J. T. S
Vol. 2 (2008), 115-127
https://doi.org/10.56424/jts.v2i00.9965

A Planar Two-Layer Unsteady State Model for Mucociliary Transport

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(Received : March 11, 2008)

Abstract

In this paper, a planar two-layer unsteady state fluid model is proposed to study the transport of mucus in the respiratory tracts due to cillia beating and airmotion by considering mucus as a viscoelastic fluid. The serous layer is considered to be a low viscosity Newtonian fluid. The effect of airmotion due to forced expiration and other processes is considered by prescribing shear stress at the mucus-air interface. It is shown that the mucus transport increases as the pressure drop, acceleration due to gravity and shear stress due to airmotion increase. Also, it is shown that mucus transport decreases as its density and elastic modulus increase. It has been also shown that the mucus transport decreases as the viscosity of serous layer fluid or mucus increases.

Key Words :

Mucociliary transport, unsteady state, planar model, viscoelastic mucus.

AMS 2000 Subject Classification : 76Z05, 92C10, 92C35.

1. Introduction

The mucociliary transport is one of the primary pulmonary defense mechanism. The mucociliary system essentially consists of a mucus layer, a serous layer and cilia embedded in the epithelium. The main functions of this system are to clean inspired air of contaminants and to remove entrapped particles and cellular debris from the lung through mucus transport, which is mainly attributed to the activity of cilia beating in the serous layer and to air flow interaction during forced expiration, coughing etc. (Sleigh et al. [1]). The force of gravity also plays a role in some situations (Blake [2]). The transport of mucus also depends upon the viscosity and the thickness of the serous layer and the viscoelastic properties of the mucus and its thickness (Ross and Corrsin [3], Blake and Winet [4]).

The transport of mucus in the respiratory tracts has been studied by several investigators (Sleigh et al. [1], Blake [2], King et al. [5, 6, 7] Agrawal and Verma [8], Verma [9]).

But no attention has been paid to study mucociliary transport in the respiratory tracts using mathematical models under unsteady state condition. In this paper, a planar two layer unsteady state mathematical model of mucociliary transport in the respiratory tracts has been studied.

2. Mathematical Model

The physical situation of the transport of mucus and serous layer fluid in the airways is represented by a planar two layer fluid model as shown in figure (1). The mucus is considered as a viscoelastic (Maxwell) fluid, while the serous layer is considered a Newtonian fluid. The serous layer fluid is divided into two sublayers, one in contact with the epithelium and the other, in contact with the mucus. It is assumed that cilia during beating impart a velocity at the mean level of their tips, causing the serous sublayer in contact with mucus to undergo motion. No net flow is assumed in the serous sublayer in contact with the epithelium. Effects of airmotion, force due to gravity and pressure gradients are also taken in the model.

The equations governing the motion of the serous sublayer fluid and the mucus, under unsteady state and low Reynolds number flow approximations, by taking the effect of force due to gravity into account in the direction of the flow, can be written as follows :

Region I - Serous Layer ($h_e \leq y \leq h_s$) :

$$\rho_s \frac{\partial u_s}{\partial t} = -\frac{\partial p}{\partial x} + \rho_s g \cos\theta + \mu_s \frac{\partial^2 u_s}{\partial y^2} . \tag{1}$$

Region II - Mucus Layer ($h_s \leq y \leq h_m$) :

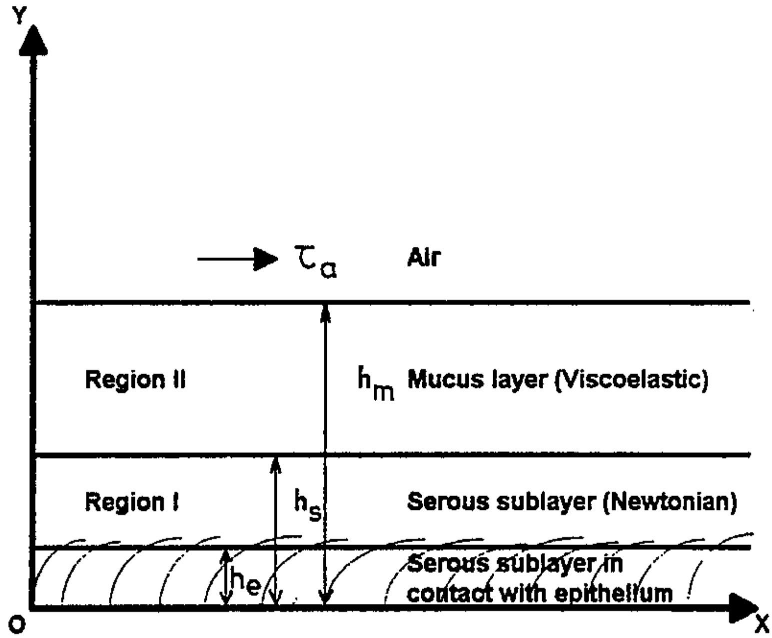

Fig. 1. A Planar model for mucus transport

$$\rho_m \frac{\partial u_m}{\partial t} = -\frac{\partial p}{\partial x} + \rho_m g \cos\theta + \frac{\partial \tau_m}{\partial y} \tag{2}$$

$$\tau_m + \lambda \frac{\partial \tau_m}{\partial t} = \mu_m \frac{\partial u_m}{\partial y} \tag{3}$$

where t is time, p is the pressure that is constant across the layers; u_s, u_m are the velocity components of the serous layer fluid and mucus in the x-direction, respectively; ρ_s, μ_s and ρ_m, μ_m are their respective densities and viscosities; τ_m is the shear stress in the mucus layer; λ ($= \mu_m/G$) is the relaxation time; G is the shear modulus; g is the acceleration due to gravity and θ is the angle by which the airway is inclined with the vertical. Here, h_e is the mean thickness measured from the surface of the epithelium to the tips of the cilia during beating i.e. the interface between the two serous sublayers; h_s is the thickness measured from the surface of the epithelium to the interface between the serous sublayer and the mucus, and h_m

is the thickness measured from the surface of the epithelium to the mucus-air interface. Equation (3) gives the relationship between the shear stress and the velocity gradient for viscoelastic fluid.

The following conditions are taken for the system (1) - (3).

Initial Conditions

$$u_s = u_m = \tau_m = 0, \qquad t = 0. \tag{4}$$

Boundary Conditions

$$u_s = U_0, \qquad y = h_e \tag{5}$$

$$\tau_m = \tau_a, \qquad y = h_m. \tag{6}$$

where U_0 is the mean velocity imparted by cilia tips during beating in the serous layer at the level $y = h_e$. The condition (6) implies that the shear stress is continuous at the mucus-air interface and incorporates the effect of airmotion similar to the analysis of Blake [2].

Matching Conditions

$$u_s = u_m = U_1, \qquad y = h_s \tag{7}$$

$$\tau_s = \tau_m, \qquad y = h_s \tag{8}$$

where U_1 is the mucus-serous sublayer interface velocity to be determined by using equation (8). The condition (7) and (8) imply that the velocities and the stresses are continuous at the mucus-serous sublayer interface.

3. Solution

Taking Laplace Transform of the system of equations (1) - (3), we get

$$\frac{d^2 \bar{u}_s}{d y^2} - K_s^2 \bar{u}_s = \frac{\bar{\phi}_s}{\mu_s} \tag{9}$$

$$\frac{d^2 \bar{u}_m}{d y^2} - K_m^2 \bar{u}_m = \frac{\bar{\phi}_m}{\lambda(S)} \tag{10}$$

$$\bar{\tau}_m = \lambda(S)\frac{d\,\bar{u}_m}{dy}, \qquad \lambda(S) = \frac{\mu_m}{(1+\lambda S)} \tag{11}$$

where $K_s^2 = \frac{S\,\rho_s}{\mu_s}$, $K_m^2 = \frac{S\,\rho_m}{\mu_m}(1+\lambda S)$, $\phi_s = \frac{\partial p}{\partial x} - \rho_s\, g\cos\theta$, $\phi_m = \frac{\partial p}{\partial x} - \rho_m\, g\cos\theta$.

Transformed conditions are :

$$\bar{u}_s = \bar{u}_m = \bar{\tau}_m = 0, \qquad t = 0 \tag{12}$$

$$\bar{u}_s = \bar{U}_0, \qquad y = h_e \tag{13}$$

$$\bar{\tau}_m = \bar{\tau}_a, \qquad y = h_m \tag{14}$$

$$\bar{u}_s = \bar{u}_m = \bar{U}_1, \qquad y = h_s \tag{15}$$

$$\bar{\tau}_s = \bar{\tau}_m, \qquad y = h_s \tag{16}$$

Solving equations (9) - (11) and using conditions (12) - (16), the velocity components can be written as :

$$\bar{u}_s = \bar{U}_1\frac{\sinh K_s(y-h_e)}{\sinh K_s(h_s-h_e)} - \bar{U}_0\frac{\sinh K_s(y-h_s)}{\sinh K_s(h_s-h_e)} - \frac{\bar{\phi}_s}{S\,\rho_s}\left[1+\frac{\sinh K_s(y-h_s)-\sinh K_s(y-h_e)}{\sinh K_s(h_s-h_e)}\right] \tag{17}$$

$$\bar{u}_m = \bar{U}_1\frac{\cosh K_m(y-h_m)}{\cosh K_m(h_m-h_s)} + \frac{K_m\,\bar{\tau}_a}{S\,\rho_m}\frac{\sinh K_m(y-h_s)}{\cosh K_m(h_m-h_s)} - \frac{\bar{\phi}_m}{S\,\rho_m}\left[1-\frac{\cosh K_m(y-h_m)}{\cosh K_m(h_m-h_s)}\right] \tag{18}$$

where $\bar{U}_1$ is given by the relation

$$\mu_s K_s\left[\bar{U}_1\coth K_s(h_s-h_e) - \bar{U}_0\,\mathrm{cosech}\,K_s(h_s-h_e) + \frac{\bar{\phi}_s}{S\,\rho_s}\tanh\frac{K_s}{2}(h_s-h_e)\right]$$

$$= \bar{\tau}_a\,\mathrm{sech}\,K_m(h_m-h_s) - \frac{S\,\rho_m}{K_m}\left(\bar{U}_1 + \frac{\bar{\phi}_m}{S\,\rho_m}\right)\tanh K_m(h_m-h_s) \tag{19}$$

and (–) denotes the Laplace Transform of the function and 'S' is the corresponding transform variable.

The volumetric flow rates in the two layers are defined as :

$$\overline{Q}_s = \int_{h_e}^{h_s} \overline{u}_s \, dy, \qquad \overline{Q}_m = \int_{h_s}^{h_m} \overline{u}_m \, dy,$$

which after using (17) and (18), are found as :

$$\overline{Q}_s = \left(\frac{\overline{U}_0 + \overline{U}_1}{K_s}\right) \tanh \frac{K_s}{2} (h_s - h_e) - \frac{\overline{\phi}_s}{S\,\rho_s} \left[(h_s - h_e) - \frac{2}{K_s} \tanh \frac{K_s}{2} (h_s - h_e)\right] \tag{20}$$

$$\overline{Q}_m = \frac{\overline{U}_1}{K_m} \tanh K_m (h_m - h_s) + \frac{\overline{\tau}_a}{S\,\rho_m} [1 - \operatorname{sech} K_m (h_m - h_s)]$$

$$- \frac{\overline{\phi}_m}{S\,\rho_m} \left[(h_m - h_s) - \frac{1}{K_m} \tanh K_m (h_m - h_s)\right]. \tag{21}$$

Now, it can be seen by using the equation of fluid continuity that $\overline{Q}_s$ and $\overline{Q}_m$ are constants. Therefore, from equations (20) and (21), we note that $-\frac{\partial p}{\partial x}$ is also constant. Hence, replacing it by the pressure drop over the length L of the cilia beating zone, we get the expressions for the flow rates as follows :

$$\overline{Q}_s = \left(\frac{\overline{U}_0 + \overline{U}_1}{K_s}\right) \tanh \frac{K_s}{2} (h_s - h_e) + \frac{\overline{\phi}_{so}}{S\,\rho_s} \left[(h_s - h_e) - \frac{2}{K_s} \tanh \frac{K_s}{2} (h_s - h_e)\right] \tag{22}$$

$$\overline{Q}_m = \frac{\overline{U}_1}{K_m} \tanh K_m (h_m - h_s) + \frac{\overline{\tau}_a}{S\,\rho_m} [1 - \operatorname{sech} K_m (h_m - h_s)]$$

$$+ \frac{\overline{\phi}_{mo}}{S\,\rho_m} \left[(h_m - h_s) - \frac{1}{K_m} \tanh K_m (h_m - h_s)\right]. \tag{23}$$

where $\overline{U}_1$ is given by

$$\mu_s K_s [\overline{U}_1 \coth K_s (h_s - h_e) - \overline{U}_0 \operatorname{cosech} K_s (h_s - h_e) - \frac{\overline{\phi}_{so}}{S\,\rho_s} \tanh \frac{K_s}{2} (h_s - h_e)]$$

$$= \bar{\tau}_a \operatorname{sech} K_m (h_m - h_s) - \frac{S\rho_m}{K_m} \left(U_1 - \frac{\bar{\phi}_{mo}}{S\,\rho_m}\right) \tanh K (h_m - h_s) \tag{24}$$

and $$\bar{\phi}_{so} = \frac{\Delta p}{L} + \rho_s\, g \cos\theta, \quad \bar{\phi}_{mo} = \frac{\Delta p}{L} + \rho_m\, g \cos\theta \tag{25}$$

where $\Delta p = p_0 - p_L$, $p = p_0$ at $x = 0$, $p = p_L$ at $x = L$. It is noted here, that the effect of gravity is similar to that of the pressure drop.

Since it is not easy to find the inverse transform of $\bar{Q}_s$ and $\bar{Q}_m$, its expressions can be approximated using reasonable assumptions (small thickness of serous layer fluid and large mucus thickness) i.e.

$$K_s (h_s - h_e) << 1 \quad \text{and} \quad K_m (h_m - h_s) >> 1.$$

In such a case, equations (22) and (23) can be simplified as :

$$\bar{Q}_s = (h_s - h_e)\left[\frac{1}{S} - \left(\frac{h_s - h_e}{2\,\mu_s}\right)\frac{(\rho_m G)^{1/2}}{S^{1/2}(S+\alpha)^{1/2}}\right] U_0$$

$$+ \frac{(h_s - h_e)^2}{2\,\mu_s}\left(\frac{G}{\rho_m}\right)^{1/2}\left[\frac{1}{S^{3/2}(S+\alpha)^{1/2}} - \left(\frac{h_s - h_e}{\mu_s}\right)\frac{(\rho_m G)^{1/2}}{S\,(S+\alpha)}\right]\phi_{mo}$$

$$+ \frac{(h_s - h_e)^3}{4\,\mu_s}\left[\frac{1}{S} - \left(\frac{h_s - h_e}{\mu_s}\right)\frac{(\rho_m G)^{1/2}}{S^{1/2}(S+\alpha)^{1/2}}\right]\phi_{so} \tag{26}$$

$$\bar{Q}_m = \left(\frac{G}{\rho_m}\right)^{1/2}\left[\frac{1}{S^{3/2}(S+\alpha)^{1/2}} - \left(\frac{h_s - h_e}{\mu_s}\right)\frac{(\rho_m G)^{1/2}}{S\,(S+\alpha)}\right] U_0$$

$$+ \left(\frac{G}{\rho_m}\right)\left[\left(\frac{h_s - h_e}{\mu_s}\right)\left\{\frac{1}{S^2(S+\alpha)} - \left(\frac{h_s - h_e}{\mu_s}\right)\frac{(\rho_m G)^{1/2}}{S^{3/2}(S+\alpha)^{3/2}}\right\}\right.$$

$$\left[-\frac{1}{(\rho_m G)^{1/2}}\,\frac{1}{S^{5/2}(S+\alpha)^{1/2}}\right]\phi_{mo} + \frac{1}{\rho_m S^2}\left[\tau_a + (h_m - h_s)\,\phi_{mo}\right]$$

$$+\frac{(h_s-h_e)^2}{2\,\mu_s}\left(\frac{G}{\rho_m}\right)^{1/2}\left[\frac{1}{S^{3/2}(S+\alpha)^{1/2}}-\left(\frac{h_s-h_e}{\mu_s}\right)\frac{(\rho_m\,G)^{1/2}}{S\,(S+\alpha)}\right]\phi_{so} \quad (27)$$

where $\alpha=\frac{1}{\lambda}=\frac{G}{\mu_m}$.

Now taking Inverse Laplace Transform of (26) and (27), the expressions of Q_s and Q_m can be written as :

$$Q_s = (h_s-h_e)\left[1-\left(\frac{h_s-h_e}{2\,\mu_s}\right)(\rho_m\,G)^{1/2}\exp\left(-\frac{\alpha\,t}{2}\right)I_0\left(\frac{\alpha\,t}{2}\right)\right]U_0$$

$$+\frac{(h_s-h_e)^2}{2\,\mu_s}\,[\left(\frac{G}{\rho_m}\right)^{1/2} t\exp\left(-\frac{\alpha t}{2}\right)\left\{I_0\left(\frac{\alpha\,t}{2}\right)+I_1\left(\frac{\alpha\,t}{2}\right)\right\}$$

$$-\frac{\mu_m}{\mu_s}(h_s-h_e)\,\{1-\exp(-\alpha\,t)\}]\,\phi_{mo}$$

$$+\frac{(h_s-h_e)^3}{4\,\mu_s}\left[1-\left(\frac{h_s-h_e}{\mu_s}\right)(\rho_m\,G)^{1/2}\exp\left(-\frac{\alpha\,t}{2}\right)I_0\left(\frac{\alpha\,t}{2}\right)\right]\phi_{so} \quad (28)$$

$$Q_m=\frac{t}{\rho_m}\left[\tau_a+\left\{(h_m-h_s)+\frac{\mu_m}{\mu_s}(h_s-h_e)\right\}\phi_{mo}\right]+\left\{U_0+\frac{(h_s-h_e)^2}{2\,\mu_s}\,\phi_{so}\right\}$$

$$\left[\left(\frac{G}{\rho_m}\right)^{1/2} t\exp\left(-\frac{\alpha t}{2}\right)\left\{I_0\left(\frac{\alpha\,t}{2}\right)+I_1\left(\frac{\alpha\,t}{2}\right)\right\}-\frac{\mu_m}{\mu_s}(h_s-h_e)\{1-\exp(-\alpha t)\}\right]$$

$$+\left(\frac{G}{\rho_m}\right)^{1/2}\frac{\phi_{mo}}{\alpha^2}\,[\frac{8}{\rho_m}\left\{\left(1+\frac{\alpha t}{4}\right)^2\exp\left(-\frac{\alpha t}{2}\right)-1\right\}$$

$$-\left(\frac{G}{\rho_m}\right)^{1/2}\left(\frac{h_s-h_e}{\mu_s}\right)\{1-\exp(-\alpha t)\}+\frac{2\mu_m}{\mu_s}(h_s-h_e)\,\alpha^2\,t\exp\left(-\frac{\alpha t}{2}\right)I_1\left(\frac{\alpha t}{2}\right)] \quad (29)$$

where I_0 and I_1 are modified Bessel functions of order zero and unity respectively.

4. Results and Discussion

To study the effect of various parameters on mucus flow rate quantitatively, the expression for Q_m given by (29), can be written in non-dimensional form as :

$$Q_m^* = \frac{t^*}{\rho_m^*}\left[\tau_a^* + \{(1-h_s^*) + \frac{\mu_m^*}{\mu_s^*}(h_s^* - h_e^*)\}\,\phi_{mo}^*\right] + \left\{1 + \frac{(h_s^* - h_e^*)^2}{2\,\mu_s^*}\,\phi_{so}^*\right\}$$

$$\left[\left(\frac{G^*}{\rho_m^*}\right)^{1/2} t^*\exp\left(\frac{-\alpha^* t^*}{2}\right)\left\{I_0\left(\frac{\alpha^* t^*}{2}\right) + I_1\left(\frac{\alpha^* t^*}{2}\right)\right\} - \frac{\mu_m^*}{\mu_s^*}(h_s^* - h_e^*)\{1 - \exp(-\alpha^* t^*)\}\right]$$

$$+\left(\frac{G^*}{\rho_m^*}\right)^{1/2}\frac{\phi_{mo}^*}{\alpha^{*2}}\,[\frac{8}{\rho_m^*}\left\{\left(1+\frac{\alpha^* t^*}{4}\right)^2\exp\left(-\frac{\alpha^* t^*}{2}\right) - 1\right\}$$

$$-\left(\frac{G^*}{\rho_m^*}\right)^{1/2}\left(\frac{h_s^* - h_e^*}{\mu_s^*}\right)\{1 - \exp(-\alpha^* t^*)\}$$

$$+\frac{2\mu_m^*}{\mu_s^*}(h_s^* - h_e^*)\,\alpha^{*2}\,t^*\exp\left(-\frac{\alpha^* t^*}{2}\right) I_1\left(\frac{\alpha^* t^*}{2}\right)] \quad (30)$$

by using the following parameters :

$$\mu_s^* = \frac{\mu_s}{\mu_0},\quad \mu_m^* = \frac{\mu_m}{\mu_0},\quad \rho_m^* = \rho_m\, h_m \frac{U_0}{\mu_0},\quad G^* = \frac{G\, h_m}{\mu_0\, U_0},\quad t^* = \frac{t\, U_0}{h_m},$$

$$\phi_{so}^* = \phi_{so}\frac{h_m^2}{\mu_0\, U_0},\quad \phi_{mo}^* = \phi_{mo}\frac{h_m^2}{\mu_0\, U_0},\quad \tau_a^* = \frac{\tau_a\, h_m}{\mu_0\, U_0},\quad h_e^* = \frac{h_e}{h_m},\quad h_s^* = \frac{h_s}{h_m},$$

$$Q_m^* = \frac{Q_m}{h_m\, U_0},$$

where μ_0 is the viscosity of the serous sublayer fluid in contact with epithelium.

Various graphs are plotted in figures (2)-(5), by using following set of parameters which have been calculated by using typical values of various characteristics related to airways (King et al. [7]).

$$\mu_s^* = 1-7, \quad \mu_m^* = 25-5000, \; \rho_m^* = 0.2-0.5\,, \; G^* = 10-100, \; t^* = 0.1,$$

$$\phi_{so}^* = 1.0, \quad \phi_{mo}^* = 2-25\,, \quad \tau_a^* = 1-10, \quad h_e^* = 0.1, \quad h_s^* = 0.2-0.6$$

It is noted from figure (2), that for various set of parameters (as shown in the figure), mucus transport decreases as the viscosity of serous layer fluid or mucus increases. However, increase in mucus viscosity at higher values does not have any significant effect on its transport. These results are in line with the experimental observations of King et al. [5, 6], Ross and Corrsin [3] and analytical results of King et al. [7], Agarwal and Verma [8] and Verma [9].

From figure (3), it is seen that mucus transport increases as the shear stress induced by air-motion at the mucus-air interface increases.

Figure (4) shows that mucus transport increases as the pressure drop or force due to gravity increases. However, increase in mucus viscosity at higher values does not have any significant effect on its transport. These results are in line with the analytical results shown earlier by King et al. [7], Agarwal and Verma [8] and Verma [9].

It may be pointed out from figure (5), that mucus transport decreases as elastic modulus increases. It is also noted in this figure, that mucus transport decreases as its density increases for fixed elastic modulus, but the relative decrease being larger at lower values of elastic modulus. The decrease in mucus transport with increase in its elastic modulus verifies the observations made by King et al. [5, 6].

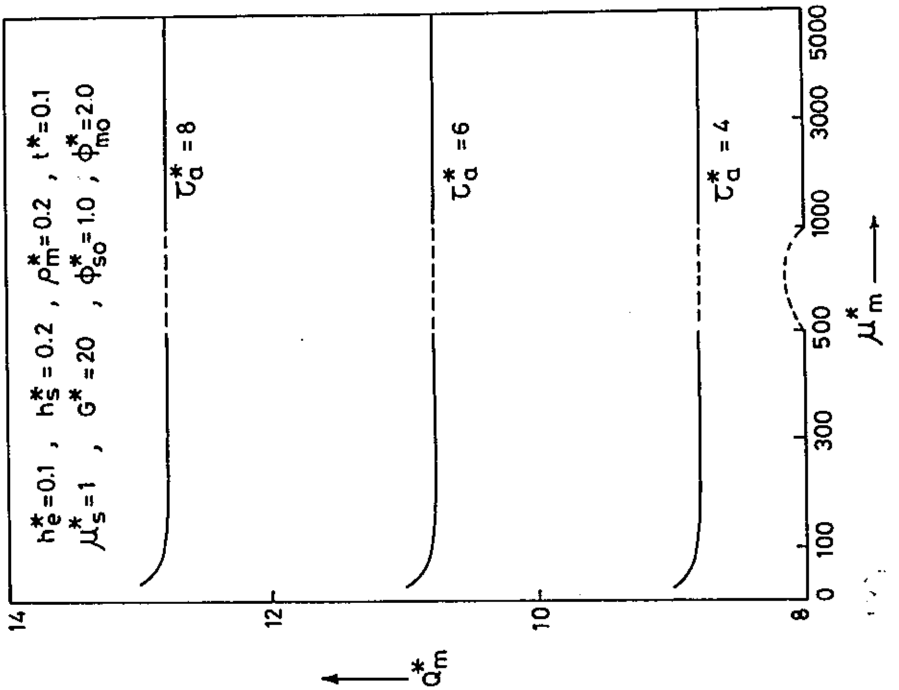

Fig. 3. Variation of Q_m^* with μ_m^* for different τ_a^*

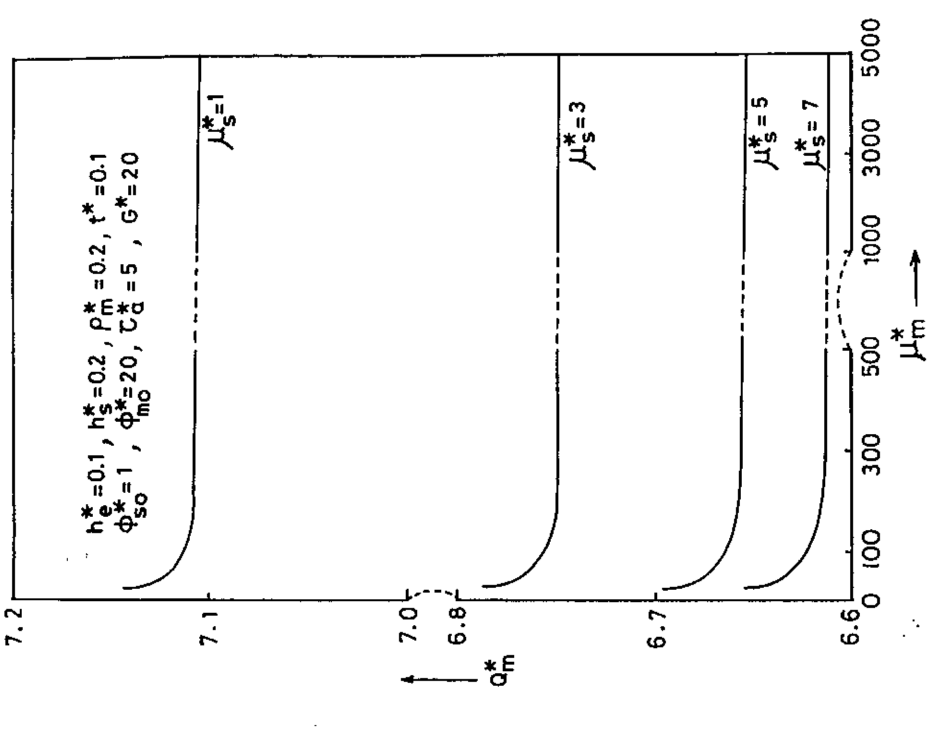

Fig. 2. Variation of Q_m^* with μ_m^* for different μ_s^*

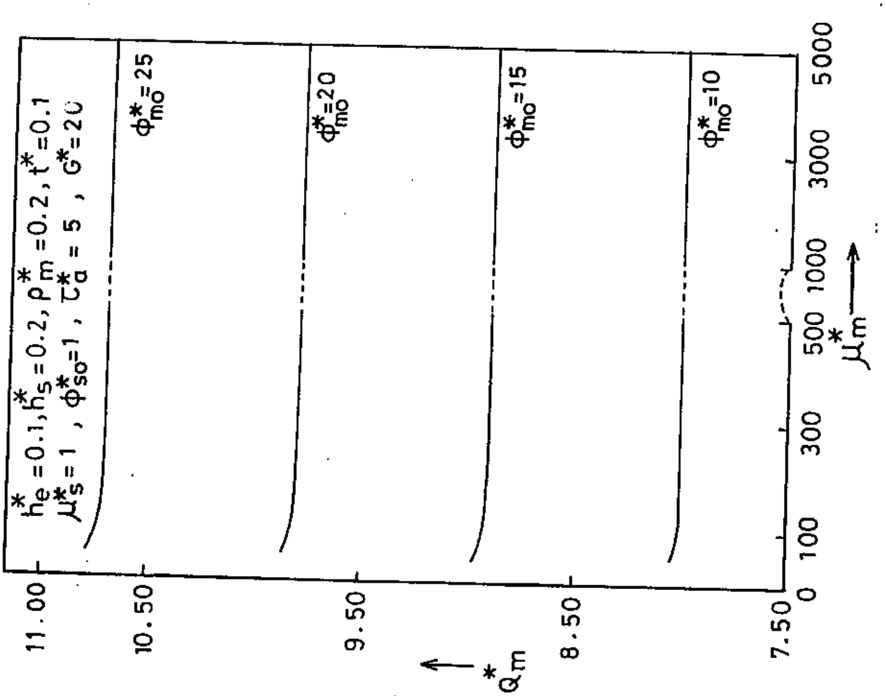

Fig. 4. Variation of Q_m^* with μ_m^* for different ϕ_{mo}^*

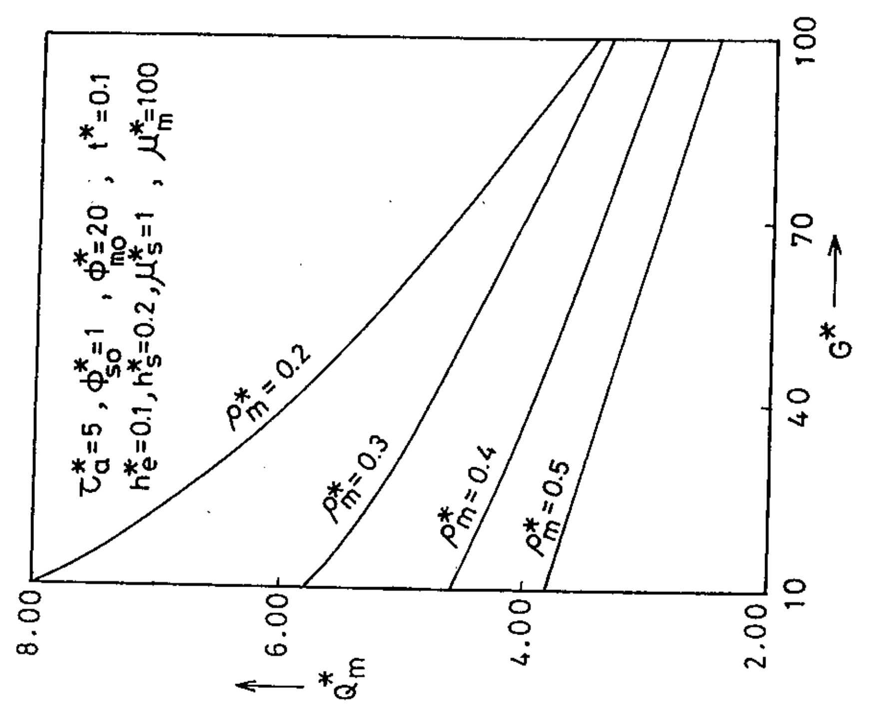

Fig. 5. Variation of Q_m^* with G^* for different ρ_m^*

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