J. T. S
Vol. 2 (2008), 75 - 81
https://doi.org/10.56424/jts.v2i00.9961

Absolute Nevanlinna Summability of a Series associated with the Derived Fourier Series

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(Received : September 22, 2008)

Abstract

In this paper we have proved a theorem on absolute Nevanlinna of a series associated with the derived Fourier series, which generalizes various known result. However, our theorem in as follows :

Theorem : Let $\alpha \geq 0$, $1 < p \leq \alpha$, $\alpha > \frac{1}{p}$, and let the function q_α satisfy th conditions

$$\int_0^1 q_\delta(t)\, dt = 1 \quad \text{and} \quad \psi_\alpha(t) = 0$$

where $Q_\delta(t) = \int_{1-t}^1 q_\delta^{(p)}(x)\, dx$ and $\int_0^\pi t^{-\alpha-1} |d\psi_\alpha(t)| < \infty$.

Then at t = x the derived series of a Fourier series of f is summable by the method $|N(q_n)|$.

Keywords and phrases : Absolute Nevanlinna Summability, Derived Fourier Series.

2000 Mathematics Subject Classification : 40D025, 40E05, 40F05, 40G05, 42C05 and 42C10.

1. Definitions and Notations

Given a series Σu_n, let

$$F(w) = \sum_{n < w} u_n.$$

Let $q_\delta = q_\delta(t)$ be defined for $0 \le t < 1$. The $N(q_\delta)$ transform $N(F, q_\delta)$ of F is defined by

$$N(F, q_\delta)(w) = \int_0^1 q_\delta(t) F(wt)\, dt.$$

The series Σu_n is said to be summable by the method $N(q_\delta)$ to the sum s if

$$\lim_{w \to \infty} N(F, q_\delta)(w) = s.$$

It is said to be absolutely summable by the method $N(q_\delta)$ and we shall write

$$\Sigma u_n \in |N(q_\delta)|$$

if

$$N(F, q_\delta)(\omega) \in BV(A, \infty).$$

For some $A \ge 0$, which is indeed equivalent to

$$\int_A^\infty \left| \sum_{n<\omega} q_\delta\left(\frac{n}{\omega}\right) n u_n \right| \frac{d\omega}{\omega^2} < \infty$$

for the regularity, we need

$$\int_0^1 q_\delta(t)\, dt = 1.$$

The parameter δ will be a non-negative real number. We have further two sets of restriction on q_δ. One for $0 \le \delta \le 1$ and the other for $\delta \ge 1$.

In the case $0 \le \delta < 1$, $q_\delta(t)$ is increasing for $0 < t < 1$.

In the case $\delta \ge 1$, q_δ satisfies following : $q_\delta(t)$ is decreasing for $0 < t < 1$ with $p = [\delta]$, the integral part of δ,

$$\left(\frac{d}{dt}\right)^{p-1} q_\delta(t) \in A \subset [0, 1]$$

$$\left[\left(\frac{d}{dt}\right)^k q_\delta(t)\right]_{t=1} = 0, \quad k = 0, 1, \ldots\ldots, (p-1)$$

$$(-1)^p \left(\frac{d}{dt}\right)^p q_\delta (t) \geq 0 \text{ and is increasing.}$$

Also, for $\delta \geq 0$, $p = [\delta]$, we assume

$$\frac{Q_\delta (t)}{t^{\delta - p + 1}} \in L(0, 1)$$

where $$Q_\delta(t) = \int_{1-t}^{1} q_\delta^{(p)} (x)\, dx.$$

2. Let f(t) be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$ and let

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \tag{2.1}$$

the first differentiated series of (2.1) at $t = x$ is

$$\sum_{n=1}^{\infty} n\, (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x). \tag{2.2}$$

The series $$\sum \frac{s_n (x)}{n} \tag{2.3}$$

will be called the associated derived Fourier series, where $s_n(x)$ denotes the n^{th} partial sum of the series (2.2).

We shall use the following notations :

$$\psi(t) = \frac{1}{2} \{f(x + t) - f(x - t)\}$$

$$g(t) = \frac{\psi (t)}{t}$$

$$S(w, t, p, r) = \sum_{n < w} (w - n)^r n^p \cos (nt + \theta),$$

θ independent of n.

$$g(n, t) = \int_t^{\pi} (y - t)^{h - \alpha} \cos \left(ny - \frac{h\pi}{2}\right) dy,$$

where h = [α], the integral part of α

$$H^{\alpha}(n, t, \alpha) = \frac{1}{\gamma(\alpha+1)} \int_t^{\pi} \frac{d}{dv} g(n, v)\, dv$$

and $\qquad H(n, t) = H^*(n, t, 0).$

3. Generalizing the theorem's of Bosanquet [1], [2], Samal [9] has proved the following theorem.

Theorem A. Let $1 > c > 0$. Let the function q_c satisfy the conditions $\int_0^1 q_\delta(t)\, dt = 1$ and $0 \le \delta < 1$, $q_\delta(t)$ is increasing for $0 < t < 1$ and let $Q_c(t)/t^{c+1} \in L(0, 1)$. Then

$$\int_0^{\pi} t^{-c}\, |d\phi(t)| < \infty = \Sigma|N(q_c)|.$$

In 2000 Dikshit [4] extended the above result for a absolute Nevanlinna summability of Fourier series as follows :

Theorem B. Let $\alpha \ge 0$ and let the functions q_α satisfy the conditions

$$\int_0^1 q_\delta(t)\, dt = 1$$

for $\delta \ge 0$, $p = [\delta]$, we assume

$$\frac{Q_\delta(t)}{t^{\delta - p + 1}} \in L(0, 1)$$

where $\qquad Q_\delta(t) = \int_{1-t}^{1} q_\delta^{(p)}(x)\, dx$

with $\delta = \alpha$. If $\Phi_\alpha(t) \in BV(0, \pi)$, then $t = x$, the Fourier series of f is summable by the method $|N(q_\alpha)|$.

Main Theorem. We shall prove the following theorem.

Theorem. Let $\alpha \ge 0$, $1 < p \le \alpha$, $\alpha > 1/P'$ and let the function q_α satisfy the conditions

$$\int_0^1 q_\delta(t)\,dt = 1 \quad \text{and} \quad \psi_\alpha(t) = 0 \tag{4.1}$$

where $Q_\delta(t) = \int_{1-t}^{1} q_\delta^{(p)}(x)\,dx$ and $\int_0^\pi t^{-\alpha-1}\,|d\psi_\alpha(t)| < \infty$ then at t = x the derived series of a Fourier series of f is summable by the method $|N(q_\alpha)|$.

5. Proof of the theorem

Let $J_n^\mu(x)$ denote the sum of the first n terms of the series (2.2) at the point t = x, then we have

$$J_n^\mu(x) = -\frac{1}{2\pi}\int_0^\pi \frac{d}{dt}\left\{\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}t}\right\} Q_\delta(t)\,N(F, q_\delta)(\omega)\,dt$$

where $N(F, q_\delta)(\omega)$ is the Nevanlinna mean of the sequence {sin nt}. Now, on integration by parts, we obtain

$$J_n^\mu(x) = \frac{1}{2\pi}\int_0^\pi \left\{\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\} N(F, q_\delta)(\omega)\,\frac{d}{dt}\{Q_\delta(t)\}\,dt$$

$$= \left[\sum_{\rho=1}^{h} (-1)^{\rho-1} Q_\delta(t)\left(\frac{d}{dt}\right)^{\rho} N(F, q_\delta)(\omega)\right]_0^\pi$$

$$+ (-1)^{h+1}\int_0^\pi Q_\delta(t)\left(\frac{d}{dt}\right)^{h+1} N(F, q_\delta)(\omega)\,dt$$

$$= 0(n^{\alpha-\beta}) + (-1)^{h+1} M_n$$

where
$$M_n = \int_0^\pi Q_\delta(t)\left(\frac{d}{dt}\right)^{h+1} N(F, q_\delta)(\omega)\,dt$$

$$= \frac{1}{\Gamma(h+1-\alpha)}\int_0^\pi \left(\frac{d}{dt}\right)^{h+1} N(F, q_\delta)(\omega)\,dt \int_0^t (t-u)^{h-\alpha} Q_\delta(u)\,du$$

$$= \frac{1}{\Gamma(h+1-\alpha)} \int_0^t Q_\delta(u)\, du \int_0^\pi (t-u)^{h-\alpha} \left(\frac{d}{dt}\right)^{h+1} N(F, q_\delta)(\omega)\, dt.$$

Thus to complete the proof of the theorem it is sufficient to show that

$$\Sigma \frac{|J_n^\mu(x)|}{n} < \infty. \tag{5.1}$$

To prove the above it is enough to show that

$$\int_0^\pi \Sigma \frac{|g(n,u)|}{n} Q_\delta(u)\, du < \infty \tag{5.2}$$

when $g(n,u) = \dfrac{1}{\Gamma(h+1-\alpha)} \displaystyle\int_0^\pi (t-u)^{h-\alpha} \left(\frac{d}{dt}\right)^{h+1} N(F, q_\delta)(\omega)\, dt.$

Again (5.1) holds provided that

$$\Sigma \frac{|g(n,u)|}{n} = O(u^{-\alpha-1}). \tag{5.3}$$

We now proceed to prove (5.3) writing

$$g(n,u) = \int_u^{u+\frac{1}{n}} + \int_{u+\frac{1}{n}}^{\pi}$$

Then

$$\Sigma \frac{|g(n,u)|}{n} = \sum_{n \le \frac{1}{u}} + \sum_{n > \frac{1}{u}}$$

$$= \sum_{n \le \frac{1}{u}} O(n^\alpha) + \sum_{n > \frac{1}{u}} O(n^{\alpha-\beta-1} u^{-\beta-1})$$

$$= O(u^{-\alpha-1}).$$

This completes the proof of (5.3). Thus the theorem follows from (5.2), (5.3) and the hypothesis.

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