with the wave velocity C in the positive Z direction, as follows:

$$X = x, \quad Y = y, \quad Z = z + Ct, \quad U = u, \quad V = v, \quad W = w + C. \tag{5}$$

Thus, the partial differential equations (1)-(2) and the boundary conditions (3)-(4) in the moving co-ordinates in non-dimensional form can be written as:

$$\frac{\partial^2 \overline{w}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{w}}{\partial \overline{y}^2} = \frac{1}{\mu}\frac{\partial \overline{P}}{\partial \overline{z}} \tag{6}$$

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} + \frac{\partial \overline{w}}{\partial \overline{z}} = 0 \tag{7}$$

$$\frac{\partial \overline{w}}{\partial \overline{x}} = 0, \qquad \frac{\partial \overline{w}}{\partial \overline{y}} = 0 \qquad \text{at } \overline{x} = 0, \overline{y} = 0, \tag{8}$$

$$\overline{w} = w_0 \sin 2\pi\,\overline{z} - 1, \quad \text{at } \frac{\overline{x}^2}{\overline{a}^2} + \frac{\overline{y}^2}{\overline{b}^2} = 1, \tag{9}$$

where

$$\overline{a} = 1 + \overline{\epsilon}_a \sin 2\pi\overline{z} \qquad \text{and} \qquad \overline{b} = \overline{b}_0 + \overline{\epsilon}_b \sin 2\pi\overline{z} \tag{10}$$

and

$$
\begin{aligned}
&\overline{x} = \frac{x}{a_0}, \qquad \overline{y} = \frac{y}{a_0}, \quad \overline{z} = \frac{z}{\lambda}, \qquad \overline{u} = \frac{u}{C}, \quad \overline{v} = \frac{v}{C}, \\
&\overline{w} = \frac{w}{C}, \qquad \overline{w}_0 = \frac{w_0}{C}, \quad \overline{\mu} = \frac{\mu}{\mu_0}, \qquad \overline{P} = \frac{a_0^2 P}{\mu_0 \lambda C}, \quad \overline{\epsilon}_a = \frac{\epsilon_a}{a_0} \\
&\overline{\epsilon}_b = \frac{\epsilon_b}{a_0}, \qquad \overline{a} = \frac{a}{a_0}, \quad \overline{b} = \frac{b}{a_0}, \qquad \overline{b}_0 = \frac{b_0}{a_0}.
\end{aligned}
\tag{11}
$$

Solving (6) and using boundary conditions (8) and (9), w is obtained as (dropping the bars for convenience):

$$
w = w_0 \sin 2\pi z - 1 - \frac{1}{2\mu}\frac{dP}{dz}\left(\frac{a^2 b^2}{a^2 + b^2}\right)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{12}
$$

Now, the dimensionless flux ($\overline{q}$) is given by $\overline{q} = q/\pi a_0^2 C$, where q is the flux in the moving co-ordinate system, which is given by

$$
q = 4 \int_0^1 \int_0^{b(1-x^2/a^2)^{1/2}} w \, dx \, dy \tag{13}
$$

which on integration and using (12) yields

$$
q = ab(w_0 \sin 2\pi z - 1) - \frac{1}{4\mu}\frac{dP}{dz}\left(\frac{a^3 b^3}{a^2 + b^2}\right). \tag{14}
$$

Since, the pressure drop ($\triangle P = P_0 - P_1$) across one wavelength (λ) is same in each case whether measured in stationary system or in moving co-ordinate system, therefore, it can be calculated as follows:

$$
\triangle P = -\int_0^1 \frac{dP}{dz} \, dz \tag{15}
$$

Then, from (14) and (15), we have

$$
q = \frac{\triangle P}{I_1} + \frac{I_2}{I_1} \tag{16}
$$

where

$$
I_1 = \int_0^1 \frac{4\mu(a^2 + b^2)}{a^3 b^3} \, dz \tag{17}
$$

and

$$
I_2 = \int_0^1 \frac{4\mu(a^2 + b^2)}{a^2 b^2} (w_0 \sin 2\pi z - 1) \, dz \tag{18}
$$

For $b = \alpha a$, $\alpha \leq 1$, the flux can be calculated and is given by

$$
q = \frac{\alpha^3 (1 - \epsilon_a^2)^{7/2} \triangle P}{2\mu(1 + \alpha^2)(2 + 3\epsilon_a^2)} - \frac{2\alpha(1 - \epsilon_a^2)^2}{(2 + 3\,\epsilon_a^2)}(1 + \epsilon_a w_0). \tag{19}
$$

The flux q is related to the flux Q of the stationary co-ordinate system as follows:

$$Q = 4 \int_0^a \int_0^{b(1-x^2/a^2)^{1/2}} (w+1)\, dx\, dy = q + ab. \tag{20}$$

The time averaged flux $\overline{Q}$ for a complete time period $T = \lambda/C$ is given as

$$\overline{Q} = \frac{1}{T} \int_0^T Q\, dt. \tag{21}$$

The final expression of $\overline{Q}$ can be obtained as follows:

$$\overline{Q} = \frac{\alpha^3(1-\epsilon_a^2)^{7/2} \triangle P}{2\mu(1+\alpha^2)\,(2+3\epsilon_a^2)} + \frac{\alpha(16-\epsilon_a^2)\,\epsilon_a^2}{2(2+3\epsilon_a^2)} - \frac{2\alpha(1-\epsilon_a^2)^2\,\epsilon_a w_0}{(2+3\,\epsilon_a^2)}. \tag{22}$$

Again, when $\alpha = 1$ and therefore, $b = a$, then from (22), the result of circular case model can be obtained. In this case, for $\triangle P = 0$, and $w_0 = 0$, the expression for $\overline{Q}$ corresponds to Barton and Raynor [10] model.

4. Results and Discussion

From equation (22), it is noted that for $\triangle P > 0$ and $w_0 < 0$, $\overline{Q}$ increases as α increases for a fixed value of ϵ_a, which implies that the flux is maximum, when the tube is circular for the same set of parameters. From thios equation, it is also seen that $\overline{Q}$ varies linearly with $\triangle P$ and w_0. We further note that $\overline{Q}$ always increases as ϵ_a increases. These results are also observed by plotting equation (22) in the Fig. 2 and Fig. 3 for typical set of parameters. The study

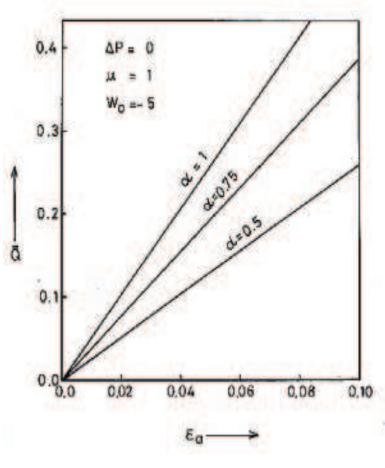

Fig. 2. Variation of $\overline{Q}$ with ϵ_α for different α

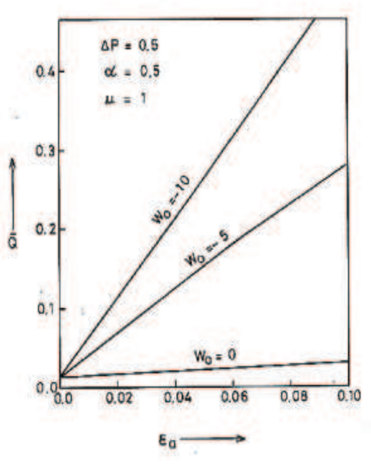

Fig. 3. Variation of $\overline{Q}$ with ϵ_α for different w_0

presented here, suggests one of the reasons of why in physiological system, tubes are generally circular.

Acknowledgement

The author is thankful to Professor Manju Agarwal, Department of Mathematics & Astronomy, Lucknow University, Lucknow for her valuable suggestions.

References

1. Shapiro, A. H., Jaffrin, M. Y. and Weinberg, S. L. : Peristaltic pumping with long wavelengths at low Reynolds number, J. Fluid Mech., 37 (1969), 799-825.
2. Fung, Y. C. and Yih, C. S. : Peristaltic transport, J. Appl. Mech., 35 (1968), 669-675.
3. Eytan, O. and Elad, D. : Analysis of Intra-Uterine fluid motion induced by uterine contraction, Bull. Math. Biology, 61 (1999), 221-238.
4. Mishra, M. and Rao, A. R. : Peristaltic transport of a non-Newtonian fluid in an asymmetric channel, ZAMP, 54 (2003), 532-550.
5. Rao, A. R. and Mishra, M. : Non-linear and curvature effects on peristaltic flow of a viscous fluid in an asymmetric channel, Acta Mech., 168 (2004), 35-59.
6. Shukla, J. B. and Gupta, S. P. : Peristaltic transport of a power law fluid with variable consistency, J. Biomech. Engineering, 104 (1982), 182.
7. Misra, J. C. and Pandey, S. K. : Peristaltic transport of physiological fluids, Biomathematics edited by J. C. Misra, World Scientific Publishing Co. Pvt. Ltd. Singapore, (2006), 167-193.
8. Subba Reddy, M. V., Rao, R. C. and Sreenadh, S. : Peristaltic motion of a power law fluid in an asymmetric channel, Int. J. Non-linear Mech., 42 (2007), 1153-1161.
9. Lykoudis, P. S. and Roos, R. : The fluid mechanics of the ureter from a lubrication point of view, J. Fluid Mech., 43 (1970), 661-674.
10. Barton, C. and Raynor, S. : Peristaltic flow in tubes, Bull. Math. Biophy., 30 (1968), 663.