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On Finsler Spaces with a Quartic Metric

T. N. Pandey and V. K. Chaubey

Department of Mathematics and Statistics
D. D. U. Gorakhpur University, Gorakhpur-273009, U.P., India
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Abstract

The purpose of the present paper is to study spaces with a quartic metric from the standpoint of Finsler geometry. The paper deals with Berwald and Landsbergs spaces among quartic Finsler spaces. A Finsler connectionis defined in a quartic Finsler space from the standpoint of the generalized metric space.

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0. Introduction

The so-called quartic metric on a differentiable manifold with the local co-ordinates x^i is defined by

$$L^4(x, y) = a_{ijkh}(x)\, y^i y^j y^k y^h , \quad (y^i = \dot{x}^i) \tag{0.1}$$

where $a_{ijkh}(x)$ are components of a symmetric tensor field of (0, 4)-type, depending on the position x alone, and a Finsler space with a quartic metric is called quartic Finsler space.

We have had few papers studying cubic Finsler spaces ([1], [5], [6], [7], [8], [9]) although there are various papers on the geometry of spaces with a cubic metric as a generalization of Euclidean or Riemannian geometry. The purpose of the present paper is to study spaces with a quartic metric from the standpoint of Finsler geometry.

Section 1 is devoted to developing a fundamental treatment of quartic Finsler spaces and a characterization of such spaces is given in terms of well-known tensors in Finsler geometry. Section 2 is devoted to finding Berwald space and landsberg spaces among quartic Finsler spaces. In Section 3, a characteristic

Finsler connection is defined in a quartic Finsler space from the standpoint of the generalized metric space due to A. Moor [4]. In Section 4 some distinctive quartic Finsler space are treated they are decomposable in a sense.

1. Characterization of Quartic Metric

We consider an n-dimensional Finsler space F^n with a quartic $L^4(x, y)$ defined by (0.1). Putting

$$L^3 a_i (x, y) = a_{ijkh} y^j y^k y^h, \quad L^2 a_{ij} (x, y) = a_{ijkh} y^k y^h, \quad La_{ijk} (x, y) = a_{ijkh} y^h, \tag{1.1}$$

The normalized supporting element $l_i = \dot{\partial}_i L$, the angular metric tensor $h_{ij} = L \dot{\partial}_i \dot{\partial}_j L$ and the fundamental tensor $g_{ij} = \dot{\partial}_i \dot{\partial}_j L^2/2 = h_{ij} + l_i l_j$ are respectively given by the equations,

$$\text{(a)} \quad l_i = a_i, \qquad \text{(b)} \quad h_{ij} = 3 (a_{ij} - a_i a_j), \qquad \text{(c)} \quad g_{ij} = 3a_{ij} - 2a_i a_j \tag{1.2}$$

The problem appearing first in treating special Finsler metric of an interesting cocrete form is to find the inverse matrix (g^{ij}) of the matrix g_{ij} (see, for instance, § 30 of [2]). In case of quartic the problem is easy as follows :

Definition : A quartic Finsler space or some domain of the space is called regular if the intrinsic metric tensor a_{ij} (x, y) has non-vanishing determinant.

Then by inverse matrix (a^{ij}) of a_{ij} the contravariant component g^{ij} of the fundamental tensor are written as

$$g^{ij} = (a^{ij} + 2a^i a^j)/3 \tag{1.3}$$

where $a^i = a^{ir} a_r$, $a^i a_i = 1$, $l_i = y^i/L$, $a^h_{ij} a_h = a_{ij}$.

It is easy to show that

$$\dot{\partial}_j a_i = 3 (a_{ij} - a_i a_j)/L, \qquad \dot{\partial}_k a_{ij} = 2 (a_{ijk} - a_{ij} a_k)/L.$$

Therefore it follows from (1.2)(c) that the covariant components $C_{ijk} = \dot{\partial}_k g_{ij}/2$ of the (h) hv-torsion tensor of the Cartan connection $C\Gamma$ are written as,

$$LC_{ijk} = 3 (a_{ijk} - a_{ij} a_k - a_{jk} a_i - a_{ki} a_j + 2 a_i a_j a_k). \tag{1.4}$$

It is well-known that a Finsler space is Riemannian, iff $C_{ijk} = 0$. This characterization of Riemannian metric is nothing but the equation $\dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2 = 0$. Similarly a quartic metric $L^4(x, y)$ is characterized by the equation $\dot{\partial}_h \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^4 = 0$. By direct computation we get generally,

$$\dot{\partial}_h \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k \dot{\partial}_m L^4 = 8\,[L T_{hijk}|_m + l_m T_{hijk} + l_h T_{ijkm} + l_i T_{hjkm} + l_j T_{hikm} + l_k T_{hijm} + L^2 A_{hijkm} + L B_{hijkm} + C_{hijkm}] \tag{1.5}$$

where

$$A_{hijkm} = C^{\lambda}_{jm} (C_{\lambda h\mu} C^{\mu}_{ik} + C_{\lambda i\mu} C^{\mu}_{hk} + C_{\lambda k\mu} C^{\mu}_{hi}) + C^{\lambda}_{km} (C_{\lambda h\mu} C^{\mu}_{ij} + C_{\lambda i\mu} C^{\mu}_{hj} + C_{\lambda j\mu} C^{\mu}_{hi}) + C^{\lambda}_{im} (C_{\lambda h\mu} C^{\mu}_{jk} + C_{\lambda j\mu} C^{\mu}_{hk} + C_{\lambda k\mu} C^{\mu}_{hj}) + C^{\lambda}_{hm} (C_{\lambda i\mu} C^{\mu}_{jk} + C_{\lambda j\mu} C^{\mu}_{ik} + C_{\lambda k\mu} C^{\mu}_{ij}).$$

$$B_{hijkm} = C^{\lambda}_{hi} T_{\lambda jkm} + C^{\lambda}_{jk} T_{\lambda him} + C^{\lambda}_{hj} T_{\lambda ikm} + C^{\lambda}_{hk} T_{\lambda ijm} + C^{\lambda}_{hm} T_{\lambda ijk} + C^{\lambda}_{ij} T_{\lambda hkm} + C^{\lambda}_{jm} T_{\lambda hik} + C^{\lambda}_{ki} T_{\lambda hjm} + C^{\lambda}_{km} T_{\lambda hij} + C^{\lambda}_{im} T_{\lambda hjk}.$$

$$C_{hijkm} = 2h_{hi} C_{jkm} + h_{km} C_{hij} + 2h_{hj} C_{ikm} + 2h_{jk} C_{him} + h_{im} C_{hjk} + h_{jm} C_{hik} + 2h_{ki} C_{hjm} + 2h_{ij} C_{hkm} + h_{hm} C_{ijk} + 2h_{hk} C_{ijm}.$$

In terms of well-known T-tensor T_{hijk} (§28.20, [2]) given in [3]. The equation (1.5) is rewritten in the form, if we put,

$$U_{hijkm} = L\, T_{hijk}|_m + l_m T_{hijk} + l_h T_{ijkm} + l_i T_{hjkm} + l_j T_{hikm} + l_k T_{hijm}$$

is

$$U_{hijkm} + L^2 A_{hijkm} + L B_{hijkm} + C_{hijkm} = 0. \tag{1.5)$'$}$$

Theorem 1. A Finsler space is one with a quartic metric, iff the equation (1.5)′ holds identically.

Corollary. [3]. A two dimensional Finsler space with a quartic metric is characterized by the equation $I_{;2;2} + 10\, II_{;2} + 4I\,(3I^2 + 4) = 0$, where I is the main scalar and $I_{;2} = Lm^i \dot{\partial}_i I$ and $I_{;2;2} = Lm^i \dot{\partial}_i I_{;2}$.

Let us specialize the characterization theorem of three-dimensional case, similarly to corollary. In a strongly non-Riemannian Finsler space F^3 ([10], [11], [12], § 29 of [2]), we can refer to the Moor frame (l^i, m^i, n^i) and the main scalars H, I, J appear.

Theorem 2. A strongly non-Riemannian Finsler space of three dimensions is a quartic Finsler space, iff the following equations hold good :

1. $C_{222;2;2} + 12(H^3 + J^2 I) + 10(HC_{222;2} - JC_{322;2}) + 16H = 0$

2. $C_{222;2;3} - 12(H^2 J + JI^2) + (6HC_{222;3} - 6JC_{322;3} - 4JC_{222;2} + 4IC_{322;2}) - 12J = 0$

3. $C_{222;3;2} - (12H^2 J + 6I^2 J + 3J^3 - 3J^2 I) + \{2H(2C_{223;2} + C_{222;3}) - 2J(2C_{323;2}$
$+ 2C_{222;2} + C_{322;3}) + 4IC_{322;2}\} - 9J = 0.$

4. $C_{222;3;3} + (9J^2 H + 6I^3 + 3J^3 - 3J^2 I + 3H^2 I) + \{3H(2C_{223;2} - J(3C_{323;3} + 5C_{222;3})$
$+ I(5C_{322;3} + C_{222;3} + C_{323;2})\} + H + 6I = 0$

5. $C_{233;2;2} + (10J^2 H + 2H^2 I - 8J^2 I + 4I^3) + \{3HC_{232;2} - J(5C_{232;2} + 3C_{333;2})$
$+ I(6C_{332;2} + C_{222;2})\} + 2H + 4I = 0$

6. $C_{233;2;3} + (4J^2 I + 10I^2 J - 8HIJ - 2J^3) + \{HC_{233;3} - J(3C_{232;3} + C_{333;3})$
$+ I(4C_{332;3} + 2C_{333;2} + 2C_{223;2} + C_{222;3})\} - 2J = 0$

7. $C_{233;3;2} + (9I^2 J + 6J^2 I - 6HIJ - 3J^2) + \{HC_{233;3} - J(2C_{233;2} + C_{333;3} + C_{223;3})$
$+ I(5C_{333;2} + 3C_{223;2} + C_{323;3})\} - 5J = 0$

8. $C_{233;3;3} + (21IJ^2 + 3I^2 H) + \{I(4C_{333;3} + 4C_{223;3} + 2C_{233;2}) + 2JC_{333;2}\} + 9I = 0$

9. $C_{333;2;2} + (9I^2 J - 9J^3 - 6HIJ) + \{HC_{233;3} - J(2C_{233;2} + C_{333;3} + C_{232;3})$
$+ I(3C_{232;2} + 5C_{333;2} + C_{332;3})\} - 5J = 0$

10. $C_{333;2;3} + (21IJ^2 + 3I^2 H) + \{H(4C_{232;2} + 2C_{222;3}) - J(4C_{323;2} + 4C_{222;2}$
$+ 2C_{322;3}) + 4IC_{322;2}\} + 9I = 0$

11. $C_{333;3;2} + 24J^2I + \{I(6C_{233;2} + 4C_{333;3}) + J(6C_{333;2} - 4C_{233;3})\} + 12I = 0$

12. $C_{333;3;3} + 12(J^3 - I^2J) + 10IC_{233;3} + 10JC_{333;3} + 16J = 0$

13. $C_{223;2;2} + (3J^3 - 12H^2J - 9I^2J) + \{H(4C_{232;2} + 2C_{222;3}) - J(4C_{222;2}$
$+ 4C_{332;2} + 2C_{322;3}) + 4IC_{322;2}\} - 9J = 0$

14. $C_{223;2;3} + (3H^2I - 6IJ^2 + J^2H + 6I^3) + \{3HC_{232;3} - J(5C_{222;3} + 3C_{322;3})$
$+I(5C_{322;3} + C_{222;2} + C_{332;2})\} + H + 6I = 0$

15. $C_{223;3;2} + (10HJ^2 + 4I^3 - 8J^2I + 2H^2I) + \{H(2C_{232;3} + C_{233;2}) + I(4C_{323;2}$
$+ C_{222;2} + 2C_{322;3}) - J(3C_{223;2} + 2C_{222;3} + 2C_{332;3}$
$+ C_{333;2})\} - 2H + 4I = 0$

16. $C_{223;3;3} + (10I^2J - 8HIJ - 6J^3) + \{HC_{233;3} - J(6C_{223;2} - 3C_{322;3} + C_{333;3})$
$+ I(6C_{323;2} + 3C_{222;3})\} - 2J = 0.$

That is v-scalar derivatives $C_{222;2;2}$ ...., $C_{223;3;3}$ of main scalars H, I, J are functions of main scalars.

Remark. It is known (§29 of [2]) that the first v-covariant derivatives vanish identically and in this case second v-covariant derivatives also vanishes identically.

2. Certain important tensors of quartic Finsler spaces

The h-and v-covariant derivatives $X_{i|j}$, $X_i|_j$ of a covariant vector field X_i w.r.t. CΓ are defined by

$$X_{i|j} = \dot{\partial}_j X_i - (\dot{\partial}_r X_i) N^r_j - X_r F^r_{ij}, \qquad X_i|_j = \dot{\partial}_j X_i - X_r C^r_{ij}.$$

where $(F^i_{jk}, N^i_j (F^i_{0j}), C^i_{jk})$ are connection coefficients of CΓ and suffix 0 means the contraction by the supporting element y^i.

As to a Finsler space with a quartic (0.1) it follows first from (1.2)(a).,(c). That,

$$a_{i|j} = 0 \qquad a_{ij|k} = 0. \tag{2.1}$$

Because $l_{i|j} = 0$ and $g_{ij|k} = 0$. These are remarkable identities as it will be seen in the following. Then the h-covariant differentiation of (1.4) leads us the simple equation

$$LC_{ijk|h} = 3a_{ijk|h}\,. \tag{2.2}$$

Therefore the (v) hv-torsion tensor P_{ijk} given by (§ 17.22 of [2]) written as,

$$LP_{ijk} = LC_{ijk|0} = 3a_{ijk|0}. \tag{2.3}$$

As a consequence of these equations, the equation (§ 17.23 of [2]) expressing the hv-curvature tensor P_{hijk} yields

$$2L^2P_{hijk} = 6L\,(a_{ijk|h} - a_{hjk|i}) - 3(a^{\lambda}_{ij}\,a_{\lambda hk|0} - a^{\lambda}_{hj}\,a_{\lambda ik|0}) + 3(a_i\,a_{hjk|0} - a_h\,a_{ijk|0}) \tag{2.4}$$

Definition (§ (25) of [2]). 1. A Finsler space is called a Berwald space (or affinely connected space), if the tensor $C_{ijk|h}$ vanishes identically.

2. A Finsler space is called a Landsberg space, if the (v) hv-torsion tensor P_{ijk} vanishes identically.

It is noted that the condition $P_{ijk} = 0$ is equivalent to $P_{hijk} = 0$ (§ Theorem 25.3 of [2]). From (2.2) and (2.3), we have

Theorem 3. A quartic Finsler space is a Berwald space (resp. Landsberg space), iff the tensor $a_{ijk|h}$ (resp. $a_{ijk|0}$) vanishes identically, where the h-covariant differentiation is the one with respect to the Cartan connection.

3. A characteristic Finsler connection in a quartic Finsler space

First of all we remember equation (1.2)(c) giving the fundamental tensor g_{ij} of a quartic Finsler space F^n. The tensor is different from the intrinsic metric tensor a_{ij} in a regular F^n. Nevertheless we have

$$L^2(x, y) = g_{ij}\,(x, y)\,y^i\,y^j = a_{ij}\,(x, y)\,y^i\,y^j. \tag{3.1}$$

This is very interesting equation, F^n is regarded as a generalized metric space of line-element in Moor's sense [4], because there is generally no such a function M(x, y) that a_{ij} is given by $a_{ij} = \dot{\partial}_i\,\dot{\partial}_j\,M^2/2$. A. Moor has developed various interesting results on the geometry of generalized metric space of line-element.

In viewpoint of (3.1) it seems natural to us to consider the problem determining a Finsler connection based on the intrinsic metric tensor $a_{ij}(x, y)$.

Theorem 4. In a regular quartic Finsler space Fn a Finsler connection $*C\Gamma = (*F^i_{jk}, *N^i_j, *C^i_{jk})$ is uniquely determined from the intrinsic metric tensor $a_{ij}(x, y)$ by the following five axioms :

1. It is h-metrical : $a_{ij|k} = 0$
2. It is v-metrical : $a_{ij}|_k = 0$
3. It is h-symmetric : $*T^i_{jk} = *F^i_{jk} - *F^i_{kj} = 0$
4. It is v-symmetric : $*S^i_{jk} = *C^i_{jk} - *C^i_{kj} = 0$
5. Its deflection tensor vanishes : $y^i_{|j} = *N^i_j - *F^i_{0j} = 0,$

where $_|$ and $|$ denote respectively the h- and v-covariant differentiations with respect to $*C\Gamma$. Then the connection coefficients $*F^i_{jk}$ and $*N^i_j$, coincide with F^i_{jk} and N^i_j respectively and $*C^i_{jk} = C^i_{jk} + \frac{2}{3L} h_{jk} l^i$, where $C\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ is the Cartan connection.

Remark. In theorem 4 a Finsler connection is the concept given in (§ 9 of [2]). It is noteworthy that the above system of axioms is similar to the one for $C\Gamma$ (Definition 17.2 of [2]). It is, however, remarked that in case of $C\Gamma$ the identities $(\dot{\partial}_k g_{ij}) y^i = (\dot{\partial}_k g_{ij}) y^k = 0$, $\dot{\partial}_k g_{ij} = \dot{\partial}_j g_{ik}$ are full used, but for the intrinsic metric tensor a_{ij} such identities do not hold except $(\dot{\partial}_k a_{ij}) y^k = 0$. We shall show another proof in the following.

Proof. The axioms (2) and (4) lead us immediately to,

$$*C^i_{jk} = a^{ir} (\dot{\partial}_k a_{jr} + \dot{\partial}_j a_{kr} - \dot{\partial}_r a_{jk})/2. \tag{3.2}$$

That is the coefficients $*C^i_{jk}$ of v-covariant differentiation are Christoffel symbols constructed from $a_{ij}(x, y)$ with respect to y^i. Differentiation of (1.4) by y^k yields

$$\dot{\partial}_k a_{ij} = \frac{2}{3} [C_{ijk} + \frac{h_{ij} l_k + h_{jk} l_i}{2L}]. \tag{3.3}$$

Thus (3.2), (3.3) and (1.3) give the relation

$$*C^i_{jk} = C^i_{jk} + \frac{2}{3L} h_{jk} l^i. \tag{3.4}$$

Secondly, we consider the difference $D^i_{jk} = *F^i_{jk} - F^i_{jk}$, since deflection vanishes so, $*F^i_{jk} = F^i_{jk}$. Then the axiom (3) means $D^i_{jk} = D^i_{kj}$ and (5) does $D^i_{0k} = *N^i_j - N^i_k$. Then $*N^i_j = N^i_j$.

4. Decomposable quartic metrics

Theorem 5. Let F^n be a quartic Finsler space with the metric $L^4 = \alpha^2 \overline{\alpha}^2$ where $\alpha^2 = \alpha_{ij} y^i y^j$ and $\overline{\alpha}^2 = \overline{\alpha}_{ij} y^i y^j$ are non degenerate. The space F^n is Berwald space if there exists a covariant vector fields $\gamma_k(x)$ such as $\alpha_{ij|k} = \alpha_{ij} \gamma_k$ and $\overline{\alpha}_{ij|k} = -\overline{\alpha}_{ij} \gamma_k$ holds good, conversely, it is Berwald space, if

$$\alpha_{ij|k} = \alpha_{ij} \gamma_k, \ \overline{\alpha}_{ij|k} = -\overline{\alpha}_{ij} \gamma_k \text{ and } (\dot{\partial}_m a_{ijk})_{|h} = 0 \text{ holds good.}$$

Proof. The symmetric coefficients $a_{ijk}(x)$ of L^4 are given by

$$6a_{ijkm} = \alpha^2_{ij} \overline{\alpha}^2_{km} + \alpha^2_{km} \overline{\alpha}^2_{ij} + \alpha^2_{ik} \overline{\alpha}^2_{jm} + \alpha^2_{jm} \overline{\alpha}^2_{ik} + \alpha^2_{im} \overline{\alpha}^2_{kj} + \alpha^2_{kj} \overline{\alpha}^2_{im}. \tag{4.1}$$

The sufficiency is obvious from (4.1), Theorem (3) and $(\dot{\partial}_m a_{ijk})_{|h} = 0$. Next, we obtain from $L^4 = \alpha^2 \overline{\alpha}^2$ and $L_{|i} = 0$.

$$\alpha^2_{|i} \overline{\alpha}^2 + \alpha^2 \overline{\alpha}^2_{|i} = 0. \tag{4.2}$$

If the space F^n is a Berwald space, then the connection coefficients F^i_{jk} of $C\Gamma$ are function of position x alone (§ Proposition (25.1) of [2]), hence α^2 and $\overline{\alpha}^2$ ($\alpha^2_{|i}$ and $\overline{\alpha}^2_{|i}$) are polynomials of the second degree in y^i. Since α^2 is assumed to be non-degenerate, (4.2) implies that there exists a vector fields γ_k such as $\alpha_{ij|k} = \alpha_{ij} \gamma_k$ and $\overline{\alpha}_{ij|k} = -\overline{\alpha}_{ij} \gamma_k$, from which the equations stated in Theorem 5 are concluded.

Theorem 6. Let F^n be a quartic Finsler space with the metric $L^4 = \alpha^2 \beta\overline{\beta}$ where $\alpha^2 = \alpha_{ij} y^i y^j$ is non-degenerate, $\beta = \beta_i y^i$ and $\overline{\beta} = \overline{\beta}_i y^i$. The space F^n is a Berwald space if there exists a covariant vector fields γ_k (x) such as $\alpha_{ij|k} = \alpha_{ij} \gamma_k$ and $\beta_{i|j} = -\beta_i \gamma_j/2$, $\overline{\beta}_{i|j} = -\overline{\beta}_i \gamma_j/2$ holds good, conversely, it is Berwald space if $\alpha_{ij|k} = \alpha_{ij} \gamma_k$, $\beta_{i|j} = -\beta_i \gamma_j/2$ and $\overline{\beta}_{i|j} = -\beta_i \gamma_j/2$ and $(\dot{\partial}_m a_{ijk})_{|h} = 0$ holds good.

Proof. The symmetric coefficients a_{ijk} (x) of L^4 are given by

$$\begin{aligned} 12a_{ijkm} = {} & \alpha_{ij} \beta_k \overline{\beta}_m + \alpha_{ij} \beta_m \overline{\beta}_k + \alpha_{km} \beta_i \overline{\beta}_j + \alpha_{km} \beta_j \overline{\beta}_i + \alpha_{jk} \beta_i \overline{\beta}_m \\ & + \alpha_{jk} \beta_m \overline{\beta}_i + \alpha_{im} \beta_j \overline{\beta}_k + \alpha_{im} \beta_k \overline{\beta}_j + \alpha_{ik} \beta_j \overline{\beta}_m + \alpha_{ik} \beta_m \overline{\beta}_j \\ & + \alpha_{jm} \beta_i \overline{\beta}_k + \alpha_{jm} \beta_k \overline{\beta}_i . \end{aligned} \tag{4.3}$$

The sufficiency is obvious from (4.3), Theorem (3) and $(\dot{\partial}_m a_{ijk})_{|h} = 0$. Next, we obtain from $L^4 = \alpha^2 \beta\overline{\beta}$ and $L_{|i} = 0$

$$\alpha^2_{|h} \beta\overline{\beta} + \alpha^2 \beta_{|h} \overline{\beta} + \alpha^2 \beta\overline{\beta}_{|h} = 0. \tag{4.4}$$

If the space F^n is a Berwald space, then the connection coefficients F^i_{jk} of $C\Gamma$ are function of position x alone (§ Proposition (25.1) of [2]), hence α^2 and $\alpha^2_{|i}$ (resp. β and $\beta_{|h}$, $\overline{\beta}$ and $\overline{\beta}_{|h}$) are polynomials of the second (resp. first) degree in y^i. Since α^2 is assumed to be non-degenerate, (4.4) implies that there exists a vector fields $\gamma_i(x)$ such as $\alpha^2_{|i} = \alpha^2 \gamma_i$, $\beta_{|i} = -\beta\gamma_i/2$, $\overline{\beta}_{|i} = -\overline{\beta}\gamma_i/2$ and, from which the equations stated in Theorem 6 are concluded.

Theorem 7. Let F^3 be three-dimensional Finsler space with 1-form metric $L^4 = a^1 a^2 a^3 a^4$. F^3 is a Berwald space if there exist four covariant vector field γ^α_i (x) = (α = 1, 2, 3, 4) such as $\gamma^1_i + \gamma^2_i + \gamma^3_i + \gamma^4_i = 0$, $a^\alpha_{i|j} = a^\alpha_i \gamma^\alpha_j$ (not sum in α), where we put $a^\alpha = a^\alpha_i (x) y^i$, ($\alpha$ = 1, 2, 3, 4), conversely, it is Berwald space if $a^\alpha_{i|j} = a^\alpha_i \gamma^\alpha_j$, $\gamma^1_i + \gamma^2_i + \gamma^3_i + \gamma^4_i = 0$ and $(\dot{\partial}_m a_{ijk})_{|h} = 0$ holds good.

Proof. This L is a quartic metric with the symmetric coefficients,

$$
\begin{aligned}
24a_{ijkm} = {} & a_i^1 a_j^2 a_k^3 a_m^4 + a_i^1 a_j^2 a_m^3 a_k^4 + a_j^1 a_i^2 a_k^3 a_m^4 + a_j^1 a_i^2 a_m^3 a_k^4 + a_k^1 a_m^2 a_i^3 a_j^4 + \\
& + a_k^1 a_m^2 a_j^3 a_i^4 + a_m^1 a_k^2 a_i^3 a_j^4 + a_m^1 a_k^2 a_j^3 a_i^4 + a_j^1 a_k^2 a_i^3 a_m^4 + a_j^1 a_k^2 a_m^3 a_i^4 + \\
& + a_k^1 a_j^2 a_i^3 a_m^4 + a_k^1 a_j^2 a_m^3 a_i^4 + a_i^1 a_m^2 a_j^3 a_k^4 + a_i^1 a_m^2 a_k^3 a_j^4 + a_m^1 a_i^2 a_j^3 a_k^4 + \\
& + a_m^1 a_i^2 a_k^3 a_j^4 + a_k^1 a_i^2 a_j^3 a_m^4 + a_k^1 a_i^2 a_m^3 a_j^4 + a_i^1 a_k^2 a_j^3 a_m^4 + a_i^1 a_k^2 a_m^3 a_j^4 + \\
& + a_j^1 a_m^2 a_i^3 a_k^4 + a_j^1 a_m^2 a_k^3 a_i^4 + a_m^1 a_j^2 a_i^3 a_k^4 + a_m^1 a_j^2 a_k^3 a_i^4 . \qquad (4.5)
\end{aligned}
$$

Therefore the sufficiency of the condition is obvious from (4.5) and Theorem (3) and $(\dot{\partial}_m a_{ijk})_{|h} = 0$. Next we obtain from $L^4 = a^1 a^2 a^3 a^4$

$$a^1_{|i} a^2 a^3 a^4 + a^1 a^2_{|i} a^3 a^4 + a^1 a^2 a^3_{|i} a^4 + a^1 a^2 a^3 a^4_{|i} = 0. \qquad (4.6)$$

In the similar way to the proof of theorem 5 it follows from (4.6) and linear independence that there exist γ_i^α (x) satisfying $a^\alpha_{|j} = a^\alpha \gamma_j^\alpha$. Then (4.6) implies $\gamma_i^1 + \gamma_i^2 + \gamma_i^3 + \gamma_i^4 = 0$.

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