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ON A VECTOR FIELD ANALOGOUS TO CONCURRENT VECTOR FIELD IN A FINSLER SPACE

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SUMMARY : Concurrent vector fields in a Finsler space were first of all defined and studied by Tachibana [6], followed by Matsumoto [2] and others. Recently in 2004, Rastogi and Dwivedi [4] studied the existence of concurrent vector fields in a Finsler space of n-dimensions and showed that the definition in its present form is unsuitable. Further they gave a modified definition of a concurrent vector field in a Finsler space of n-dimensions as follows:

DEFINITION 1 : A vector field $X^i(x)$ in a Finsler space c is said to be concurrent vector field in F^n if it satisfies i) $X^i A_{ijk} = \alpha h_{jk}$ and ii) $X^i|_j = -\delta^i_j$, where α is an arbitrary non-zero scalar function of x and y.

In this paper an attempt has been made to study, vector fields X^i in F^2, whose v - covariant derivative satisfies a relation of type $X^i|_j = \rho(x, y)\, h^i_j$. It is interesting to note that such vector fields do exist in F^2. We have called such vector fields neo-concurrent vector fields as these vector fields seem to be analogous to concurrent vector fields in a Finsler space. In this paper we have proved that a Finsler space with neo-concurrent vector field is a P* - Finsler space Izumi [1].

1. INTRODUCTION

Let F^n be an n - dimensional Finsler sapce with metric function L(x,y), metric tensor $g_{ij}(x,y)$, angular metric tensor h_{ij} and torsion tensor C_{ijk}. The h- and v-covariant derivatives of a vector field X^i are defined as Rund [5].

$$X_j^i|_k = \delta_k X_j^i + X_j^m F^i_{mk} - X_m^i F^m_{jk} \tag{1.1}$$

and

$$X_j^i|_k = \Delta_k X_j^i + X_j^m C^i_{mk} - X_m^i C^m_{jk}, \tag{1.2}$$

where $\delta_k = \partial_k - N_k^m \Delta_m$, ∂_j and Δ_j respectively denote partial differentiation with respect to x^i and y^i.

The two torsion tensors A_{ijk} and P_{ijk} are defined as

$$A_{ijk} = LC_{ijk}, 2C_{ijk} = \Delta_k g_{ij}, P_{ijk} = A_{ijk|0} = A_{ijk|r} 1^r, 1^i = y^i / L \tag{1.3}$$

The second and third curvature tensors are given as

$$P_{ijkh} = \varsigma_{(i,j)} \{A_{jkh|i} + A_{ikr} P^r{}_{jh}\} \tag{1.4}$$

and

$$S_{ijkh} = \varsigma_{(h,k)} \{A_{ihr} A^r{}_{jk}\} \tag{1.5}$$

where $\varsigma_{(j,k)}$ means interchange of indices j and k and subtraction.

2. TWO DIMENSIONAL FINSLER SPACE

In a two dimensional Finsler space F^2, it is known that [3] $g_{ij} = l_i l_j + m_i m_j$, $h_{ij} = m_i m_j$, $l_{i|j} = 0$, $m_{i|j} = 0$, $l^i|_j = L^{-1} m_i m_j$. Let $X^i(x)$ be a vector field in F^2, which is a function of x alone, then, we can easily write

$$X^i|_j = X^r C^i_{rj}. \tag{2.1}$$

Substituting the value of $C^i_{rj} = Cm^i m_r m_j$ in (2.1) we get

$$X^i|_j = \rho(x,y)h^i_j, \tag{2.2}$$

where $\rho(x,y) = X^r C_r$.

With the help of equations (2.1) and (2.2) we define following:

DEFINITION 2.1 : A vector field $X^i(x)$, in F^2, satisfying (2.2) shall be called neo-concurrent vector field.

In F^2, $l_i m^i = 0$, implies $l_i m^i|_j = -L^{-1} m_j$. Similarly $m_i m^i|_j = -Cm_j$. Thus we can express

$$m^i|_j = -L^{-1} l^i m_j - Cm^i m_j \quad \text{or} \quad m_i|_j = -L^{-1} l_i m_j + Cm_i m_j \tag{2.3}$$

and

$$\Delta_j m_i = -L^{-1} l_i m_j + 2Cm_i m_j, \tag{2.4}$$

Let us assume that we take

$$X^i = \alpha(x,y) l^i + \beta(x,y) m^i, \tag{2.5}$$

where $\alpha(x,y)$ and $\beta(x,y)$ are scalar functions to be determined.

Multiplying (2.5) by m_i and using $X^r C_r = \rho(x,y)$, we get $C\beta(x,y) = \rho(x,y)$ and $\alpha^2 = |X|^2 - (\rho/C)^2$. Differentiating equation (2.5) we get

$$X^i|_j = \alpha L^{-1} h^i_j + \alpha|_j l^i + \beta|_j m^i + \beta(-L^{-1} l^i m_j - Cm^i m_j), \tag{2.6}$$

which by virtue of (2.3) leads to

$$\alpha|_j = \beta L^{-1} m_j, \beta|_j = (2\beta C - \alpha L^{-1}) m_j. \tag{2.7}$$

From equation (2.7), we can easily obtain $\alpha\alpha|_j + \beta\beta|_j = 2C\beta^2 m_j$, which implies

$$(\alpha^2 + \beta^2)|_j = 4C\beta^2 m_j \tag{2.8}$$

Hence we have:

THEOREM 2.1 : In a two-dimensional Finsler space F^2, if a noe-concurrent vector field X^i is expressed as (2.5), its coefficients α and β satisfy (2.8).

Remark : Here we shall not be taking the case of a unit vector X^i as it will lead to a situation where either $X^i = 1^i$, which is against the hypothesis or the space is Riemannian.

3. THREE DIMENSIONAL FINSLER SPACE

In a three dimensional Finsler space F^3, it is known that [3] $g_{ij} = l_i l_j + m_i m_j + n_i n_j$, $h_{ij} = m_i m_j + n_i n_j$, $l_{i|j} = 0$, $m_{i|j} = n_i h_j$, $n_{i|j} = -m_i h_j$, $l_i|_j = L^{-1}(m_i m_j + n_i n_j)$. Let X^i be a vector field in F^3, which is a function of x alone, then we can again get (2.1). Since we know that [3]

$$C_{ijk} = C_{(1)} m_i m_j m_k - C_{(2)}(m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k)$$
$$+ C_{(3)}(m_i n_j n_k + m_j n_k n_i + m_k n_i n_j), \tag{3.1}$$

therefore, with the help of (2.1) and (3.1) we get

$$X^i|_j = (2\rho C^{-1} C_{(1)} - \varphi C_{(2)}) m^i m_j + (2\rho C^{-1} C_{(3)} + \varphi C_{(2)}) n^i n_j$$
$$- (2\rho C^{-1} C_{(2)} - \varphi C_{(3)})(m^i n_j + n^i m_j), \tag{3.2}$$

where $X^r C_r = 2\rho$ and $X^r n_r = \varphi$.

Comparing (2.2) and (3.1), we get

$$2\rho C^{-1} C_{(1)} - \varphi C_{(2)} = \rho, \quad 2\rho C^{-1} C_{(3)} + \varphi C_{(2)} = \rho, \quad 2\rho C^{-1} C_{(2)} - \varphi C_{(3)} = 0 \tag{3.3}$$

which easily yields

$$C_{(1)} = C(2\rho^2 + C^2\varphi^2)/(4\rho^2 + C^2\varphi^2), \quad C_{(2)} = C\varphi C_{(3)}/2\rho,$$
$$C_{(3)} = 2\rho^2 C/(4\rho^2 + C^2\varphi^2). \tag{3.4}$$

Hence we have:

THEOREM 3.1 : In a three dimensional Finsler space F^3, having a neo-concurrent vector field satisfying (2.2), $C_{(1)}, C_{(2)}$ and $C_{(3)}$ are given by (3.4).

Let us assume that X^i is a vector field which is expressible as

$$X^i = \alpha(x,y)l^i + \beta(x,y)m^i + \gamma(x,y)n^i, \tag{3.5}$$

where $\alpha(x,y)$, $\beta(x,y)$ and $\gamma(x,y)$ are to be determined.

Differentiating equation (3.5), using (2.1), (3.1) and (3.2) we obtain

$$\beta(x,y) = (4\rho C^{-1} - \varphi)/2 + C_{(2)}(C_{(3)} - C_{(2)})\varphi/2C_{(3)}^2,$$

$$\gamma(x,y) = (C_{(2)}^2 + C_{(3)}^2)\varphi/2C_{(2)}C_{(3)} \tag{3.6}$$

such that

$$(C_{(2)}^2 - C_{(3)}^2)(C_{(3)}^2 + C_{(1)}C_{(2)}) + C_{(1)}C_{(2)}^2 C_{(3)} = 0. \tag{3.7}$$

Furthermore, the value of $\alpha(x,y)$ is obtained from $|X^2| = \alpha^2 + \beta^2 + \gamma^2$, where X is the magnitude of the vector X^i. Hence we have:

THEOREM 3.2 : In a three dimensional Finsler space F^3, having a neo-concurrent vector field satisfying (2.2) and (3.1), $C_{(1)}, C_{(2)}$ and $C_{(3)}$ satisfy equation (3.7), while $\beta(x,y)$ and $\gamma(x,y)$ are given by (3.6).

4. NEO-CONCURRENT VECTOR FIELDS IN F^n

DEFINITION 4.1 : A vector field $X^i(x)$ shall be called neo-concurrent vector field in a Finsler space of n-dimensions F^n, if it satisfies

$$X^i|_j = \rho(x,y)h^i_j, \tag{4.1}$$

where $\rho(x,y)$ is an arbitrary non-zero scalar function of x and y.

From equations (1.2) and (4.1), we can obtain (2.1) which implies

$$\rho(x,y)=(n-1)^{-1}X^r C_r. \qquad (4.2)$$

From (4.1) and (4.2) we can obtain

$$X^i|_j=(n-1)^{-1}X^r C_r h^i_j. \qquad (4.3)$$

Similarly from (1.1), we can obtain

$$X^i{}_{|j}=\partial_j X^i+X^m F^i{}_{mj}. \qquad (4.4)$$

Hence we have :

THEOREM 4.1 : The v- and h-covariant derivatives of a neo-concurrent vector field X^i are respectively given by (4.3) and (4.4).

Differentiating equation (2.2) partially with respect to y^k, using X^r as a function of x, $C^i{}_{rj}$ as homogeneous function of degree -1 in y and h^i_j as homogeneous function of degree zero in y, we get $\rho(x,y)$ to be homogeneous function of degree - 1 in y. Hence we have:

THOREM 4.2 : For a neo-concurrent vector field X^i satisfying (2.2), the scalar $\rho(x,y)$ is homogeneous function of degree -1 in y.

From equations (4.1) and (4.4) on contraction for i and j we can obtain

$$X^i|_i=(n-1)\rho(x,y)=X^r C_r. \qquad (4.5)$$

Since $\rho(x,y)\neq 0$, therefore from equation (4.5) we can easily obtain

COROLLARY 1 : The divergence of a neo-concurrent vector field will not be zero.

5. CURVATURE TENSORS

From equation (4.4), we can further obtain by virtue of (2.1) and (2.2)

$$\varsigma_{(j,k)}\{X^i|_j|_k-h^i_j(\Delta_k\rho+\rho L^{-1}l_k)\}=0. \qquad (5.1)$$

Substituting the value of

$$\varsigma_{(j,k)}\{X^i|_j|_k-X^m\Delta_k C^i{}_{mj}\}=X^m L^{-2} S^i{}_{mkj}, \qquad (5.2)$$

from [5] in equation (5.1), we obtain on simplification

$$\varsigma_{(j,k)}\{h^i_j(\Delta_k\rho+\rho L^{-1}l_k)-X^m\Delta_k C^i_{mj}\}=X^m L^{-2}S^i_{mkj}. \tag{5.3}$$

Hence we have:

THEOREM 5.1 : The third curvature tensor of a neo-concurrent vector field in a Finsler space F^n satisfies (5.3).

If in equation (5.3), $\Delta_k\rho+\rho L^{-1}l_k=0$, we can obtain

$$\varsigma_{(j,k)}\{X^m(\Delta_j C^i_{mk}+L^{-2}S^i_{jmk})\}=0. \tag{5.4}$$

Conversely, if equation (5.4) is satisfied, $(n-2)(\Delta_k\rho+\rho l_k L^{-1})=0$, i.e., either $n=2$ or $\Delta_k\rho+\rho L^{-1}l_k=0$. Hence we have:

THEOREM 5.2 : In a Finsler space F^n $(n>2)$, the necessary and sufficient condition for equation (5.4) to be satisfied is given by $\Delta_k\rho+\rho l_k L^{-1}=0$.

Using equations (1.1) and (1.2), we obtain on simplification

$$X^i\big|_{j|k}-X^i{}_{|k}\big|_j=X^r(C^i_{rj|k}+F^i_{rm}C^m_{jk}-\Delta_j F^i_{kr})+(\partial_r X^i)C^r_{jk}. \tag{5.5}$$

Since we know from Ricci identity [3]

$$X^i\big|_{j|k}-X^i{}_{|k}\big|_j=X^i_{|h}C^h_{kj}+X^i\big|_h P^h_{kj}-X^h P^i_{hkj}, \tag{5.6}$$

therefore, comparing equations (5.5) and (5.6) and using (2.2) we obtain

$$X^r(P^i_{rkj}+C^i_{rj|k}-\Delta_j F^i_{kr})=\rho(x,y)P^i_{kj}. \tag{5.7}$$

Hence we have:

THEOREM 5.3 : A neo-concurrent vector field X^i, in a Finsler space F^n satisfies equation (5.7).

Similarly from Ricci identity [3]

$$X^i_{|k|j}-X^i_{|j|k}=X^h R^i_{hkj}-X^i\big|_h R^h_{kj}, \tag{5.8}$$

therefore, we can obtain by virtue of (4.1) and (2.2) the following relation

$$X^i_{|k|j} - X^i_{|j|k} = X^h R^i_{hkj} - \rho(x,y)(R^i_{kj} - l^i l_h R^h_{kj}), \tag{5.9}$$

which leads to

THEOREM 5.4 : The necessary and sufficient condition for a neo-concurrent vector field X to satisfy $X^i_{|k|j} = X^i_{|j|k}$ is that the curvature tensor R^i_{hkj} is satisfying $X^h R^i_{hkj} = \rho(x,y)(R^i_{kj} - l^i l_h R^h_{kj})$.

Since we know that $X^r C^i_{rj} = \rho h^i_j$, therefore taking h-covariant derivative of this equation we can obtain

$$X^r C^i_{rj|k} + X^r_{|k} C^i_{rj} = \rho_{|k} h^i_j. \tag{5.10}$$

Multiplying equation (5.10) by y^k and using equation (1.3) and theorem 4.2, we get

$$X^r P^i_{rj} + X^r_{|0} C^i_{rj} = \rho_{|0} h^i_j. \tag{5.11}$$

Let F^n be a P^*-Finsler space Izumi [1] satisfying $P^i_{rj} = \theta C^i_{rj}$, for some suitable θ, then for $\varphi = \rho_{|0} - \rho\theta$, equation (5.11) on simplification gives

$$X^r_{|0} C^i_{rj} = \varphi h^i_j. \tag{5.12}$$

Conversely, if equation (5.12) is satisfied, then equation (5.11) gives $X^r(P^i_{rj} - \theta C^i_{rj}) = 0$. Hence we have:

THEOREM 5.5 : If X^r is a neo-concurrent vector field in F^n, its covariant derivative $X^r_{|0}$ is neo-concurrent vector field in a P^*-Finsler space. Conversely, if both X^r and $X^r_{|0}$ are neo-concurrent in F^n, X^r satisfies $X^r(P^i_{rj} - \theta C^i_{rj}) = 0$.

In case of a Berwald space [3], $C^i_{jh|k} = 0$ which on application in equation (5.10) gives

$$X^r_{|k} C^i_{rj} = \rho_{|k} h^i_j. \tag{5.13}$$

From equation (5.13), we can easily obtain

$$X^r{}_{|0} C^i{}_{rj} = \rho_{|0} h^i{}_j, \tag{5.14}$$

which when substituted in (5.11) leads to $X^r P^i{}_{rj} = 0$. Hence we have:

THEOREM 5.6 : In an n-dimensional Berwald space, a neo-concurrent vector field X^i satisfies $X^r P^i{}_{rj} = 0$.

In case of a Landsberg space [3], $P_{ijkh} = 0$, therefore equation (5.7) reduces to

$$X^k (C^i{}_{rj|k} - \Delta_j F^i{}_{kr}) = 0. \tag{5.15}$$

Hence we have:

THEOREM 5.7 : An n-dimensional Landsberg space, having neo-concurrent vector field X^i, satisfies (5.15).

REFERENCES

1. Izumi, H. : On P* -Finsler space I., Memo, Defence Academy., 16(1976), 133-138.
2. Matsumoto, M. : Finsler spaces admitting concurrent vector field, Tensor, N.S., 28(1974), 239-249.
3. Matsumoto, M. : Foundations of Finsler Geometry and special Finsler spaces, Kaiseisha Press, Otsu, Japan, 1986.
4. Rastogi, S.C. and Dwivedi, A.K. : On the existence of concurrent vector fields in a Finsler space, Tensor, N.S., 65(2004), 48-54.
5. Rund, H. : The differential geometry of Finsler spaces, Springer-Verlag, 1959.
6. Tachibana, S. : On Finsler spaces which admit concurrent vector field, Tensor, N.S., 1(1950), 1-5.