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T3-Like Finsler Spaces

S. C. Rastogi

Set Vishambhar Nath Institute of Engineering, Research and Technology,
Safedabad, Barabanki-225003, U.P., India
e-mail : sureshrastogi@rediffmail.com
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Abstract

In 1972, T-tensor in a Finsler space of n-dimensions was introduced and studied simultaneously by H. Kawaguchi [3] and Matsumoto [5]. Several papers related with T-tensor, since then, have been published by various authors namely Hashiguchi [1], Matsumoto [6], Matsumoto and Shimada [7, 8], Rastogi [10, 11] and others. The purpose of the present paper is to study some properties of T-tensor in a Finsler space of three dimensions. Furthermore, we have defined and studied Finsler spaces Fn, whose T-tensor is of special form and called them T3-like Finsler spaces.

1. Introduction

Let (l^i, m^i, n^i) be the Moor's frame of a Finsler space of three dimensions F^3, where l^i is normalized supporting element : $y^i = Ll^i$, m^i is normalized torsion vector given by $C^i = Cm^i$ and n^i is a unit vector orthogonal to both m^i and l^i. The metric and angular metric tensors in F^3 are given by [6] :

$$g_{ij} = l_i l_j + m_i m_j + n_i n_j, \qquad h_{ij} = m_i m_j + n_i n_j \tag{1.1}$$

while the Cartan's C-tensor is given by Matsumoto [6] as follows :

$$\begin{aligned} C_{ijk} = C_{(1)} m_i m_j m_k - C_{(2)} (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j) \\ + C_{(3)} (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) + C_{(2)} n_i n_j n_k. \end{aligned} \tag{1.2}$$

The T-tensor in a Finsler space of n-dimensions is given by

$$T_{ijkh} = LC_{ijk}l_h + l_i C_{jkh} + l_j C_{ikh} + l_k C_{ijh} + l_h C_{ijk}, \tag{1.3}$$

where $|_h$ denotes v-covariant derivative Matsumoto [6].

2. Cartan's C-Tensor in F^3

Let $C_{ijk}\, m^k = {}'C_{ij}$ and $C_{ijk}\, n^k = {}^*C_{ij}$, then from equation (1.2) we can obtain

$$'C_{ij} = C_{(1)}\, m_i\, m_j - C_{(2)}\,(m_i\, n_j + m_j\, n_i) + C_{(3)}\, n_i\, n_j \tag{2.1}$$

and
$$^*C_{ij} = C_{(2)}\,(n_i\, n_j - m_i\, m_j) + C_{(3)}(m_i\, n_j + m_j\, n_i), \tag{2.2}$$

i.e., these tensors are symmetric in lower indices.

Further from equations (2.1) and (2.2), we can obtain

$$'C_{ij}\, m^j = C_{(1)}\, m_i - C_{(2)}\, n_i\,, \quad {}^*C_{ij}\, m^j = -\, C_{(2)}\, m_i + C_{(3)}\, n_i\,, \tag{2.3}$$

$$'C_{ij}\, m^j\, m^i = C_{(1)}, \quad 'C_{ij}\, m^j\, n^i = -\, C_{(2)}, \quad 'C_{ij}\, n^j\, n^i = C_{(3)} \tag{2.4}$$

and
$$^*C_{ij}\, m^j\, n^i = -\, C_{(2)}\,, \quad {}^*C_{ij}\, m^j\, n^i = C_{(3)}, \quad {}^*C_{ij}\, n^j\, n^i = C_{(2)}. \tag{2.5}$$

In a three dimensional Finsler space F^3, v-covariant derivatives of the vectors l^i, m^i and n^i are given in Matsumoto [6] as follows :

$$L l^i|_j = h^i_j\,, \quad L m^i|_j = -\, l^i\, m_j + n^i\, v_j, \quad L n^i|_j = -\, l^i\, n_j - m^i\, v_j \tag{2.6}$$

where $v_j = v_{2)3\gamma}\, e_{\gamma)j}$, therefore from equations (2.1) and (2.2) we can respectively obtain

$$\begin{aligned} 'C_{ij}|_h = {} & C_{(1)}|_h\, m_i\, m_j - C_{(2)}\,|_h\,(m_i\, n_j + m_j\, n_i) + C_{(3)}|_h\, n_i\, n_j \\ & + L^{-1}[C_{(1)}\{m_i\,(-\, l_j\, m_h + n_j\, v_h) + m_j\,(-\, l_i\, m_h + n_i\, v_h)\} \\ & + C_{(2)}\,\{m_i\,(l_j\, n_h + m_j\, v_h) + m_j\,(l_i\, n_h + m_i\, v_h) + n_i(l_j\, m_h - n_j\, v_h) \\ & + n_j\,(l_i\, m_h - n_i\, v_h)\} - C_{(3)}\{n_i(l_j\, n_h + m_j\, v_h) + n_j\,(l_i\, n_h + m_i\, v_h)\}] \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} ^*C_{ij}|_h = {} & C_{(2)}|_h\, n_i\, n_j - m_i\, m_j) + C_{(3)}\,|_h\,(m_i\, n_j + m_j\, n_i) \\ & + L^{-1}[C_{(2)}\{m_i\,(\, l_j\, m_h - n_j\, v_h) + m_j\,(\, l_i\, m_h - n_i\, v_h) \\ & - n_i\,(l_j\, n_h + m_j\, v_h) - n_j\,(l_i\, n_h + m_i\, v_h)\} - C_{(3)}\{m_i(l_j\, n_h + m_j\, v_h) \end{aligned}$$

$$+ m_j (l_i n_h + m_i v_h) + n_i(l_j m_h - n_j v_h) + n_j (l_i n_h - n_i v_h)\}]. \qquad (2.8)$$

Equations (2.7) and (2.8) respectively give

$$'C_{ij}|_h l^j + L^{-1}\, 'C_{ih} = 0,$$

$$'C_{ij}|_h l^h = C_{(1)}|_0 m_i m_j - C_{(2)}|_0(m_i n_j + m_j n_i) + C_{(3)}|_0 n_i n_j,$$

$$'C_{ij}|_h m^j = L^{-1} l_i (C_{(2)} n_h - C_{(1)} m_h) + m_i (C_{(1)}|_h + 2L^{-1} C_{(2)} v_h)$$
$$+ n_i\{L^{-1}(C_{(1)} - C_{(3)}) v_h - C_{(2)}|_h\},$$

$$'C_{ij}|_h n^j = - L^{-1} l_i (C_{(2)} m_h + C_{(3)} n_h) + m_i \{L^{-1}(C_{(1)} - C_{(3)}) v_h - C_{(2)}|_h\}$$
$$+ n_i (C_{(3)}|_h - 2L^{-1}C_{(2)} v_h)$$

$$*C_{ij}|_h l^j + L^{-1}\, *C_{ih} = 0,$$

$$*C_{ij}|_h l^h = C_{(2)}|_0 (n_i n_j - m_i m_j) + C_{(3)}|_0 (m_i n_j + m_j n_i),$$

$$*C_{ij}|_h m^j = L^{-1} l_i (C_{(2)} m_h - C_{(3)} n_h) - m_i (C_{(2)}|_h + 2L^{-1} C_{(3)} v_h)$$
$$+ n_i(C_{(3)}|_h - 2L^{-1}C_{(2)} v_h),$$

$$*C_{ij}|_h n^j = - L^{-1} l_i (C_{(2)} n_h + C_{(3)} m_h) + m_i (C_{(3)}|_h - 2L^{-1}C_{(2)}v_h)$$
$$+ n_i (C_{(2)}|_h + 2L^{-1}C_{(3)} v_h) \qquad (2.9)$$

and

$$'C_{ij}|_h m^j m^h m^i = 2L^{-1}C_{(2)} v_{2)32} + C_{(1)}|_h m^h,$$

$$'C_{ij}|_h m^j m^h n^i = L^{-1}(C_{(1)} - C_{(3)})v_{2)32} - C_{(2)}|_h m^h,$$

$$'C_{ij}|_h m^j n^h m^i = 2L^{-1}C_{(2)} v_{2)33} + C_{(1)}|_h n^h,$$

$$'C_{ij}|_h m^j n^h n^i = L^{-1}(C_{(1)} - C_{(3)})v_{2)323} - C_{(2)}|_h n^h,$$

$$*C_{ij}|_h m^j m^h m^i = - 2L^{-1}C_{(3)} v_{2)32} - C_{(2)}|_h m^h,$$

$$*C_{ij}|_h m^j m^h n^i = - 2L^{-1} C_{(2)} v_{2)32} + C_{(3)}|_h m^h,$$

$$*C_{ij}|_h m^j n^h m^i = - 2L^{-1}C_{(3)} v_{2)33} - C_{(2)}|_h n^h,$$

$$*C_{ij}|_h\, m^j\, n^h\, n^i = -\,2L^{-1}\, C_{(2)}\, v_{2)33} + C_{(3)}|_h\, n^h. \tag{2.10}$$

Since $C_{(1)} + C_{(3)} = C$, therefore from equation (2.10), we can obtain

$$\{('C_{ij}|_h\, m^i + *C_{ij}|_h\, n^i)\, m^j - C|_h\}\, m^h = 0, \tag{2.11}$$

$$\{('C_{ij}|_h\, n^i + *C_{ij}|_h\, m^i)\, m^j\, m^h = L^{-1}\, C_{(1)}\, v_{2)32} \tag{2.12}$$

and

$$2\, *C_{ij}|_h\, m^j\, (C_{(3)}\, n^i - C_{(2)}\, m^i) = (C_{(2)}{}^2 + C_{(3)}{}^2)|_h. \tag{2.13}$$

Hence, we have :

Theorem (2.1). In a three dimensional Finsler space F^3, $'C_{ij}|_h$ and $*C_{ij}|_h$ satisfy equations (2.11), (2.12) and (2.13).

If we take h-covariant derivative of equations (2.1) and (2.2) and use $l^i|_j = 0$, $m^i|_j = n^i\, h_j$, $n^i|_j = -\, m^i\, h_j$, we get

$$'C_{ij|k} = (C_{(1)|k} + 2C_{(2)}\, h_k)\, m_i\, m_j + (C_{(3)|k} - 2C_{(2)}\, h_k)\, n_i\, n_j$$
$$+ \{(C_{(1)} - C_{(3)})\, h_k - C_{(2)|k}\}\, (m_i\, n_j + m_j\, n_i) \tag{2.14}$$

and

$$*C_{ij|k} = (C_{(3)|k} - C_{(2)}\, h_k)\, (n_i\, m_j + m_i\, n_j) + (C_{(2)|k} + 2C_{(3)}\, h_k)(n_i\, n_j - m_i\, m_j), \tag{2.15}$$

which leads to

$$(*C_{ij|k}\, m^j - 'C_{ij|k}\, n^j)\, n^i - C_{(2)}\, h_k = 0, \tag{2.16}$$

$$'C_{ij|k}\, (m^j\, m^i + n^j\, n^i) = C_{|k}, \tag{2.17}$$

and

$$'C_{ij|k}\, m^j\, n^i - *C_{ij|k}\, m^j\, m^i = 'C_{ij|k}\, m^j\, n^i + *C_{ij|k}\, n^j\, n^i = C\, h_k. \tag{2.18}$$

Hence, we have

Theorem (2.2). In a three dimensional Finsler space F^3, h-covariant derivative of $'C_{ij}$ and $*C_{ij}$ satisfy equations (2.16), (2.17) and (2.18).

In case of a P*-Finsler space F^3, Izumi [2], $P_{ijk} = \lambda A_{ijk}$, therefore we can obtain

$$'C_{ij|0} = \lambda\, 'C_{ij} + h_0\, *C_{ij}, \qquad *C_{ij|0} = \lambda\, *C_{ij} - h_0\, 'C_{ij}. \tag{2.19}$$

Multiplying equation (2.19) by m^j and using (2.1) and (2.2), we can obtain on simplification

$$C_{(1)|0} = \lambda C_{(1)} - 3h_0 C_{(2)}, \qquad C_{(2)|0} = \lambda C_{(2)} + (C_{(1)} - 2C_{(3)}) h_0,$$

$$C_{(3)|0} = \lambda C_{(3)} + 3 h_0 C_{(2)}. \tag{2.20}$$

Hence, we have

Theorem (2.3). In a three dimensional P*-Finsler space F^3, $C_{(1)|0}$, $C_{(2)|0}$ and $C_{(3)|0}$ are respectively given by (2.20).

Furthermore, from equation (2.19), we can obtain with the help of equations (2.4) and (2.5)

$$'C_{ij|0}\, m^j m^i = C_{(1)|0} + 2C_{(2)} h_0, \qquad 'C_{ij|0}\, m^j n^i = - C_{(2)|0} + h_0(C_{(1)} - C_{(3)}),$$

$$'C_{ij|0}\, n^j n^i = C_{(3)|0} - 2C_{(2)} h_0. \tag{2.21}$$

Hence, we have

Theorem (2.4). In a three dimensional P*-Finsler space F^3, $'C_{ij|0}$ satisfies equation (2.21).

Further from equation (2.19), we can obtain on simplification

$$'C_{ij} = (h_0{}^2 + \lambda^2)^{-1} (\lambda\, 'C_{ij|0} - h_0\, {}^*C_{ij|0}) \tag{2.22}$$

and

$$^*C_{ij} = (h_0{}^2 + \lambda^2)^{-1} (\lambda\, {}^*C_{ij|0} + h_0\, 'C_{ij|0}) \tag{2.23}$$

which lead to

Theorem (2.5). In a P*-Finsler space F^3, $'C_{ij}$ and $^*C_{ij}$ respectively satisfy equations (2.22) and (2.23).

3. v-Covariant Derivative of C-Tensor in F^3

Taking v-covariant derivative of (1.2) and using equation (2.5) we can obtain on simplification

$$C_{ijk}|_h = C_{(1)}|_h\, m_i m_j m_k - C_{(2)}|_h\, (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k)$$

$$+ C_{(3)}|_h (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j)$$
$$- L^{-1}\Sigma_{(i,j,k)}[C_{(1)}(l_i m_h - n_i v_h) m_j m_k - C_{(2)}\{l_i (m_h m_j n_k + m_j m_k n_h$$
$$+ m_h m_k n_j - n_h n_j n_k) + v_h (m_i m_j m_k - m_j m_k n_i - 2m_i n_k n_j)\}$$
$$+ C_{(3)}\{l_i(m_h n_j n_k + m_j n_k n_h + m_k n_h n_j) + 2m_i m_j v_h n_k\}], \tag{3.1}$$

which by virtue of symmetry of $C_{ijk}|_h$ in k and h easily leads to

$$\zeta_{(k,h)}[\{C_{(1)|h} m_i m_j m_k - C_{(2)|h} (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k)$$
$$+ C_{(3)|h} (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j)\} + L^{-1}[C_{(1)}\{(m_i n_j + m_j n_i). m_k v_h$$
$$+ m_i m_j (l_h m_k + n_k v_h)\} - C_{(2)}\{(m_i n_j + m_j n_i)(m_k l_h + 3n_k v_h)$$
$$+ (m_i m_j - n_i n_j)(l_h n_k + 3v_k m_h)\} + C_{(3)}\{(m_i n_j + m_j n_i).(n_k l_h + 2v_k m_h)$$
$$+ (m_i m_j - n_i n_j) 2v_k n_h + n_i n_j (l_h m_k + v_h n_k)\}] = 0. \tag{3.2}$$

Multiplying equation (3.2) by $m^i m^j m^k$, we get on simplification

$$\alpha_h - \alpha_k m^k m_h + \beta_k m^k n_h = 0, \tag{3.3}$$

where α_h and β_h are given by

$$\alpha_h = LC_{(1)}|_h + C_{(1)} l_h + 3C_{(2)} v_h \tag{3.4}$$

and
$$\beta_h = LC_{(2)}|_h + C_{(2)} l_h - (C_{(1)} - 2C_{(3)}) v_h. \tag{3.5}$$

With the help of equations (3.3), (3.4) and (3.5), we can obtain

$$\alpha_0 = \alpha_h l^h = LC_{(1)}|_0 + C_{(1)}, \qquad \beta_0 = \beta_h l^h = LC_{(2)}|_0 + C_{(2)} \tag{3.6}$$

$$\alpha_h m^h = LC_{(1)}|_h m^h + 3C_{(2)} v_{2)32}, \tag{3.7}$$

$$\alpha_h n^h = LC_{(1)}|_h n^h + 3C_{(2)} v_{2)33}, \tag{3.8}$$

$$\beta_h m^h = LC_{(2)}|_h m^h - (C_{(1)} - 2C_{(3)}) v_{2)32}, \tag{3.9}$$

$$\beta_h n^h = LC_{(2)}|_h n^h - (C_{(1)} - 2C_{(3)}) v_{2)33}, \tag{3.10}$$

and

$$LC_{(1)}|_h n^h + 3C_{(2)} v_{2)33} + LC_{(2)}|_h m^h - (C_{(1)} - 2C_{(3)}) v_{2)32} = 0. \qquad (3.11)$$

Hence, we have

Theorem (3.1). In a three dimensional Finsler space F^n, the coefficients $C_{(1)}$, $C_{(2)}$ and $C_{(3)}$ satisfy equations (3.6) to (3.11).

Multiplying equation (3.2) by $n^i n^j n^k$, we get on simplification

$$LC_{(2)}|_h + C_{(2)} l_h + 3C_{(3)} v_h - (LC_{(3)}|_k n_k - 3C_{(2)} v_{2)33}) m_h$$
$$- (LC_{(2)}|_k n^k + 3C_{(3)} v_{2)33}) n_h = 0. \qquad (3.12)$$

Multiplying equation (3.12) by m^h, we get

$$3(C_{(2)} v_{2)33} + C_{(3)} v_{2)32}) = L(C_{(3)}|_h n^h - C_{(2)}|_h m^h). \qquad (3.13)$$

Hence, we have

Theorem (3.2). In a three dimensional Finsler space F^n, the coefficients $C_{(2)}|_h$ and $C_{(3)}|_h$ satisfy equation (3.13).

Multiplying equation (3.2) by $n^i n^k m^j m^h$, we get on simplification

$$3C_{(2)} v_{2)32} + (C_{(1)} - 2C_{(3)}) v_{2)33} = L(C_{(3)}|_h m^h + C_{(2)}|_h n^h). \qquad (3.14)$$

Hence, we have

Theorem (3.3). In a three dimensional Finsler space F^3, the coefficients $C_{(2)}|_h$ and $C_{(3)}|_h$ satisfy equation (3.14).

From equations (3.13) and (3.14), on simplification we can obtain

$$L[C_{(2)} C_{(3)} \{\log (C_{(3)} C_{(2)}^{-1})\} |_h n^h - (1/2) (C_{(2)}^2 + C_{(3)}^2) |_h m^h]$$
$$= v_{2)33} (3C_{(2)}^2 + 2C_{(3)}^2 - C_{(1)} C_{(3)}). \qquad (3.15)$$

Hence, we have

Theorem (3.4). In a three dimensional Finsler space F^n, the coefficients $C_{(1)}$, $C_{(2)}$ and $C_{(3)}$ satisfy equation (3.15).

Multiplying equation (3.1) by $g^{ij}(x, y)$ and using $C_{(1)} + C_{(3)} = C$, we can obtain on simplification

$$C_k|_h = C|_h\, m_k - L^{-1}\, C\{l_k\, m_h - (1/4)\, n_k\, v_h\}. \tag{3.16}$$

Hence, we have

Theorem (3.5). In a three dimensional Finsler space F^3, scalar C satisfies (3.16).

4. P-Tensor in F3

In a three dimensional Finsler space F^3, P-tensor is expressed as

$$P_{ijk} = L[(C_{(1)|0} + 3C_{(2)}\, h_0)\, m_i\, m_j\, m_k - \{C_{(2)|0} - (C_{(1)} - 2C_{(3)})\, h_0\}$$
$$(m_i\, m_j\, n_k + m_j\, m_k\, n_i + m_k\, m_i\, n_j) + (C_{(3)|0} - 3C_{(2)}\, h_0)$$
$$(m_i\, n_j\, n_k + m_j\, n_k\, n_i + m_k\, n_i\, n_j) + (C_{(2)|0} + 3C_{(3)}\, h_0)\, n_i\, n_j\, n_k]. \tag{4.1}$$

Let $P_{ijk}\, m^k = {}'P_{ij}$ and $P_{ijk}\, n^k = {}^*P_{ij}$, then from equation (4.1), we can obtain

$${}'P_{ij} = L[(C_{(1)|0} + 3C_{(2)}\, h_0)\, m_i\, m_j - \{C_{(2)|0} - (C_{(1)} - 2C_{(3)})\, h_0\}$$
$$(m_j\, n_i + m_i\, n_j) + (C_{(3)|0} - 3C_{(2)}\, h_0)\, n_i\, n_j] \tag{4.2}$$

and

$${}^*P_{ij} = L[-\{C_{(2)|0} - (C_{(1)} - 2C_{(3)}\, h_0\}\, m_i\, m_j + (C_{(3)|0} - 2C_{(2)}\, h_0)$$
$$(m_i\, n_j + m_j\, n_i) + (C_{(2)|0} + 3C_{(3)}\, h_0)\, n_i\, n_j] \tag{4.3}$$

which are symmetric tensors in lower indices. Further from equations (4.2) and (4.3), we can obtain

$${}'P_{ij}\, m^j = L[(C_{(1)|0} + 3C_{(2)}\, h_0)\, m_i - \{C_{(2)|0} - (C_{(1)} - 2C_{(3)}\, h_0\}\, n_i] \tag{4.4}$$

$${}'P_{ij}\, n^j = L[-\{C_{(2)|0} - (C_{(1)} - 2C_{(3)})\, h_0\}\, m_i + (C_{(3)|0} - 3C_{(2)}\, h_0)\, n_i] \tag{4.5}$$

$${}'P_{ij}\, m^j\, m^i = L(C_{(1)|0} + 3C_{(2)}\, h_0), \quad {}'P_{ij}\, m^j\, n^i = -L\{C_{(2)|0} - (C_{(1)} - 2C_{(3)})\, h_0\} \tag{4.6}$$

$${}'P_{ij}\, n^j\, n^i = L(C_{(3)|0} - 3C_{(2)}\, h_0), \tag{4.7}$$

$${}^*P_{ij}\, m^j = L[-\{C_{(2)|0} - (C_{(1)} - 2C_{(3)})\, h_0\}\, m_i + (C_{(3)|0} - 3C_{(2)}\, h_0)\, n_i], \tag{4.8}$$

$${}^*P_{ij}\, n^j = L[(C_{(3)|0} - 3C_{(2)}\, h_0)\, m_i + (C_{(2)|0} + 3C_{(3)}\, h_0)\, n_j], \tag{4.9}$$

$$*P_{ij}\, m^j\, m^i = -L\{C_{(2)|0} - (C_{(1)} - 2C_{(3)})\, h_0\}, \tag{4.10}$$

$$*P_{ij}\, m^j\, n^i = L(C_{(3)|0} - 3C_{(2)}\, h_0), \tag{4.11}$$

$$*P_{ij}\, n^j\, n^i = L(C_{(2)|0} + 3C_{(3)}\, h_0). \tag{4.12}$$

From $C_{ijk}\, m^k = 'C_{ij}$ and $C_{ijk}\, n^k = *C_{ij}$, we can easily obtain

$$'P_{ij} = L('C_{ij|0} - h_0\, *C_{ij}), \qquad *P_{ij} = L\,(*C_{ij|0} + h_0\, 'C_{ij}). \tag{4.13}$$

Hence, we have

Theorem (4.1). In a three dimensional Finsler space, tensors $'P_{ij}$ and $*P_{ij}$ are related with $'C_{ij}$ and $*C_{ij}$ by equation (4.13).

In case of a P*-Finsler space F^3, with the help of equation (2.19) and (4.13), we can establish

Theorem (4.2). In a three dimensional P*-Finsler space F^3, $'P_{ij} = L\lambda\, 'C_{ij}$ and $*P_{ij} = L\lambda *C_{ij}$.

5. T-Tensor in F^3

Substituting the value of C_{ijk} and $C_{ijk}|_h$ from equations (1.2) and (3.1) in (1.3), we can obtain

$$\begin{aligned}
T_{ijk} = {} & L[C_{(1)}|_h\, m_i\, m_j\, m_k - C_{(2)}|_h\, (m_i\, m_j\, n_k + m_j\, m_k\, n_i + m_k\, m_i\, n_j - n_i\, n_j\, n_k) \\
& + C_{(3)}|_h\, (m_i\, n_j\, n_k + m_j\, n_k\, n_i + m_k\, n_i\, n_j)] \\
& + C_{(1)}\, [m_i\, m_j\, m_k\, l_h + v_h\, (m_i\, m_j\, n_k + m_j\, m_k\, n_i + m_k\, m_i\, n_j)] \\
& - C_{(2)}\, [v_h\, (m_i\, n_j\, n_k + m_j\, n_k\, n_i + m_k\, n_i\, n_j - 3\, m_i\, m_j\, m_k) \\
& + l_h\, \{n_i\, n_j\, n_k - (m_i\, m_j\, n_k + m_j\, m_k\, n_i + m_k\, m_i\, n_j)\}] \\
& + C_{(3)}\, \{l_h\, (m_i\, n_j\, n_k + m_j\, n_k\, n_i + m_k\, n_i\, n_j) \\
& - 2\, v_h\, (m_i\, m_j\, n_k + m_j\, m_k\, n_i + m_k\, m_i\, n_j)\}.
\end{aligned} \tag{5.1}$$

Equation (5.1) can also be expressed as

$$T_{ijkh} = m_i\, m_j\, m_k\, \alpha_h - \Sigma_{(i,j,k)} \{m_i\, m_j\, n_k\, \beta_h - m_i\, n_j\, n_k\, \gamma_h\} + n_i\, n_j\, n_k\, \delta_n \quad (5.2)$$

where

$$\gamma_h = LC_{(3)}|_h + C_{(3)}\, l_h + C_{(2)}\, v_h, \qquad \delta_h = LC_{(2)}|_h + C_{(2)}\, l_h. \quad (5.3)$$

such that

$$\gamma_0 = \gamma_h\, l^h = LC_{(3)}|_0 + C_{(3)}, \qquad \delta_0 = \delta_h\, l^h = \beta_0 = LC_{(2)}\,|_0 + C_{(2)}$$

$$\gamma_h\, m^h = LC_{(3)}\,|_h\, m^h + C_{(2)}\, v_{2)32}, \qquad \delta_h\, m^h = LC_{(2)}\,|_h\, m^h$$

$$\gamma_h\, n^h = LC(3)\, |h\, nh + C(2)\, v2)33, \quad \delta_h\, n^h = LC_{(2)}|_h\, n^h \quad (5.4)$$

Multiplying equation (5.2) by $g_{jk}\, l^h$, we can obtain $LC\,|_0 = -C$, $LC_{(1)}|_0 = -C_{(1)}$ and $LC_{(2)}|_0 = -C_{(2)}$. Hence, we have

Theorem (5.1). In a three dimensional Finsler space F^3, if T-tensor is expressed by (5.2), coefficients α_0, β_0, γ_0, δ_0, $C_{(1)}$, $C_{(2)}$ and $C_{(3)}$ satisfy $\alpha_0 + \gamma_0 = 0$, $\beta_0 = \delta_0$, $LC|_0 = -C$, $LC_{(1)}|_0 = -C_{(1)}\,|_0 = -C_{(1)}$ and $LC_{(2)}\,|_0 = -C_{(2)}$.

From equation (5.2), we can obtain by virtue of $T_{ijkh}\, g^{kh} = T_{ij}$

$$T_{ij} = \{LC_{(1)}|_h\, m^h + 3C_{(2)}v_{2)32} - LC_{(2)}|_h\, n^h + (C_{(1)} - 2C_{(3)})\, v_{2)33}\}\, m_i\, m_j$$
$$+ \{LC_{(3)}|_h\, n^h + C_{(2)}v_{2)33} - LC_{(2)}|_h\, m^h + (C_{(1)} - 2C_{(3)})\, v_{2)32}\}.$$
$$(m_i\, n_j + m_j\, n_i) + \{LC_{(3)}|_h\, m^h + C_{(2)}v_{2)32} + LC_{(2)}|_h\, n^h\}\, n_i\, n_j \quad (5.5)$$

which satisfies $T_{ij}\, l^i = 0$,

$$T_{ij}\, m^i = \{LC_{(1)}|_h\, m^h + 3C_{(2)}v_{2)32} - LC_{(2)}|_h\, n^h + (C_{(1)} - 2C_{(3)})\, v_{2)33}\}m_j$$
$$+\{LC_{(3)}|_h\, n^h + C_{(2)}v_{2)33} - LC_{(2)}|_h\, m^h + (C_{(1)} - 2C_{(3)})\, v_{2)32}\}n_j \quad (5.6)$$

and

$$T_{ij}\, n^i = \{LC_{(3)}|_h\, m^h + C_{(2)}v_{2)32} + LC_{(2)}|_h\, n^h\}n_j$$
$$+ \{LC_{(3)}|_h\, n^h + C_{(2)}v_{2)33} - LC_{(2)}|_h\, m^h + (C_{(1)} - 2C_{(3)})\, v_{2)32}\}m_j \quad (5.7)$$

From equation (5.5), we can further obtain by virtue of $T_{ij}\, g^{ij} = T$

$$T = LC|_h\, m^h + 4C_{(2)}\, v_{2)32} + (C_{(1)} - 2C_{(3)})\, v_{2)33}. \quad (5.8)$$

6. T3-Like Finsler Spaces

From equation (5.2), we can write tensor T_{ijkh} of a 3-dimensional Finsler space F^3 in the following form

$$T_{ijkh} = \Sigma_{(i,j,k)}\{a_{hk}\, h_{ij} + b_{hk}\, m_i\, m_j\}, \tag{6.1}$$

where a_{hk} and b_{hk} are the second order tensors defined by

$$a_{hk} = \gamma_h\, m_k + (1/3)\,\delta_h\, n_k, \quad b_{hk} = ((1/3)\,\alpha_h - \gamma_h)\, m_k - (\beta_h + (1/3)\,\delta_h)\, n_k. \tag{6.2}$$

From equation (6.2), we can obtain

$$a_{0k} = \gamma_0\, m_k + (1/3)\,\delta_0\, n_k, \quad b_{0k} = ((1/3)\,\alpha_0 - \gamma_0)\, m_k - (\beta_0 + (1/3)\,\delta_0)\, n_k \tag{6.3}$$

and

$$a_{0k} + b_{0k} = (1/3)\,\alpha_0\, m_k - \beta_0\, n_k. \tag{6.4}$$

Comparing equations (1.3) and (6.1) and solving, we get

$$4\, a_{0k} = -\, b_{0i}\,(\delta^i_k + 2m^i\, m_k). \tag{6.5}$$

Hence, we have

Theorem (6.1). In a 3-dimensional Finsler space F^3, second order tensors a_{0k} and b_{0k} satisfy (6.4) and (6.5).

In any three dimensional Finsler space F^3 the T-tensor is defined by (6.1), which helps us give the following definition :

Definition (6.1). A Finsler space F^n $(n > 3)$, shall be called T3-like Finsler space, if for arbitrary second order tensors a_{hk} and b_{hk} satisfying $a_{h0} = 0$, $b_{h0} = 0$, its T-tensor T_{ijkh} is non-zero and is expressed by an equation of the form :

$$T_{ijkh} = \Sigma_{(i,j,k)}\,\{a_{hk}\, h_{ij} + b_{hk}\, C_i\, C_j\}. \tag{6.6}$$

It is known, Shimada [13], that for second curvature tensor P_{hijk}, the Ricci tensors defined by

$$P_{hk}^{(1)} = P^j_{\ hjk} = C_{k|h} - C^j_{\ hk|j} + P^j_{\ kr}\, C^r_{\ jh} - P^r_{\ hk}\, C_r \tag{6.7}$$

and

$$P_{hk}^{(2)} = P^j_{\ hkj} = C_{k|h} - C^j_{\ hk|j} + C^r_{\ kh}\, C_{r|0} - P^j_{\ hr}\, C^r_{\ kj} \tag{6.8}$$

are non-symmetric such that

$$P_{h0}^{(1)} = 0, \quad P_{h0}^{(2)} = 0, \quad P_{0k}^{(1)} = C_{k|0} = P_k, P_{0k}^{(2)} = C_{k|0} = P_k. \tag{6.9}$$

If we assume that the tensor $a_{hk} = P_{hk}^{(1)}$, we can obtain by virtue of equation (6.7) and $*T_{kh} = T_{ijkh}\, g^{ij}$

$$*T_{kh} = (n+1)\, P_{hk}^{(1)} + C^2\, b_{hk} + 2\, b_{hi}\, C^i\, C_k. \tag{6.10}$$

Since $*T_{kh}$ is symmetric in k and h, from equation (6.10), we can obtain

$$\zeta_{(k,\,h)}\, \{C_{k|h} + P^j{}_{kr}\, C^r{}_{jh} - (n+1)^{-1}\, b_{ki}\, (C^2\, \delta^i_h + 2\, C^i\, C_h)\} = 0, \tag{6.11}$$

therefore, we can have

Theorem (6.2). In a T3-like Finsler space Fn (n > 3), if tensor $a_{hk} = P_{hk}^{(1)}$, equation (6.11) is satisfied.

Multiplying equation (6.11) by l^k, we get

$$L(b_{0h}\, C^2 + 2b_{0i}\, C^i\, C_h) + (n+1)\, P_h = 0, \tag{6.12}$$

and $$b_{0i}\, C^i = -\, \{(n+1)/3\}\, L^{-1}\, C^{-2}\, P_i\, C^i. \tag{6.13}$$

From equations (6.12) and (6.13), we can obtain

$$b_{0h} = (n+1)\, P_i\, L^{-1}\, C^{-2}\{(2/3)\, C^i\, C_h - \delta^i_h)\}. \tag{6.14}$$

Hence, we have

Theorem (6.3). In a T3-like Finsler space F^n (n > 3), if the tensor $a_{hk} = P_{hk}^{(1)}$, the tensor b_{0h} is given by equation (6.14).

From equation (6.10), we can obtain on simplification

$$b_{hi}\, C^i = [L\, C_i|_h\, C^i + C^2\, l_h - (n+1)\, P_{hi}^{(1)}\, C^i]/3C^2. \tag{6.15}$$

Substituting the value of $b_{hi}\, C^i$, from equation (6.15) in (6.10), we obtain on simplification

$$b_{hk} = C^{-2}[\{LC_i|_h - (n+1)\, P_{hi}^{(1)}\}(\delta^i_k - (2/3C^2)\, C^i\, C_k) + l_k\, C_h + (1/3)\, l_h\, C_k]. \tag{6.16}$$

Hence, we have

Theorem (6.4). In a T3-like Finsler space F^n $(n > 3)$, if the tensor a_{hk} is given by the Ricci tensor $P_{hk}^{(1)}$, the tensor b_{hk} is given by equation (6.16).

7. C-Reducible Finsler Space

In a C-reducible Finsler space F^3, it is known that Matsumoto [4], $C_{(1)} = (3/4)\,C$, $C_{(2)} = 0$ and $C_{(3)} = (1/4)\,C$, therefore from equations (2.1) and (2.2), we can obtain

$$'C_{ij} = (C/4)\,(3\,m_i\,m_j + n_i\,n_j) \tag{7.1}$$

and

$$*C_{ij} = (C/4)\,(m_i\,n_j + m_j\,n_i), \tag{7.2}$$

which lead to

$$'C_{ij}\,m^j = (3/4)\,Cm_i, \quad 'C_{ij}\,n^j = (1/4)\,Cn_i = *C_{ij}\,m^j, \quad *C_{ij}\,n^j = (1/4)\,Cm_i. \tag{7.3}$$

Furthermore, we can obtain

$$\begin{aligned} 'C_{ij}|_h = {} & (1/4)\,C|_h\,(3m_i\,m_j + n_i\,n_j) + L^{-1}(1/4)\,C[3\{m_i\,(-l_j\,m_h + n_j\,v_h) \\ & + m_j\,(-l_i\,m_h + n_i\,v_h)\} - \{n_i\,(l_j\,n_h + m_j\,v_h) + n_j\,(l_i\,n_h + m_i\,v_h)\}] \end{aligned} \tag{7.4}$$

and

$$\begin{aligned} *C_{ij}|_h \doteq {} & (1/4)\,[C|_h\,(m_i\,n_j + m_j\,n_i) - L^{-1}C\{m_i\,(l_j\,n_h + m_j\,v_h) \\ & + m_j\,(l_i\,n_h + m_i\,v_h) + n_i\,(l_j\,m_h - n_j\,v_h) + n_j\,(l_i\,n_h - n_i\,v_h)\}]. \end{aligned} \tag{7.5}$$

From equations (3.1) and (3.2), we can obtain

$$\begin{aligned} C_{ijk}|_h = {} & (1/4)\,C|_h\,\{3m_i\,m_j\,m_k + (m_i\,n_j\,n_k + m_j\,n_k\,n_i + m_k\,n_i\,n_j) \\ & - L^{-1}\,\Sigma_{(i,\,j,\,k)}\,(1/4)\,C[3(l_i\,m_h - n_i\,v_h)\,m_j\,m_k \\ & + \{l_i\,(m_h\,n_j\,n_k + m_j\,n_k\,n_h + m_k\,n_h\,n_j) + 2m_i\,m_j\,v_h\,n_k\}], \end{aligned} \tag{7.6}$$

which by virtue of symmetry of $C_{ijk}|_h$ in k and h, easily leads to

$$\begin{aligned} \zeta_{(k,\,h)}\,[\{ & C_{(1)}|_h\,m_i\,m_j\,m_k + C_{(3)}\,|_h\,(m_i\,n_j\,n_k + m_j\,n_k\,n_i + m_k\,n_i\,n_j)\} \\ & + L^{-1}[C_{(1)}\{(m_i\,n_j + m_j\,n_i).m_k\,v_h + m_i\,m_j\,(l_h\,m_k + n_k\,v_h)\} \\ & + C_{(3)}\{(m_i\,n_j + m_j\,n_i).(n_k\,l_h + 2v_k\,m_h) + (m_i\,m_j - n_i\,n_j)\,2v_k\,n_h \end{aligned}$$

$$+ n_i n_j (l_h m_k + v_h n_k)\}] = 0. \tag{7.7}$$

Equations (3.4) to (3.11) for a C-reducible Finsler space reduce to

$$\alpha_h = (3/4)(LC|_h + Cl_h), \qquad \beta_h = (1/4)\, Cv_h,$$

$$\alpha_0 = \alpha_h l^h = (3/4)(LC|_0 + C), \quad \beta_0 = \beta_h l^h = 0, \quad \alpha_h m^h = (3/4)\, L\, C|_h m^h$$

$$\alpha_h n^h = (3/4)\, LC|_h n^h, \quad \beta_h m^h = (1/4)\, C\, v_{2)32}, \quad \beta_h n^h = (1/4)\, C\, v_{2)33}$$

$$3LC|_h n^h + Cv_{2)32} = 0. \tag{7.8}$$

For a C-reducible Finsler space equations (3.12), (3.13) and (3.14) can be expressed as

$$3C(v_h - v_{2)33}\, n_h) = LC|_k n^k m_h, \tag{7.9}$$

$$L\, C|_h n^h - 3C\, v_{2)32} = 0 \tag{7.10}$$

and

$$L\, C|_h m^h = C\, v_{2)33}. \tag{7.11}$$

From equations (7.8) and (7.10), we can obtain

$$C|_h n^h = 0 \qquad \text{or} \qquad v_{2)32} = 0. \tag{7.12}$$

Hence, we have

Theorem (7.1). In a 3-dimensional C-reducible Finsler space $LC|_h m^h = Cv_{2)33}$ and $C|_h n^h = 0$.

From equation (4.2) for a C-reducible Finsler space F^3, we get

$$T_{ijkh} = (1/4)\, \Sigma_{(i,j,k)}[(LC|_h + Cl_h)(m_i m_j m_k + m_i n_j n_k) + Cv_h m_i m_j n_k], \tag{7.13}$$

which leads to

$$T_{ij} = C\, v_{2)33}\, (m_i m_j + (1/4)\, n_i n_j), \tag{7.14}$$

$$T_{ij}\, mi = C\, v_{2)33}\, m_j, \quad T_{ij}\, n^i = (1/4)\, C\, v_{2)33}\, n_j \tag{7.15}$$

and

$$T = (5/4)\, C\, v_{2)33}. \tag{7.16}$$

Hence, we have

Theorem (7.2). In a 3-dimensional C-reducible Finsler space Tijkh is given by (7.13) and $T = (5/4)\, C\, v_{2)33}$.

8. P-Reducible Finsler Spaces

A Finsler space is defined as P-reducible by Matsumoto and Shimada [7] and studied by Rastogi and Kawaguchi [9] and Rastogi [12], if its torsion tensor $P_{ijk} = A_{ijk}|_0$, is expressible as

$$P_{ijk} = (n+1)^{-1}\,(A_{k|0}\, h_{ij} + A_{i|0}\, h_{jk} + A_{j|0}\, h_{ki}). \tag{8.1}$$

In three dimensional Finsler space P_{ijk} on simplification can be expressed as

$$\begin{aligned} P_{ijk} = L[(C_{(1)|0} + 3C_{(2)}\, h_0)\, m_i\, m_j\, m_k + (1/4)\, C_{|0}\,(m_i\, n_j\, n_k + m_j\, n_k\, n_i \\ + m_k\, n_i\, n_j) + (1/4)\, Ch_0\,(3n_i\, n_j\, n_k + m_i\, m_j\, n_k + m_j\, m_k\, n_i + m_k\, m_i\, n_j)]. \end{aligned} \tag{8.2}$$

Multiplying (8.2) by g^{jk} and using $P_i = LC_{i|0}$ and $C_{|0} = C_{i|0}\, m^i$, we get

$$3\, C_{|0} = 4C_{(1)|0} + 12\, C_{(2)}\, h_0 \qquad \text{or} \qquad 4C_{(3)|0} - C_{|0} = 12\, C_{(2)}\, h_0 \tag{8.3}$$

Hence, we have

Theorem (8.1). In a P-reducible Finsler space F^3, coefficients $C_{(1)}$, $C_{(2)}$ and $C_{(3)}$ are related by equation (8.3).

From equation (2.1), we can easily obtain

$$\begin{aligned} L\, {}'C_{ij|0} = (1/4)\,[(3m_i\, m_j + n_i\, n_j)\{C_{(1)|0} + 3C_{(2)}\, h_0 + (1/4)\, C_{|0}\} \\ + Ch_0\,(m_j\, n_i + m_i\, n_j)]. \end{aligned} \tag{8.4}$$

From equation (8.4), we can obtain

$$\begin{aligned} &{}'C_{ij|0}\, l^j = 0, \qquad L\, {}'C_{ij|0}\, m^j = (1/4)\,(3\, C_{|0}\, m_i + C\, h_0\, n_i), \\ &L\, {}'C_{ij|0}\, n^j = (1/4)\,(C_{|0}\, n_i + C\, h_0\, m_i). \end{aligned} \tag{8.5}$$

Similarly from equation (2.2), we can obtain

$$L\, {}^*C_{ij|0} = C_{|0}\,(m_i\, n_j + m_j\, n_i) + 4C\, h_0\, m_i\, m_j - L\, h_0\, {}'C_{ij}, \tag{8.6}$$

which implies

$$*C_{ij|0}\, l^j = 0, \qquad L\, *C_{ij|0}\, m^j = m_i\, (4C - C_{(1)}\, h_0 + n_i\, (C_{|0} - C_{(2)}\, h_0),$$

$$L\, *C\, C_{ij|0}\, n^j = (C_{|0} + Lh_0\, C_{(2)}\,)\, m_i - L\, h_0\, C_{(3)}\, n_i. \tag{8.7}$$

Hence, we have

Theorem (8.2). In a P-reducible Finsler space F^3, tensors $'C_{ij}$ and $*C_{ij}$ satisfy equations (8.6) and (8.7).

From equation (1.3) with the help of equations (8.1), we can obtain on simplification

$$LT_{ijkh|0} = L^2\, C_{ijk}|_{h|0} + (n+1)^{-1}\, \{A_{i|0}\, \Sigma_{(j,\,k,\,h)}\, (l_j\, h_{kh}) + A_{j|0}\, \Sigma_{(i,\,k,\,h)}\, (l_i\, h_{kh})$$
$$+ A_{k|0}\, \Sigma_{(i,\,j,\,h)}\, (l_i\, h_{jh}) + A_{h|0}\, \Sigma_{(i,\,j,\,k)}\, (l_i\, h_{jk})\}, \tag{8.8}$$

which gives

Theorem (8.3). In a P-reducible Finsler space F^3, T-tensor satisfies equation (8.8).

In a T3-like Finsler space equation (8.8) gives on simplification

$$L\Sigma_{(i,\,j,\,k)}\{a_{hk|0}\, h_{ij} + b_{hk|0}\, C_i\, C_j\} - L^2\, C_{ijk}\, |_{h|0} - (n+1)^{-1}\, [A_{i|0}\{\Sigma_{(j,\,k,\,h)}\, (l_j\, h_{kh})$$
$$- (n+1)(b_{hk}\, C_j + b_{hj}\, C_k)\} + A_{j|0}\{\Sigma_{(i,\,k,\,h)}(l_i\, h_{kh}) - (n+1)(b_{hk}\, C_i + b_{hi}\, C_k)\}$$
$$+ A_{k|0}\{\Sigma_{(i,\,j,\,h)}\, (l_i\, h_{hj}) - (n+1)(b_{hi}\, C_j + b_{hj}\, C_i)\} + A_{h|0}\, (l_i\, h_{jk})] = 0. \tag{8.9}$$

Hence, we have

Theorem (8.4). If a T3-like Finsler space is also a P-reducible Finsler space it satisfies (8.9).

From equation (8.9), we can obtain

$$3\, a_{hk|0} + b_{hi|0}\, (C^2\, \delta^i_k + 2C^i\, C_k) - L\, C_k|_{h\,|0} - l_k\, C_{h|0} - l_h\, C_{k|0}$$
$$= -\, 2(n+1)^{-1}\, C_{i|0}\, (b_{hk}\, C^i + b^i_{\ h}\, C_k - b_{hj}\, C^j\, \delta^i_k). \tag{8.10}$$

Hence, we have

Theorem (8.5). In a T3-like P-reducible Finsler space arbitrary tensors a_{hk} and b_{hk} are related by equation (8.10).

References

1. Hashiguchi, M. : On conformal transformations of Finsler metrics, J. Math. Kyoto Univ., 16 (1976), 25-50.

2. Izumi, H. : On P*-Finsler space I., Memo. Defence Academy, 16 (1976), 133-138.

3. Kawaguchi, H. : On the Finsler spaces with the vanishing second curvature tensor, Tensor, N. S. 26 (1972), 250-254.

4. Matsumoto, M. : On C-reducible Finsler space, Tensor, N. S., 24 (1972), 29-37.

5. Matsumoto, M. : V-transformation of Finsler spaces. I. Definitions, infinitesimal transformations and isometries, J. Math. Kyoto Univ., 12 (1972), 479-512.

6. Matsumoto, M. : Foundations of Finsler Geometry and special Finsler spaces, Kaiseisha Press, Otsu, 520, Japan, 1986.

7. Matsumoto, M. and Shimada, H. : On Finsler spaces with the curvature tensor P_{hijk} and S_{hijk} satisfying special conditions, Rep. on Math. Phys., 12 (1977), 77-87.

8. Matsumoto, M. and Shimada, H. : On Finsler spaces with 1-form metric, Tensor, N. S., 32 (1978), 161-169.

9. Rastogi, S. C. and Kawaguchi, H. : A geometrical meaning of the P-reducible condition in Finsler spaces, Tensor, N. S., 51 (1992), 251-256.

10. Rastogi, S. C. : On some new tensors and their properties in a Finsler space, J.T.S.I., 8, 9, 10 (1990-92), 12-21.

11. Rastogi, S. C. : T2 and TR2-like Finsler spaces, J. Nat. Acad. Math., 17 (2003), 1-8.

12. Rastogi, S. C. : On certain P-reducible Finsler spaces, Ganita, 56, 1 (2005), 55-64.

13. Shimada, H. : On the Ricci tensors of particular Finsler spaces, J. Korean Math. Soc., 14, 1 (1977), 41-63.