J. T. S
Vol. 2 (2008), 91- 99
https://doi.org/10.56424/jts.v2i00.9963

On a Pseudo Normal Generalized Quasi-Sasakian Manifold of the First kind

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(Received : February 27, 2008, Revised : July 20, 2008)

In 1989, Mishra defined Pseudo normal generalized quasi-sasakian manifold of the first kind. In the present paper some properties of these manifolds are studied. Finally we studied invariant submanifolds of Pseudo normal generalized quasi-sasakian manifold of first kind.

1. Introduction

Let M be an n-dimensional C^{∞}-manifold and let there exist on M a vector valued linear function ϕ of type (1, 1), a vector field ξ and a 1-form η such that

$$\text{(a)} \quad \phi^2 X = -X + \eta(X)\,\xi, \qquad \text{(b)} \quad \eta(\xi) = 1 \tag{1.1}$$

for arbitrary vector field X. Then M is called an almost contact manifold and structure (ϕ, ξ, η) is called an almost contact structure (Sasaki [5]).

It follows from (1.1) that the following hold in M : rank $(\phi) = n - 1$, n is odd, i.e., $n = 2m + 1$ and

$$\text{(a)} \quad \eta(\phi X) = 0, \qquad \text{(b)} \quad \phi\xi = 0. \tag{1.2}$$

In addition, if in M, there exist a metric tensor g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y), \tag{1.3}$$

which is equivalent to $g(\phi X, \phi Y) = -\,g(\phi^2 X, Y)$ and $g(X, \xi) = \eta(X)$, then M is called an almost contact metric manifold and (ϕ, ξ, η, g) an almost contact metric structure (Sasaki [5]).

The fundamental 2-form of F of an almost contact metric is defined by

$$F(X, Y) = g(\phi X, Y). \tag{1.4}$$

Thus, we have

(a) $F(\phi X, \phi Y) = F(X, Y)$, (b) $F(X, Y) = -F(Y, X)$. (1.5)

On an almost contact metric manifold, we have

(a) $(D_X F)(Y, \xi) = -(D_X \eta)(\phi Y)$ (b) $g((D_X \phi)Y, Z) = (D_X F)(Y, Z)$, (1.6)

$$(D_X F)(\phi Y, \phi Z) = -(D_X F)(Y, Z) + \eta(Y)(D_X \eta)(\phi Z) - \eta(Z)(D_X \eta)(\phi Y). \tag{1.7}$$

Further that the Nijenhuis tensor N is given by

$$N(X, Y) = (D_{\phi X} Y)(Y) - (D_{\phi Y} \phi)(X) - \phi(D_X \phi)(Y) + \phi(D_Y \phi)(X). \tag{1.8}$$

Hence,

$$N(X, Y, Z) = (D_{\phi X} F)(Y, Z) - (D_{\phi Y} F)(X, Z) + (D_X F)(Y, \phi Z) - (D_Y F)(X, \phi Z), \tag{1.9}$$

where $N(X, Y, Z) = g(N(X, Y), Z)$ and D is the Riemannian connection of g.

If an almost contact metric manifold M satisfies (Mishra [3]).

$$(D_{\phi X} F)(\phi Y, Z) + (D_X F)(Y, Z) = \eta(X)(D_Y \eta)(\phi Z) + \eta(Y)(D_{\phi X} \eta)(Z), \tag{1.10}$$

then M is called a Pseudo normal generalized quasi-sasakian manifold of the first kind.

If in an almost contact metric manifold M (Mishra [2])

$$(D_{\phi X} F)(\phi Y, Z) + (D_X F)(Y, Z) = \eta(Y)(D_{\phi X} \eta)(Z) - \eta(X)(D_{\phi Y} \eta)(Z), \tag{1.11}$$

then M is called a Pseudo normal nearly co-symplectic manifold.

2. Properties

Theorem (2.1) : On a Pseudo normal generalized quasi-sasakian manifold of the first kind, we have

(a) $D_X \xi = \phi(D_{\phi X} \xi)$, (b) $D_\xi \xi = 0$,

(c) $(D_\xi \phi)(Y) = -\phi(D_Y \xi)$, (d) $(D_X \eta)(Y) = -(D_{\phi X} \eta)(\phi Y)$,

(e) $(D_\xi \eta) = 0$, (f) $(D_X \eta)(Y) = -(D_Y \eta)(X)$,

(g) $(D_{\phi X} \eta)(Y) = (D_X \eta)(\phi Y) + \eta(D_{\phi X} \xi)\,\eta(Y)$ (2.1)

where X, Y are any vector fields on M.

Proof. From (1.10), we put $Y = \xi$

$$(D_{\phi X}F)(\phi\xi, Z) + (D_X F)(\xi, Z) = \eta(X)(D_\xi \eta)(\phi Z) + \eta(\xi)(D_{\phi X}\eta)(Z),$$

i.e.
$$g(D_X\xi, \phi Z) = g((D_{\phi X}\xi), Z).$$

Now, we have

$$-\phi(D_X \xi) = (D_{\phi X} \xi). \tag{2.2}$$

Operating ϕ in (2.2), we obtain (2.1)a.

We put $X = \xi$ in (2.1)a, we obtain (2.1)b. Again we put $X = \xi$ in (1.10), we have

$$(D_\xi F)(Y, Z) = (D_Y \eta)(\phi Z). \tag{2.3}$$

Thus, we obtain (2.1)c.

Now we have

$$(D_X \eta)(Y) + (D_{\phi X} \eta)(\phi Y) = g(D_X \xi, Y) + g(D_{\phi X} \xi, \phi Y)$$

$$= g((D_X \xi - \phi(D_{\phi X} \xi), Y). \tag{2.4}$$

From (2.1)a and (2.4), we obtain (2.1)d.

Setting $X = \xi$ in (2.1)d, we obtain (2.1)e.

Now we have

$$(D_X \eta)(Y) + (D_Y \eta)(X) = g(D_X \xi, Y) + g(D_Y\xi, X)$$

$$= g(\phi X, Y) + g(\phi Y, X)$$

$$= g(\phi X, Y) - g(\phi X, Y) = 0$$

we obtain (2.1)(f).

Now, we have

$$(D_X\eta)(\phi Y) - (D_{\phi X}\eta)(Y) = g(D_X\xi, \phi Y) - g(D_{\phi X}\xi, Y)$$
$$= -g(\phi D_X\xi, Y) - g(D_{\phi X}\xi, Y)$$
$$= -g[(\phi(D_X\xi) + (D_{\phi X}\xi), Y] \quad (2.5)$$

In view of (2.1)a, we get

$$\phi(D_X\xi) = \phi^2(D_{\phi X}\xi)\xi$$

i.e.

$$\phi(D_X\xi) + (D_{\phi X}\xi) = h(D_{\phi X}\xi) \quad (2.6)$$

From (2.5) and (2.6), we obtain (2.1)g.

Theorem (2.2) : The pseudo normal generalized quasi-sasakian manifold of the first kind is Pseudo normal nearly co-symplectic manifold if

$$(D_Y\eta)(\phi Z) = -(D_{\phi Y}\eta)(Z) \quad (2.7)$$

Proof. From (1.10) and (2.7), we get (1.11).

Theorem (2.3) : A Pseudo normal generalized quasi-sasakian manifold of the first kind is integrable if,

$$(D_{\phi^2 X}\phi)(\phi Y) + (D_{\phi X}\phi)(\phi^2 Y) = (D_{\phi^2 Y}\phi)(\phi X) + (D_{\phi Y}\phi)(\phi^2 X) \quad (2.8)$$

Proof. The condition for an almost contact metric manifold to be completely integrable (Mishra [4]) is that

$$N(\phi X, \phi Y, \phi Z) = 0. \quad (2.9)$$

Operating ϕ on X in (1.10), we find

$$(D_{\phi X}F)(Y, Z) = -(D\phi^2{}_X F)(\phi Y, Z) + \eta(Y)(D\phi^2{}_X\eta)(Z)$$
$$= (D_X F)(\phi Y, Z) - \eta(X)(D_\xi F)(\phi Y, Z) - \eta(Y)(D_X\eta)(Z). \quad (2.10)$$

Again operating ϕ on Z in (1.10), we find

$$(D_XF)(Y, \phi Z) = -(D_{\phi X}F)(\phi Y, \phi Z) + \eta(X)(D_Y\eta)(\phi^2 Z) + \eta(Y)(D_{\phi X}\eta)(\phi Z). \tag{2.11}$$

Adding (2.10) and (2.11), we have

$$\begin{aligned}(D_{\phi X}F)(Y, Z) + (D_XF)(Y, \phi Z) = (D_XF)(\phi Y, Z) - \eta(X)(D_\xi F)(\phi Y, Z)\\ - \eta(Y)(D_X\eta)(Z) - (D_{\phi X}F)(\phi Y, \phi Z)\\ - \eta(X)(D_Y\eta)(Z) + \eta(Y)(D_{\phi X}\eta)(\phi Z).\end{aligned} \tag{2.12}$$

Similarly,

$$\begin{aligned}(D_{\phi Y}F)(X, Z) + (D_YF)(X, \phi Z) = (D_YF)(\phi X, Z) - \eta(X)(D_\xi F)(\phi X, Z)\\ - \eta(X)(D_Y\eta)(Z) - (D_{\phi Y}F)(\phi X, \phi Y)\\ - \eta(Y)(D_X\eta)(Z) + \eta(X)(D_{\phi Y}\eta)(\phi Z).\end{aligned} \tag{2.13}$$

From (2.12), (2.13) and (1.9), we have

$$\begin{aligned}N(X, Y, Z) = (D_XF)(\phi Y, Z) - (D_YF)(\phi X, \phi Z) - (D_{\phi X}F)(\phi Y, \phi Z)\\ + (D_{\phi Y}F)(\phi X, \phi Z) - \eta(X)[(D_\xi F)(\phi Y, Z) + (D_{\phi Y}\eta)(\phi Z)]\\ + \eta(Y)[(D_{\phi X}\eta)(\phi Z) + (D_\xi F)(\phi X, Z)].\end{aligned} \tag{2.14}$$

Using (1.7) and (2.3), we have

$$\begin{aligned}N(X, Y, Z) = (D_{\phi X}F)(Y, Z) - (D_{\phi Y}F)(X, Z) + (D_XF)(\phi Y, Z)\\ - (D_YF)(\phi X, Z) - \eta(X)(D_{\phi Y}\eta)(\phi Z) + \eta(Y)(D_{\phi X}\eta)(\phi Z)\\ + \eta(Z)[(D_{\phi X}\eta)(\phi Y) - (D_{\phi Y}\eta)(\phi X)].\end{aligned} \tag{2.15}$$

Operating ϕ on X, Y, Z, we have

$$\begin{aligned}N(\phi X, \phi Y, \phi Z) = (D_{\phi^2 X}F)(\phi Y, \phi Z) - (D_{\phi^2 Y}F)(\phi X, \phi Z)\\ + (D_{\phi X}F)(\phi^2 Y, \phi Z) - (D_{\phi Y}F)(\phi^2 X, \phi Z).\end{aligned} \tag{2.16}$$

In view of (2.9) and (2.16), we have

$$\begin{aligned}(D_{\phi^2 X}F)(\phi Y, \phi Z) + (D_{\phi X}F)(\phi^2 Y, \phi Z) = (D_{\phi^2 Y}F)(\phi X, \phi Z)\\ + (D_{\phi Y}F)(\phi^2 X, \phi Z).\end{aligned} \tag{2.17}$$

From (1.6)(b) and (2.17), we obtain (2.8), which proves the theorem.

Theorem (2.4) : On a Pseudo normal generalized quasi-Sasakian manifold of the first kind, we have

(a) $N(X, Y, Z) + N(Y, X, Z) = 0,$

(b) $N(X, Y, Z) + N(Y, Z, X) + N(Z, X, Y) = 2[(D_{\phi X}F)(Y, Z) + (D_{\phi Y}F)(Z, X)$
$+ (D_{\phi Z}F)(X, Y) + (D_X F)(\phi Y, Z) + (D_Y F)(\phi Z, X) + (D_Z F)(\phi X, Y)],$

(c) $N(\phi X, \phi Y, Z) + N(\phi Y, \phi X, Z) = 0.$

Proof. (a) From (2.15), we have

$$N(Y, X, Z) = (D_{\phi Y}F)(X, Z) - (D_{\phi X}F)(Y, Z) + (D_Y F)(\phi X, Z) - (D_X F)(\phi Y, Z) - \eta(Y)(D_{\phi X}\eta)(\phi Z) + \eta(X)(D_{\phi Y}\eta)(\phi Z) + \eta(Z)[(D_{\phi Y}\eta)(\phi X) - (D_{\phi X}\eta)(\phi Y)]. \tag{2.18}$$

Adding (2.15) and (2.18), we obtain the result (a).

Similarly other results can be proved.

Theorem (2.5) : On a Pseudo normal generalized quasi-sasakian manifold of first kind, we have

$$N(\phi X, \phi^2 Y) + N(\phi Y, \phi^2 X) = 0.$$

Proof. Applying ϕ on Y in (1.10), we have

$$(D_{\phi X}F)(\phi^2 Y, Z) + (D_X F)(\phi Y, Z) = \eta(X)(D_{\phi Y}\eta)(\phi Z).$$

Using (2.1)(a), we have

$$(D_{\phi X}\phi)(Y) - (D_X\phi)\phi Y = \eta(X)(D_Y\xi) + \eta(Y)(D_X\xi). \tag{2.19}$$

Applying ϕ on Y in (1.8), we have

$$N(X, \phi Y) = (D_{\phi X}\phi)(\phi Y) + (D_Y\phi)(X) - \eta(Y)(D_\xi\phi)(X) - \phi(D_X\phi)(\phi Y) + \phi(D_{\phi Y}\phi)(X). \tag{2.20}$$

Similarly

$$N(Y, \phi X) = (D_{\phi Y}\phi)(\phi X) + (D_X\phi)(Y) - \eta(X)(D_\xi\phi)(Y) - \phi(D_Y\phi)(\phi X) + \phi(D_{\phi X}\phi)(Y). \quad (2.21)$$

Adding (2.20) and (2.21), we have

$$N(X, \phi Y) + N(Y, \phi X) = (D_{\phi X}\phi)(\phi Y) + (D_{\phi Y}\phi)(\phi X) + (D_Y\phi)(X) + (D_X\phi)(Y) + \eta(Y)(D_X\xi) + \eta(X)(D_Y\xi) - \phi(D_X\phi)(\phi Y) + \phi(D_{\phi Y}\phi)(X) - \phi(D_Y\phi)(\phi X) + \phi(D_{\phi X}\phi)(Y).$$

Using (1.10), we obtain

$$N(X, \phi Y) + N(Y, \phi X) = \eta(X)\phi\,(D_Y\xi) + \eta(Y)(D_{\phi X}\xi) + \eta(Y)(D_X\xi) - \eta(Y)\phi\,(D_X\xi) + \eta(X)(D_{\phi Y}\xi) + \eta(X)(D_Y\xi) + 2\phi[\eta(X)\,(D_Y\xi) + \eta(Y)(D_X\xi)].$$

Operating ϕ on X, Y, we have

$$N(\phi X, \phi^2 Y) + N(\phi Y, \phi^2 X) = 0.$$

3. Invariant Submanifolds of Pseudo normal generalized quasi-Sasakian manifold of first kind.

Let $\bar{M}$ $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be an almost contact metric manifold. Let M be an invariant submanifold of $\bar{M}$, i.e. an odd dimensional submanifold such that $\bar{\xi}$ is tangent to M everywhere on M and $\bar{\phi}X$ is tangent of M for any tangent vector X to M. Consequently for any normal vector Y to M also $\bar{\phi}Y$ is normal to M. An invariant submanifold M has the induced structure tensor (ϕ, ξ, η, g).

Let $\bar{D}$ and D be the covariant differentiation with respect to the Riemannian connection defined by $\bar{g}$ and g respectively. Then the Gauss formula is given by

$$\bar{D}_X Y = D_X Y + B(X, Y), \quad (3.1)$$

where X and Y are vector fields tangent to M and B is second fundamental form.

Finally, we assume that $\overline{M}$ is a pseudo normal generalized quasi-sasakian manifold of first kind. Then for any vector fields X and Y tangent to M, we have

$$(\overline{D}_X\overline{\phi})(Y) + (\overline{D}_{\phi X}\overline{\phi})(\phi Y) = \overline{\eta}(Y)(\overline{D}_{\phi X}\xi) - \overline{\eta}(X)\phi(\overline{D}_Y\xi). \tag{3.2}$$

Theorem (3.1) : Any invariant submanifold M with induced structure tensor of a pseudo normal generalized quasi-sasakian manifold of first kind $\overline{M}$ is also pseudo normal generalized quasi-sasakian manifold of first kind.

Proof. If $\overline{M}$ is a pseudo normal nearly co-sympletic manifold, then from (3.2), we have

$$(\overline{D}_X\overline{\phi})(Y) + (\overline{D}_{\phi X}\overline{\phi})(\phi Y) = \overline{\eta}(Y)(\overline{D}_{\phi X}\xi) - \overline{\eta}(X)\phi(\overline{D}_Y\xi).$$

Thus, using (3.1) and comparing tangential and normal components, we find

$$(D_X\phi)(Y) + (D_{\phi X}\phi)(\phi Y) = \eta(Y)(D_{\phi X}\xi) - \eta(X)\phi(D_Y\xi) \tag{3.3}$$

and

$$B(X, \phi Y) - \overline{\phi}(B(X, Y)) - B(\phi X, Y) - \overline{\phi}(B(\phi X, \phi Y)) = 0. \tag{3.4}$$

From (3.3), we conclude that M is pseudo normal generalized quasi-sasakian manifold of the first kind.

Theorem (3.2) : If M is an invariant submanifold of a pseudo normal generalized quasi-sasakian manifold of the first kind $\overline{M}$, then its second fundamental form B satisfied

$$B(X, \xi) = 0. \tag{3.5}$$

Proof. If we interchange the X and Y in (3.4) and subtract the result from (3.4), then by symmetry of B, we get

$$B(\phi X, Y) = B(X, \phi Y), \tag{3.6}$$

from which

$$B(\phi X, \xi) = 0. \tag{3.7}$$

Finally in consequence of (2.1)(a), we have

$$\overline{D}_X\xi = \phi\,(\overline{D}_{\phi X}\,\xi),$$

consequently, using (3.1) and comparing the normal parts, we get

$$B(X, \xi) = \phi(B(\phi X, \xi)). \tag{3.8}$$

By virtue of (3.7), (3.8) gives (3.5).

Theorem (3.3) : Any covariant submanifold M of a pseudo normal generalized quasi-sasakian manifold of the first kind $\overline{M}$ is minimal.

Proof. Proof follows on the line of the proof of similar result in Endo [1].

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