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On the Hypersurfaces of Almost r-Contact Manifold

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Abstract

In the present paper, we have studied the structure induced on hypersurfaces of almost r-contact Riemannian manifold and proved that on this hypersurface with $(\phi, \xi_\alpha, \eta^\alpha)$ connection if 1-form $\tilde{\eta}^\alpha$, $\alpha \in (r)$ is covariant almost analytic vector field, then we have

$$b(\tilde{Y})\, h(\tilde{X}, \tilde{\xi}_\alpha) = b(\tilde{X})\, h(\tilde{Y}, \tilde{\xi}_\alpha)$$

provided $$(\tilde{\nabla}_{\tilde{X}} b)\, \tilde{Y} = (\tilde{\nabla}_{\tilde{Y}} b)\, \tilde{X} \quad \text{and } a_\alpha \neq 0$$

where b is 1-form, $\tilde{\nabla}$ is induced Riemannian connection on the hypersurface (M_{n-1}) and h is second fundamental tensor.

Keywords : Hypersurface, r-Contact manifold, Connection.

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1. Introduction

Let M be an n-dimensional Riemannian manifold with a positive definite metric g. If on M there exist a tensor field ϕ of type (1, 1), r vector field $\xi_1, \xi_2, \ldots\, \xi_r$ $(r < n)$ and r 1-forms $\eta^1, \eta^2, \ldots, \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \qquad \alpha, \beta \in (r) = \{1, 2, \ldots, r\} \tag{1.1}$$

$$\phi^2 = -I + \eta^\alpha \otimes \xi_\alpha, \qquad \alpha \in (r) \tag{1.2}$$

$$\eta^\alpha(X) = g(X, \xi_\alpha), \qquad \alpha \in (r) \tag{1.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \sum_{\alpha} \eta^{\alpha}(X)\, \eta^{\alpha}(Y) \tag{1.4}$$

where X, Y are vector fields on M and $a^{\alpha} b_{\alpha} \overset{\text{def}}{=} \Sigma\, a^{\alpha} b_{\alpha}$, then $\Sigma = (\phi, \xi_{\alpha}, \eta^{\alpha}, g)_{\alpha \in (r)}$ is said to be an almost r-contact Riemannian structure on M is almost r-contact Riemannian manifold.

From (1.1) through (1.4), we also have

$$\phi(\xi_{\alpha}) = 0, \qquad \alpha \in (r) \tag{1.5}$$

$$\eta^{\alpha} \circ \phi = 0, \qquad \alpha \in (r) \tag{1.6}$$

$$'F(X, Y) \overset{\text{def}}{=} g(\phi X, Y) = -\, 'F(X, Y). \tag{1.7}$$

Theorem 1.1. An almost r-contact structure on M is normal iff

$$N(X, Y) = N_{\phi}(X, Y) - 2\, d\eta^{\alpha}(X, Y)\, \xi_{\alpha} = 0,$$

where N_{ϕ} is the Nijenhuis tensor field for ϕ and $\alpha \in (r)$.

For a Riemannian connection ∇ on M, the tensor N may be expressed in the following way

$$N_{\phi}(X, Y) = (\nabla_{\phi Y} \phi)(X) - (\nabla_X \phi)(\phi Y) - (\nabla_{\phi X} \phi)(Y) + (\nabla_Y \phi)(\phi X)$$
$$+ \eta^{\alpha}(X)\, \nabla_Y \xi_{\alpha} - \eta^{\alpha}(Y)\, \nabla_X \xi_{\alpha}. \tag{1.8}$$

2. Hypersurface of almost r-contact Riemannian manifold

Let M_n be a n-dimensional Riemannian manifold with a positive definite metric g and let M_{n-1} be a hypersurface immersed in M_n.

If J* denotes the differential of the immersion J of M_{n-1} into M_n and $\tilde{X}$ is a vector field on M_{n-1}.

Remark. From now, on all objects on M_{n-1} will be denoted with the mark '~' over them e.g. $\tilde{\phi}$, $\tilde{X}$ etc.

Let N be the unit normal field to M_{n-1}. The induced metric g on M_{n-1} is defined by

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(\tilde{X}, \tilde{Y}) \tag{2.1}$$

we have

$$\tilde{g}(\tilde{X}, N) = 0 \quad \text{and} \quad \tilde{g}(N, N) = 1. \tag{2.2}$$

If ∇ is the Riemannian connection in M_n, then the Gauss and Wiengarten formulas are given respectively by

$$\nabla_{\tilde{X}} \tilde{Y} = \tilde{\nabla}_{\tilde{X}} \tilde{Y} + h(\tilde{X}, \tilde{Y}) N \tag{2.3}$$

$$\nabla_{\tilde{X}} N = -H(\tilde{X}) \tag{2.4}$$

where $\tilde{\nabla}$ is the Riemannian connection in M_{n-1} and h is the second fundamental tensor satisfying

$$h(\tilde{X}, \tilde{Y}) = h(\tilde{Y}, \tilde{X}) = \tilde{g}(H(\tilde{X}), \tilde{Y}). \tag{2.5}$$

Now suppose that $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is an almost r-contact Riemannian structure on M_n. Then every vector field X on M_n is decomposed as

$$X = \tilde{X} + u(X) N \tag{2.6}$$

where u is 1-form on M_n, and for every vector field $\tilde{X}$ on M_{n-1} and the normal N, we have

$$\phi \tilde{X} = \tilde{\phi} \tilde{X} + b(\tilde{X}) N \tag{2.7}$$

$$\phi N = \tilde{N} + \lambda N \tag{2.8}$$

where $\tilde{\phi}$ is a tensor field of type (1, 1) on a hypersurface M_{n-1}, b is a 1-form on M and λ is a scalar function on M.

For each $\alpha \in (r)$, we have

$$\xi_\alpha = \tilde{\xi}_\alpha + a_\alpha N, \qquad \alpha \in (r) \tag{2.9}$$

$$a_\alpha = u(\xi_\alpha) = \eta^\alpha(N), \qquad \alpha \in (r) \tag{2.10}$$

We define 1-form $\tilde{\eta}^\alpha$ as follows :

$$\tilde{\eta}^\alpha(\tilde{X}) = \eta^\alpha(\tilde{X}), \qquad \alpha \in (r) \tag{2.11}$$

from (2.7), (2.8), (2.9), (2.10) and (2.11), we have

$$\tilde{\phi}^2 \tilde{X} + b(\tilde{X}) \tilde{N} = -\tilde{X} + \tilde{\eta}^{\alpha}(\tilde{X}) \tilde{\xi}_{\alpha} \tag{2.12}$$

and

$$b(\tilde{\phi} X) + \lambda b(\tilde{X}) = a_{\alpha} \tilde{\eta}^{\alpha}(\tilde{X}) \tag{2.13}$$

$$\tilde{\phi} \tilde{N} + \lambda \tilde{N} = \sum_{\alpha} a_{\alpha} \tilde{\xi}_{\alpha} \tag{2.14}$$

$$b(\tilde{N}) + \lambda^2 = \sum_{\alpha} a_{\alpha}^2 - 1 \tag{2.15}$$

$$\tilde{\phi}\tilde{\xi}_{\alpha} = -a_{\alpha} \tilde{N} \qquad \alpha \in (r) \tag{2.16}$$

$$b(\tilde{\xi}_{\alpha}) + \lambda a_{\alpha} = 0 \qquad \alpha \in (r) \tag{2.17}$$

$$(\tilde{\eta}^{\alpha} \circ \tilde{\phi}) \tilde{X} + b(\tilde{X}) a_{\alpha} = 0, \qquad \alpha \in (r) \tag{2.18}$$

$$\delta_{\beta}^{\alpha} = \tilde{\eta}^{\alpha}(\tilde{\xi}_{\beta}) + a_{\beta} a_{\alpha} \tag{2.19}$$

$$\tilde{\eta}^{\alpha}(\tilde{X}) = \tilde{g}(\tilde{X}, \tilde{\xi}_{\alpha}). \tag{2.20}$$

Since $g(\phi\tilde{X}, \phi\tilde{Y}) = g(\tilde{X}, \tilde{Y}) - \sum_{\alpha} \eta^{\alpha}(\tilde{X}) \eta^{\alpha}(\tilde{Y}),$

therefore

$$g(\tilde{\phi} \tilde{X}, \tilde{\phi} \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}) - \sum_{\alpha} \tilde{\eta}^{\alpha}(\tilde{X}) \tilde{\eta}^{\alpha}(\tilde{Y}) - b(\tilde{X}) b(\tilde{Y}). \tag{2.21}$$

Since $'F(\tilde{X}, \tilde{Y}) = g(\phi\tilde{X}, \tilde{Y}).$

We find

$$'F(\tilde{X}, \tilde{Y}) = '\tilde{F}(\tilde{X}, \tilde{Y}) \tag{2.22}$$

again since $0 = g(\tilde{\phi} \tilde{X}, N)$

$$0 = -b(\tilde{X}) - \tilde{g}(\tilde{X}, \tilde{N})$$

i.e.

$$\tilde{g}(\tilde{X}, \tilde{N}) = -b(\tilde{X}). \tag{2.23}$$

Differentiating covariantly (2.7) and (2.8) along M_{n-1}, using (2.3) and (2.4), we have

$$(\nabla_{\tilde{Y}}\, \phi)\, \tilde{X} = (\tilde{\nabla}_{\tilde{Y}}\, \tilde{\phi})\, \tilde{X} + \{(\tilde{\nabla}_{\tilde{Y}}\, b)\, \tilde{X} - \lambda\, h\, (\tilde{X}, \tilde{Y}) + h(\tilde{\phi}\, \tilde{X}, \tilde{Y})\}\, N$$
$$- h\, (\tilde{X}, \tilde{Y})\, \tilde{N} + b(\tilde{X})\{\tilde{\nabla}_{\tilde{Y}}\, N + h(N, \tilde{Y})\, N\} \qquad (2.24)$$

$$(\nabla_{\tilde{Y}}\, \phi)\, N = \tilde{\nabla}_{\tilde{Y}}\, \tilde{N} + \tilde{\phi}\, (H(\tilde{Y})) - \lambda\, H(\tilde{Y}) + N\tilde{Y}(\lambda) \qquad (2.25)$$

We also have from (2.9) and (2.10)

$$\nabla_{\tilde{Y}}\, \xi_\alpha = \tilde{\nabla}_{\tilde{Y}}\, \tilde{\xi}_\alpha + \{h\, (\tilde{Y}, \tilde{\xi}_\alpha) + \tilde{Y}\, (a_\alpha)\}\, N - H(\tilde{Y})\, a_\alpha \qquad (2.26)$$

from above

$$(\nabla_{\tilde{Y}}\, \eta^\alpha)\, \tilde{X} = (\tilde{\nabla}_{\tilde{Y}}\, \eta^\alpha)\, \tilde{X} - a_\alpha\, h\, (\tilde{X}, \tilde{Y}) \qquad (2.27)$$

and since $(\nabla_{\tilde{Z}}\, 'F)\, (\tilde{X}, \tilde{Y}) = \tilde{g}\, ((\nabla_{\tilde{Z}}\, \phi)\tilde{X}, \tilde{Y})$

$$(\nabla_{\tilde{Z}}\, 'F)\, (\tilde{X}, \tilde{Y}) = (\tilde{\nabla}_{\tilde{Z}}\, '\tilde{F})\, \tilde{X} - b\, (\tilde{Y})\, h\, (\tilde{X}, \tilde{Z}) - b(\tilde{X})\, h(\tilde{Y}, \tilde{Z})\,. \qquad (2.28)$$

3. Hypersurfaces of almost r-contact manifold with $(\phi, \xi_\alpha, \eta^\alpha)$ connections

If almost r-contact manifold satisfying $(\phi, \xi_\alpha, \eta^\alpha)$ connection, then

$$\nabla\, \phi = 0,$$

$$\nabla\, \xi_\alpha = 0, \quad \alpha \in (r)$$

$$\nabla\, \eta^\alpha = 0, \quad \alpha \in (r)$$

where ∇ denotes the covariant differentiation with respect to a symmetric affine connection on M_n.

Hence we can state the following theorem :

Theorem 3.1. On the hypersurfaces of almost r-contact manifold with $(\phi, \xi_\alpha, \eta^\alpha)$ connection, we have

(a) $(\tilde{\nabla}_{\tilde{X}}\, \tilde{\phi})\, \tilde{Y} = h\, (\tilde{X}, \tilde{Y})\, \tilde{N} + H(\tilde{X})\, b(\tilde{Y}) - \{(\tilde{\nabla}_{\tilde{X}}\, b)\tilde{Y} - \lambda h(\tilde{X}, \tilde{Y}) + h(\tilde{X}, \tilde{\phi}\tilde{Y})\}N$

(b) $(\tilde{\nabla}_{\tilde{X}}\, \tilde{N}) = \lambda H(\tilde{X}) - \tilde{\phi}\, H(\tilde{X}) - N\tilde{X}\, (\lambda)$

(c) $(\tilde{\nabla}_Z {}'\tilde{F})(\tilde{X}, \tilde{Y}) = b(\tilde{X})\, h(\tilde{Y}, \tilde{Z}) + b(\tilde{Y})\, h(\tilde{X}, \tilde{Z})$

(d) $(\tilde{\nabla}_{\tilde{Y}}\ \tilde{\xi}_\alpha) = H(\tilde{Y})\, a_\alpha - \{h(\tilde{Y}, \tilde{\xi}_\alpha) + \tilde{Y}(a_\alpha)\}\, N$

(e) $(\tilde{\nabla}_{\tilde{Y}}\ \tilde{\eta}^\alpha)\,\tilde{X} = a_\alpha\, h(\tilde{X}, \tilde{Y})$.

Further if $\tilde{\xi}_\alpha$, $\alpha \in (r)$ is contravariant almost analytic vector fields on the hypersurface M_{n-1}, then

$$\pounds_{\tilde{\xi}_\alpha} \tilde{\phi} = 0,$$

where £ denotes the Lie derivative, then

$$[\tilde{\xi}_\alpha, \tilde{\phi}\tilde{X}] = \tilde{\phi}\,[\tilde{\xi}_\alpha, \tilde{X}]$$

or
$$(\tilde{\nabla}_{\tilde{\xi}_\alpha} \tilde{\phi})\,\tilde{X} = \tilde{\nabla}_{\tilde{\phi}\tilde{X}}\,\tilde{\xi}_\alpha - \tilde{\phi}\,(\nabla_{\tilde{X}}\,\tilde{\xi}_\alpha)$$

then from theorem 3.1(a), we have the following theorem :

Theorem 3.2. On the hypersurfaces of almost r-contact manifold with $(\phi, \xi_\alpha, \eta^\alpha)$ connection if $\tilde{\xi}_\alpha$, $\alpha \in (r)$ is contravariant almost analytic vector field then we have

$$'\tilde{F}(\tilde{\nabla}_{\tilde{X}}\,\tilde{\xi}_\alpha, \tilde{Y}) = b(\tilde{X})\, h(\tilde{\xi}_\alpha, \tilde{Y}) - b(\tilde{Y})\, h(\tilde{\xi}_\alpha, \tilde{X}) - a_\alpha\, h(\tilde{X}, \tilde{Y}).$$

4. Killing vector field

A vector field ξ_α, $\alpha \in (r)$ is killing vector field with respect to Riemannian connection ∇ then we have

$$\pounds_{\xi_\alpha}\, g = 0,$$

i.e.
$$(\nabla_{\tilde{Y}}\ \eta^\alpha)\,\tilde{X} + (\nabla_{\tilde{X}}\ \eta^\alpha)\,\tilde{Y} = 0$$

from (2.27)
$$(\tilde{\nabla}_{\tilde{Y}}\,\tilde{\eta}^\alpha)\,\tilde{X} + (\tilde{\nabla}_{\tilde{X}}\,\tilde{\eta}^\alpha)\,\tilde{Y} = 2a_\alpha\, h(\tilde{X}, \tilde{Y})$$

this leads to the following theorem :

Theorem 4.1. On the hypersurfaces of almost r-contact manifold if ξ_α, $\alpha \in (r)$ is killing vector field with respect to Riemannian connection.

We have

$$(\tilde{\nabla}_{\tilde{Y}} \tilde{\eta}^{\alpha})\tilde{X} + (\tilde{\nabla}_{\tilde{X}} \tilde{\eta}^{\alpha})\tilde{Y} = 2a_{\alpha}\, h(\tilde{X}, \tilde{Y}).$$

5. Covariant almost analytic vector field

An 1-form η^{α} (r) is said to be covariant almost analytic if

$$\tilde{\eta}^{\alpha}\,((\tilde{\nabla}_{\tilde{X}} \tilde{\phi})\tilde{Y} - (\tilde{\nabla}_{\tilde{Y}} \tilde{\phi})\tilde{X}) = (\tilde{\nabla}_{\tilde{\phi}\tilde{X}} \tilde{\eta}^{\alpha})\tilde{Y} - (\tilde{\nabla}_{\tilde{X}} \tilde{\eta}^{\alpha})\tilde{\phi}\tilde{Y}$$

using (2.24) and (2.27), we have

$$(\nabla_{\phi\tilde{X}} \eta^{\alpha})\tilde{Y} - (\nabla_{\tilde{X}} \eta^{\alpha})\tilde{\phi}\tilde{Y} = \eta^{\alpha}((\nabla_{\tilde{X}} \phi)\tilde{Y} - (\nabla_{\tilde{Y}} \phi)\tilde{X}) - a_{\alpha}(\tilde{\nabla}_{\tilde{X}} b)\tilde{Y}$$

$$- (\tilde{\nabla}_{\tilde{Y}} b)\tilde{X}\} + b(\tilde{Y})\,\tilde{\eta}^{\alpha}(H(\tilde{X})) - b(\tilde{X})\,\tilde{\eta}^{\alpha}(H(\tilde{Y})).$$

Further with respect to $(\phi, \xi_{\alpha}, \eta^{\alpha})$ connections above becomes

$$a_{\alpha}\{(\tilde{\nabla}_{\tilde{X}} b)\tilde{Y} - (\tilde{\nabla}_{\tilde{Y}} b)\tilde{X}\} = -b(\tilde{X})\,\tilde{g}(H(\tilde{Y}), \tilde{\xi}_{\alpha}) + b(\tilde{Y})\,\tilde{g}(H(\tilde{X}), \tilde{\xi}_{\alpha})$$

$$= b(\tilde{Y})\,h(\tilde{X}, \tilde{\xi}_{\alpha}) - b(\tilde{X})\,h(\tilde{Y}, \tilde{\xi}_{\alpha}).$$

Hence we can state the following theorem :

Theorem 5.1. On the hypersurfaces of almost r-contact manifold with $(\phi, \xi_{\alpha}, \eta^{\alpha})$ connection if 1-form $\tilde{\eta}^{\alpha}$, $\alpha \in (r)$ is covariant almost analytic vector field then

$$b(\tilde{Y})\,h(\tilde{X}, \tilde{\xi}_{\alpha}) = b(\tilde{X})\,h(\tilde{Y}, \tilde{\xi}_{\alpha})$$

provided $\quad (\tilde{\nabla}_{\tilde{X}} b)\tilde{Y} = (\tilde{\nabla}_{\tilde{Y}} b)\tilde{X}, \quad a_{\alpha} \neq 0.$

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