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Generalised Quasi-Sasakian Manifold admitting Semi-Symmetric Non Metric Connection

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Abstract

Agashe and Chafle [4] defined a semi-symmetric non-,etric connection in a Riemannian manifold. Further, the properties of semi-symmetric non-metric connection in almost contact metric manifold has been studied by Ojha and Prasad [5]. The purpose of this paper is to introduce a semi-symmetric non-metric connection in generalised quasi-Sasakian manifold.

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1. Introduction

An odd-dimensional differentiable manifold M^{2n+1} on which there are defined a tensor field F of type (1, 1), a vector field T, a 1-form A and a metric tensor field g, satisfying

$$\text{(a)} \qquad \overline{\overline{X}} = -X + A(X)\,T.$$

$$\text{(b)} \qquad \overline{T} = 0$$

$$\text{(c)} \qquad 'F(X, Y) \overset{\text{def}}{=} g(\overline{X}, Y) = -\,'F(Y, X) \tag{1.1}$$

where

$$\overline{X} \overset{\text{def}}{=} FX \tag{1.2}$$

for arbitrary vector fields X, Y is called an almost contact metric manifold. On an almost contact metric manifold, we have

$$(D_X{'}F)(T, Y) = (D_X A)(\overline{Y}) \tag{1.3}$$

$$\text{(a)} \quad (D_X{'}F)(\overline{Y}, \overline{Z}) + (D_X{'}F)(Y, \overline{Z}) = A(Y)(D_X A)(\overline{Z}) - A(Z)(D_X A)(\overline{Y})$$

$$\text{(b)} \quad (D_X{'}F)(\overline{Y}, Z) - (D_X{'}F)(Y, Z) = A(Y)(D_X A)(Z) + A(Z)(D_X A)(Y) \tag{1.4}$$

where D is the Riemannian Connection on M^{2n+1}.

We will now define only those almost contact metric manifolds according to need. An almost contact metric manifold satisfying

$$(D_X{'}F)(Y, Z) + (D_Y{'}F)(Z, X) + (D_Z{'}F)(X, Y) = 0 \tag{1.5}$$

is called a quasi-Sasakian manifold [2].

Putting T for X in (1.5) and using (1.3) and (1.4)(a), we have on a quasi-Sasakian manifold

$$\text{(a)} \quad (D_T{'}F)(Y, Z) = (D_Y A)(\overline{Z}) - (D_Z A)(\overline{Y})$$

$$= (D_{\overline{Y}} A)(Z) - (D_{\overline{Z}} A)(Y) + A(Y)(D_T A)(\overline{Z}) - A(Z)(D_X A)(\overline{Y})$$

$$\text{(b)} \quad (D_T{'}F)(\overline{Y}, Z) = (D_{\overline{Y}} A)(\overline{Z}) + (D_Z A)(Y)$$

$$= -(D_Y A)(Z) - (D_Z A)(Y) + A(Y)(D_T A)(Z) + A(Z)(D_T A)(Y) \tag{1.6}$$

If on an almost contact metric manifold M^{2n+1}.

$$(D_X{'}F)(Y, Z) + (D_Y{'}F)(Z, X) + (D_Z{'}F)(X, Y) = A(X)(D_Z A)(\overline{Y}) + A(Y)(D_X A)(\overline{Z})$$

$$+ A(Z)(D_Y A)(\overline{X}) \tag{1.7}$$

holds, then M^{2n+1} is called generalised quasi-Sasakian manifold of the first kind [3].

On a generalized quasi-Sasakian manifold of first kind

$$\text{(a)} \quad (D_T{'}F)(Y, Z) = (D_Y A)(\overline{Z}) = -(D_Z A)(\overline{Y}) = (D_{\overline{Y}} A)(Z) = -(D_{\overline{Z}} A)(Y)$$

$$\text{(b)} \quad (D_T{'}F)(\overline{Y}, Z) = (D_{\overline{Y}} A)(\overline{Z}) = (D_Z A)(Y) = -(D_Y A)(Z) = -(D_{\overline{Z}} A)(Y) \tag{1.8}$$

hold. An almost contact metric manifold on which

$$(D_X{'F})(Y, Z) + (D_Y{'F})(Z, X) + (D_Z{'F})(X, Y) = A(X)\{(D_Y A)(\bar{Z}) - (D_Z A)(\bar{Y})\}$$

$$+ A(Y)\{(D_Z A)(\bar{X}) - (D_X A)(\bar{Z})\} + A(Z)\{(D_X A)(\bar{Y}) - (D_Y A)(\bar{X})\} \quad (1.9)$$

holds, is called a generalised quasi-Sasakian manifold [3].

2. Semi-Symmetric Non-Metric Connection

Let D be the Riemannian connection in a generalised quasi-Sasakian manifold M^{2n+1} and B be another connection in M^n defined by

$$B_X Y = D_X Y + A(Y)\, X. \quad (2.1)$$

Then the connection B is called a semi-symmetric non-metric connection [4]. If the torsion tensor S of B is given by

$$S(X, Y) = A(Y)\, X - A(X)\, Y. \quad (2.2)$$

Put

$$B_X Y = D_X\, Y + H(X, Y), \quad (2.3)$$

where H is a tensor field of type (1, 2).

Let us define

(a) $${'S}(X, Y, Z) \overset{\text{def}}{=} g\,(S(X, Y), Z)$$

(b) $${'H}(X, Y, Z) \overset{\text{def}}{=} g\,(H(X, Y), Z).$$

Then

(c) $${'S}(X, Y, Z) = {'H}(X, Y, Z) - {'H}(Y, X, Z). \quad (2.4)$$

Further, we have

(a) $${'H}(X, Y, Z) + {'H}(X, Z, Y) = A(Y)\, g(X, Z) + A(Z)\, g(X, Y)$$

and

(b) $$(B_X g)(Y, Z) = -A(Y)\, g(X, Z) - A(Z)\, g(X, Y)$$

(c) $$(B_X A)(Y) = (D_X A)(Y) - A(X)\, A(Y). \quad (2.5)$$

Theorem (2.1). In generalised quasi-Sasakian manifold with the semi-symmetric non-metric connection B, we get

$$(B_X{'F})(Y, Z) = (D_X{'F})(Y, Z) - A(Y)\,{'F}(X, Z) - A(Z)\,{'F}(Y, X) \tag{2.6}$$

$$(B_X{'F})(Y, Z) + (B_Y{'F})(Z, X) + (B_Z{'F})(X,Y) = A(X)[(D_T{'F})(Y, Z) - 2g(Z,Y)]$$
$$+ A(Y)[(D_T{'F})(Z, X) - 2g(X, Z)] + A(Z)[(D_T{'F})(Y, Z) - 2g(Y, X)]. \tag{2.7}$$

Proof. We have

$$X({'F}(Y, Z)) = (B_X{'F})(Y, Z) + {'F}(B_XY, Z) + {'F}(Y, B_XZ)$$
$$= (D_X{'F})(Y, Z) + {'F}(D_XY, Z) + {'F}(Y, D_XZ)$$

using (2.1), in the above equation, we have

$$(B_X{'F})(Y, Z) = (D_X{'F})(Y, Z) - A(Y)\,{'F}(X, Z) - A(Z)\,{'F}(Y, X). \tag{2.8}$$

To prove (2.7), we have from (2.8)

$$(B_X{'F})(Y, Z) = (D_X{'F})(Y, Z) - A(Y)\,{'F}(X, Z) - A(Z)\,{'F}(Y, X).$$

Writing two other equation by cyclic permutation of X, Y, Z and adding the resulting equation in the above equation, we have

$$(B_X{'F})(Y, Z) + (B_Y{'F})(Z, X) + (B_Z{'F})(X, Y) = [(D_X{'F})(Y, Z) + (D_Y{'F})(Z, X) +$$
$$(D_Z{'F})(X,Y)] - 2[A(Z)\,{'F}(Y, X) + A(X)\,{'F}(Z, Y) + A(Y)\,{'F}(X, Z)]$$

using equation (1.9) and (1.6)(a)

$$(B_X{'F})(Y, Z) + (B_Y{'F})(Z, X) + (B_Z{'F})(X, Y) = A(X)[(D_T{'F})(Y, \overline{Z}) - 2g(Y, Z)]$$
$$+ A(Y)[(D_T{'F})(Z, \overline{X}) - 2g(X, Z)] + A(Z)[(D_T{'F})(X, \overline{Y}) - 2g(Y, X)].$$

Theorem (2.2). In a generalised quasi-Sasakian manifold with the semi-symmetric non-metric connection B, we get

(a) $$(B_X{'F})(\overline{\overline{Y}}, Z) + (B_X{'F})(Y, \overline{\overline{Z}}) = A(Y)\{(D_X{'F})(T, Z) - {'F}(X, Z)\}$$
$$- A(Z)\,\{(D_X{'F})(Y, T) + {'F}(X, Y)\}$$

(b) $$(D'F)(\bar{X}, \bar{Y}, \bar{Z}) = (B_{\bar{X}}\,'F)(\bar{Y}, \bar{Z}) + (B_{\bar{Y}}\,'F)(\bar{Z}, \bar{Y}) + (B_{\bar{Z}}\,'F)(\bar{X}, \bar{Y}) \qquad (2.9)$$

Proof. From equation (2.8)

$$(B_X'F)(Y, Z) = (D_X'F)(Y, Z) - A(Y)\,'F(X, Z) - A(Z)\,'F(Y, X).$$

We have

$$(B_X'F)(\bar{Y}, Z) = (D_X'F)(\bar{Y}, Z) - A(Z)\,'F(\bar{Y}, X)$$

$$(B_X'F)(\bar{\bar{Y}}, Z) = -(D_X'F)(Y, Z) + A(Y)(D_X'F)(T, Z) + A(Z)\,'F(Y, X)$$
$$- A(X)\,A(Y)\,A(Z)$$

Again

$$(B_X'F)(Y, \bar{\bar{Z}}) = -(D_X'F)(Y, Z) + A(Z)(D_X'F)(Y, T) + A(Y)\,'F(X, Z)$$
$$- A(X)\,A(Y)\,A(Z).$$

From above two equation we get equation (2.9)(a).

Further from equation (2.8), we have

$$(B_{\bar{X}}\,'F)(\bar{Y}, \bar{Z}) = (D_{\bar{X}}\,'F)(\bar{Y}, \bar{Z}).$$

Writing two other term in cyclic order of X, Y, Z and adding the resulting equation in the above equation, we obtain (2.9)(b).

Theorem (2.3). In a generalised quasi-Sasakian manifold with the semi-symmetric non-metric connection B, we get

(a) $$(B_YA)(X) - (B_XA)(Y) = 2(D_YA)(X)$$

(b) $$(B_XA)(Y) + (B_YA)(X) = -2A(X)\,A(Y) \qquad (2.10)$$

Proof. The proof is obvious.

Theorem (2.4). In a generalised quasi-Sasakian manifold with the semi-symmetric non-metric connection B, we get

$$'S(\bar{\bar{X}}, Y, \bar{Z}) + 'S(Y, \bar{\bar{Z}}, \bar{X}) - 'S(\bar{\bar{Z}}, \bar{X}, Y) = 0. \qquad (2.11)$$

Proof. From (2.4)(c), we have

$$'S(\overline{X}, Y, \overline{Z}) = 'H(\overline{X}, Y, \overline{Z}) - 'H(Y, \overline{X}, \overline{Z}). \tag{2.12}$$

Similarly,

$$'S(Y, \overline{Z}, \overline{X}) = 'H(Y, \overline{Z}, \overline{X}) - 'H(\overline{Z}, Y, \overline{X}). \tag{2.13}$$

$$'S(\overline{Z}, \overline{X}, Y) = 'H(\overline{Z}, \overline{X}, Y) - 'H(\overline{X}, \overline{Z}, Y). \tag{2.14}$$

In consequence of (2.12), (2.13), (2.14) and (2.4)(c), we get (2.11).

3. Curvature Tensor

Let R and K be the curvature tensor of connection B and D respectively, then we have

$$\text{(a)} \quad R(X, Y, Z) = B_X B_Y Z - B_Y B_X Z - B_{[X, Y]} Z$$

$$\text{(b)} \quad K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z \tag{3.1}$$

Making use of (2.1) in (3.1)(a), we have

$$R(X, Y, Z) = K(X, Y, Z) + (B_X A)(Z)\, Y - (B_Y A)(Z)\, X. \tag{3.2}$$

Let us define

$$\text{(a)} \quad \text{Ric}\,(Y, Z) \overset{\text{def}}{=} (C', K)(Y, Z)$$

$$\text{(b)} \quad '\text{Ric}\,(Y, Z) \overset{\text{def}}{=} (C', R)(Y, Z) \tag{3.3}$$

Making use of (3.3) in (3.2), we get

$$'\text{Ric}\,(Y, Z) = \text{Ric}\,(Y, Z) - (n - 1)(B_Y A)(Z). \tag{3.4}$$

Put

$$\text{(a)} \quad 'R(X, Y, Z, U) \overset{\text{def}}{=} g(R(X, Y, Z), U)$$

$$\text{(b)} \quad 'K(X, Y, Z, U) \overset{\text{def}}{=} g(K(X, Y, Z), U)$$

then

(c) $$'R(X, Y, Z, U) = 'K(X, Y, Z, U) + (B_X A)(Z)\, g(Y, U) - (B_Y A)(Z)\, g(X, U). \tag{3.5}$$

Theorem (3.1). In a generalised quasi-Sasakian manifold with the semi-symmetric non-metric connection B, we get

(a) $$'R(X, Y, \overline{Z}, \overline{U}) + 'R(\overline{X}, \overline{Y}, Z, U) = 'K(X, Y, \overline{Z}, \overline{U}) + 'K(\overline{X}, \overline{Y}, Z, U) + 2g(\overline{Y}, U)\, g(\overline{X}, \overline{Z}) - 2g(\overline{X}, U)\, g(\overline{Y}, \overline{Z})$$

(b) $$'R(\overline{X}, \overline{Y}, Z, U) - 'R(X, Y, \overline{Z}, \overline{U}) = 0$$

(c) $$'R(T, Y, \overline{Z}, \overline{U}) + 'R(T, Y, \overline{U}, \overline{Z}) = 0$$

(d) $$'R(T, Y, Z, \overline{U}) - 'R(Z, \overline{U}, T, Y) = 0. \tag{3.2}$$

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