

J. T. S.

Vol. 5 (2011), pp.67-75

<https://doi.org/10.56424/jts.v5i01.10440>

## Some Curvature Tensors of a Semi Symmetric Metric $\phi$ -Connection in an LSP-Sasakian Manifold

A. Bhattacharyya and C. Patra\*

Department of Mathematics

Jadavpur University, Kolkata-700032, India

\*MCKV Institute of Engineering, Howrah-711204, India

e-mail : aribh22@hotmail.com, \*patrachinmoy@yahoo.co.in

(Received : 12 November, 2011)

### Abstract

The purpose of this paper is to investigate the conditions for the pseudo-projective curvature tensor and quasi-conformal curvature tensor of a semi-symmetric metric  $\phi$ -connection to be the pseudo-projective curvature tensor and quasi-conformal curvature tensor of a Levi-Civita connection on LSP-Sasakian manifold. Also we shall discuss the behavior of conharmonic curvature tensor and Tachibana concircular curvature tensor with respect to semi-symmetric metric  $\phi$ -connection on LSP-Sasakian manifold.

**Key Words :** Semi-symmetric metric  $\phi$ -connection, LSP-Sasakian manifold, Quasi-conformal curvature tensor, Pseudo-projective curvature tensor, Conharmonic curvature tensor, Tachibana concircular curvature tensor.

**2000 AMS Subject Classification :** 53C25.

### 1. Introduction

The definition of semi-symmetric metric  $\phi$ -connection has been introduced by K.Yano [5]. Also he discussed some properties on semi-symmetric metric  $\phi$ -connection. Some results on semi-symmetric metric connection on Sasakian manifold developed by K.Yano and T.Imai [8]. Later V.K.Jaiswal et. al. studied some properties on curvature tensor and projective curvature tensor on semi-symmetric  $\phi$ -connection in Lorentzian Special Para-Sasakian manifold [7]. In this paper we shall show the necessary and sufficient condition for pseudo projective curvature tensor and quasi conformal curvature tensor with semi-symmetric metric  $\phi$ -connection  $\bar{D}$  to be pseudo projective curvature tensor and quasi conformal curvature tensor with Levi-Civita connection on LSP-Sasakian manifold

respectively. Also our aim is to find conditions for which conharmonic curvature tensor and Tachibana curvature tensor with semi-symmetric metric  $\phi$ -connection  $\bar{D}$  equals to conharmonic curvature tensor and Tachibana curvature tensor with Levi-Civita connection on LSP-Sasakian manifold.

## 2. Preliminaries

An  $n$ -dimensional differentiable manifold  $M$  is said to a Lorentzian Para-Sasakian (LP-Sasakian) manifold if it admits a  $(1,1)$ -tensor field  $\phi$ , a vector field  $\xi$  and 1-form  $\eta$  and a Lorentzian metric  $g$  which satisfies the following relations

$$\phi^2(X) = X + \eta(X)\xi, \quad (2.1)$$

$$\eta(\xi) = -1, \quad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.3)$$

$$g(X, \xi) = \eta(X), \quad (2.4)$$

$$(D_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (2.5)$$

and 
$$D_X \xi = \phi X, \quad (2.6)$$

for arbitrary vector fields  $X$  and  $Y$ , where  $D$  denotes the covariant differentiation with respect to  $g$  [7].

An LP-Sasakian manifold  $M$  is said to be Lorentzian Special Para-Sasakian(LSP-Sasakian) manifold if it satisfies

$$F(X, Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.7)$$

where  $F(X, Y) = g(\phi X, Y)$  is a symmetric  $(0,2)$  tensor.

Let  $D$  be the Riemannian connection in  $M$  and  $\bar{D}$  be an affine connection. Then  $\bar{D}$  is said to be metric if it satisfies [7]

$$\bar{D}_X g = 0 \quad (2.8)$$

In an LP-contact manifold  $M$  an affine connection  $\bar{D}$  is called an  $\phi$ -connection if it satisfies

$$\bar{D}_X \phi = 0 \quad (2.9)$$

Again if 
$$\bar{D}_X \xi = \phi X, \quad (2.10)$$

then the connection  $\bar{D}$  is said to be semi-symmetric metric  $\phi$ -connection if it satisfies (2.8) and (2.9).

The torsion tensor of connection  $\bar{D}$  is of the form

$$T(X, Y) = \eta(Y)\phi(X) - \eta(X)\phi(Y). \quad (2.11)$$

We find the following relations on curvature tensor and Ricci tensor on semi-symmetric metric  $\phi$ -connection on LSP-Sasakian manifold [7].

$$\hat{R}(X, Y)Z = R(X, Y)Z + g(X, Z)Y - g(Y, Z)X, \quad (2.12)$$

$$\hat{S}(X, Y) = S(X, Y) - (n - 1)g(X, Y), \quad (2.13)$$

and

$$\hat{r} = r - (n - 1), \quad (2.14)$$

where  $\hat{R}(X, Y)Z$ ,  $\hat{S}(X, Y)$  and  $\hat{r}$  are the curvature tensor, Ricci tensor and scalar curvature respectively with respect to connection  $\bar{D}$ .

From [1], the pseudo-projective curvature tensor  $\tilde{P}(X, Y)Z$ , on a  $n$ -dimensional Riemannian manifold, is defined by

$$\begin{aligned} \tilde{P}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] \\ &\quad - \frac{r}{n} \left[ \frac{a}{n-1} + b \right] [g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (2.15)$$

Again from [2] we know quasi-conformal curvature tensor on a Riemannian manifold is defined by

$$\begin{aligned} \tilde{C}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY] - \frac{r}{n} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (2.16)$$

where  $a, b$  are constants and  $r$  is the scalar curvature of the Riemannian manifold,

$$g(QX, Y) = S(X, Y) \quad (2.17)$$

and

$$g(\tilde{C}(X, Y)Z, W) = \dot{\tilde{C}}(X, Y, Z, W), \quad (2.18)$$

From [6], the conharmonic curvature tensor on a Riemannian manifold is defined by

$$\begin{aligned} H(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2} [S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY], \end{aligned} \quad (2.19)$$

where

$$g(H(X, Y)Z, W) = \acute{H}(X, Y, Z, W) \quad (2.20)$$

Also from [9] we know the Tachibana concircular curvature tensor on a Riemannian manifold is defined by

$$\begin{aligned} T(X, Y)Z &= R(X, Y)Z - \frac{r}{n(n+2)}\{g(Y, Z)X - g(X, Z)Y \\ &\quad + g(\bar{Y}, Z)\bar{X} - g(\bar{X}, Z)\bar{Y} - 2g(\bar{X}, Y)\bar{Z}\}, \end{aligned} \quad (2.21)$$

where  $\phi X = \bar{X}$ .

These results will be required in the next sections.

### 3. Relation between Pseudo-Projective Curvature Tensor with Semi-Symmetric Metric $\phi$ -Connection and with Levi-Civita Connection on LSP-Sasakian Manifold

In this section we get the relation between pseudo-projective curvature tensor on a semi-symmetric metric  $\phi$ -connection and pseudo-projective curvature tensor on Levi-Civita connection in an LSP-Sasakian manifold  $M$ .

**Theorem 3.1 :** A necessary and sufficient condition that the pseudo-projective curvature tensor of the semi-symmetric metric  $\phi$ -connection  $\bar{D}$  to be equal to pseudo-projective curvature tensor with respect to Levi-Civita connection in an LSP-Sasakian manifold is  $a + (n - 1)b = 0$ .

**Proof :** Let the pseudo projective curvature tensor of a semi symmetric metric  $\phi$ -connection  $\bar{D}$  on LSP-Sasakian manifold is

$$\begin{aligned} \hat{P}(X, Y)Z &= a\hat{R}(X, Y)Z + b[\hat{S}(Y, Z)X - \hat{S}(X, Z)Y] \\ &\quad - \frac{\hat{r}}{n} \left[ \frac{a}{n-1} + b \right] [g(Y, Z)X - g(X, Z)Y]. \end{aligned} \quad (3.1)$$

Using (2.12), (2.13) and (2.14) in (3.1), we get

$$\begin{aligned} \hat{P}(X, Y)Z &= \tilde{P}(X, Y)Z + a[g(X, Z)Y - g(Y, Z)X] \\ &\quad + (n - 1)b[g(X, Z)Y - g(Y, Z)X] + \frac{(n - 1)}{n} \left[ \frac{a}{n - 1} + b \right] \end{aligned}$$

$$\begin{aligned}
& [g(Y, Z)X - g(X, Z)Y], \\
\text{or, } \hat{P}(X, Y)Z &= \tilde{P}(X, Y)Z + \frac{\{(n-1)b + a\}(n-1)}{n} \\
& [g(X, Z)Y - g(Y, Z)X] \tag{3.2}
\end{aligned}$$

If  $a + (n-1)b = 0$ , then  $\hat{P}(X, Y)Z = \tilde{P}(X, Y)Z$ .

Conversely,

we consider  $\hat{P}(X, Y)Z = \tilde{P}(X, Y)Z$ , then from (3.2), we get

$$[g(X, Z)Y - g(Y, Z)X] \frac{\{(n-1)b + a\}(n-1)}{n} = 0.$$

$$\text{or, } (1-n)\{a + (n-1)b\} = 0.$$

$$\text{i.e., } a + (n-1)b = 0, \text{ since } n \neq 1.$$

#### 4. Relation between Quasi-Conformal Curvature Tensor with Semi-Symmetric Metric $\phi$ -Connection and with Levi-Civita Connection on LSP-Sasakian Manifold

In this section we get the relation between quasi-conformal curvature tensor on a semi-symmetric metric  $\phi$ -connection and quasi-conformal curvature tensor on Levi-Civita connection in an LSP-Sasakian manifold  $M$ .

**Theorem 4.1 :** A necessary and sufficient condition that the quasi-conformal curvature tensor of the semi-symmetric metric  $\phi$ -connection  $\bar{D}$  to be equal to quasi-conformal curvature tensor with respect to Levi-Civita connection in an LSP-Sasakian manifold is  $a + (2n-1)b = 0$ .

**Proof :** Taking scalar product of (2.16) and using (2.17) and (2.18), we get

$$\begin{aligned}
\hat{C}(X, Y, Z, W) &= a\hat{R}(X, Y, Z, W) + b[S(Y, Z)g(X, W) \\
&\quad - S(X, Z)g(Y, W) + g(Y, Z)S(X, W) - g(X, Z)S(Y, W)] \\
&\quad - \frac{r}{n} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)], \tag{4.1}
\end{aligned}$$

where

$$\hat{R}(X, Y, Z, W) = g(R(X, Y)Z, W) \tag{4.2}$$

Now we consider quasi-conformal curvature tensor with respect to semi-symmetric metric  $\phi$ -connection  $\bar{D}$  on LSP-Sasakian manifold.

$$\begin{aligned}\hat{\hat{C}}(X, Y, Z, W) &= a.g(\hat{R}(X, Y)Z, W) + b[\hat{S}(Y, Z)g(X, W) \\ &\quad - \hat{S}(X, Z)g(Y, W) + g(Y, Z)\hat{S}(X, W) - g(X, Z)\hat{S}(Y, W)] \\ &\quad - \frac{\hat{r}}{n} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].\end{aligned}\quad (4.3)$$

Using (2.12), (2.13) and (2.14) in (4.3), we obtain

$$\begin{aligned}\hat{\hat{C}}(X, Y, Z, W) &= \hat{C}(X, Y, Z, W) + a[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)] \\ &\quad - b(n-1)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ &\quad + g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + \frac{(n-1)}{n} \left[ \frac{a}{n-1} + b \right] \\ &\quad [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ \hat{\hat{C}}(X, Y, Z, W) &= \hat{C}(X, Y, Z, W) + [g(X, Z)g(Y, W) - g(Y, Z)g(X, W)] \\ &\quad \left[ a + 2(n-1)b - \frac{a}{n} - \frac{b(n-1)}{n} \right].\end{aligned}\quad (4.4)$$

If  $a + (2n-1)b = 0$ , then  $\hat{\hat{C}}(X, Y, Z, W) = \hat{C}(X, Y, Z, W)$ .

Thus

$$\tilde{C}(X, Y)Z = \hat{\hat{C}}(X, Y)Z. \quad (4.5)$$

Conversely,

we consider (4.5) holds. Then from (4.4), we get

$$[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)] \left[ a + 2(n-1)b - \frac{a}{n} - \frac{b(n-1)}{n} \right] = 0.$$

$$\text{or, } a + 2(n-1)b - \frac{a}{n} - \frac{b(n-1)}{n} = 0.$$

$$\text{or, } (n-1)\{a + b(2n-1)\} = 0.$$

Thus  $a + b(2n-1) = 0$ , since  $n \neq 1$ .

### 5. Relation between Conharmonic Curvature Tensor with Semi-Symmetric Metric $\phi$ -Connection and with Levi-Civita Connection on LSP-Sasakian Manifold

In present section we get the relation between conharmonic curvature tensor on a semi-symmetric metric  $\phi$ -connection and conharmonic curvature tensor on Levi-Civita connection in an LSP-Sasakian manifold  $M$ .

**Theorem 5.1 :** In an LSP-Sasakian manifold the conharmonic curvature tensor  $\hat{H}(X, Y, Z, W)$  of a semi symmetric metric  $\phi$ -connection  $\bar{D}$  is

$$\hat{H}(X, Y, Z, W) = \acute{H}(X, Y, Z, W) + k[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)],$$

where  $k = [1 + \frac{2(n-1)}{n-2}]$ .

**Proof :** Taking scalar product of (2.19) and using (2.17), (2.20) and (4.2), one can obtain

$$\begin{aligned} \acute{H}(X, Y, Z, W) &= \acute{R}(X, Y, Z, W) - \frac{1}{n-2}[S(Y, Z)g(X, W) \\ &\quad - S(X, Z)g(Y, W) + S(X, W)g(Y, Z) - S(Y, W)g(X, Z)] \end{aligned}$$

Now we consider the conharmonic curvature tensor of a semi symmetric metric  $\phi$ -connection  $\bar{D}$  on LSP-Sasakian manifold is

$$\begin{aligned} \hat{H}(X, Y, Z, W) &= \hat{R}(X, Y, Z, W) - \frac{1}{n-2}[\hat{S}(Y, Z)g(X, W) \\ &\quad - \hat{S}(X, Z)g(Y, W) + \hat{S}(X, W)g(Y, Z) - \hat{S}(Y, W)g(X, Z)] \end{aligned}$$

Using (2.12) and (2.13), we get

$$\begin{aligned} \hat{H}(X, Y, Z, W) &= \acute{H}(X, Y, Z, W) + g(X, Z)g(Y, W) - g(Y, Z)g(X, W) \\ &\quad - \frac{2(n-1)}{n-2}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ &= \acute{H}(X, Y, Z, W) + [g(X, Z)g(Y, W) \\ &\quad - g(Y, Z)g(X, W)][1 + \frac{2(n-1)}{n-2}]. \end{aligned}$$

$$\hat{H}(X, Y, Z, W) = \acute{H}(X, Y, Z, W) + k[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)],$$

where  $k = [1 + \frac{2(n-1)}{n-2}]$ .

## 6. Relation between Tachibana Concircular Curvature Tensor with Semi-Symmetric Metric $\phi$ -Connection and with Levi-Civita Connection on LSP-Sasakian Manifold

In this section we get the relation between Tachibana concircular curvature tensor on a semi-symmetric metric  $\phi$ -connection and Tachibana concircular curvature tensor on Levi-Civita connection in an LSP-Sasakian manifold  $M$ .

**Theorem 6.1 :** In an LSP-Sasakian manifold the Tachibana concircular curvature tensor  $\hat{T}(X, Y)Z$  of a semi symmetric metric  $\phi$ -connection  $\bar{D}$  is

$$\hat{T}(X, Y)Z = T(X, Y)Z + l[g(X, Z)Y - g(Y, Z)X] + m\{g(\bar{Y}, Z)\bar{X} - g(\bar{X}, Z)\bar{Y} - 2g(\bar{X}, Y)\bar{Z}\},$$

where  $l = 1 - \frac{n-1}{n(n-2)}$  and  $m = \frac{n-1}{n(n-2)}$ .

**Proof :** Let the Tachibana concircular curvature tensor of a semi symmetric metric  $\phi$ -connection  $\bar{D}$  on LSP-Sasakian manifold is

$$\hat{T}(X, Y)Z = \hat{R}(X, Y)Z - \frac{\hat{r}}{n(n+2)}\{g(Y, Z)X - g(X, Z)Y + g(\bar{Y}, Z)\bar{X} - g(\bar{X}, Z)\bar{Y} - 2g(\bar{X}, Y)\bar{Z}\}.$$

Using (2.12) and (2.13), we obtain

$$\begin{aligned} \hat{T}(X, Y)Z &= T(X, Y)Z + g(X, Z)Y - g(Y, Z)X + \frac{(n-1)}{n(n+2)}\{g(Y, Z)X \\ &\quad - g(X, Z)Y + g(\bar{Y}, Z)\bar{X} - g(\bar{X}, Z)\bar{Y} - 2g(\bar{X}, Y)\bar{Z}\}. \\ &= T(X, Y)Z + [g(X, Z)Y - g(Y, Z)X] \left[1 - \frac{n-1}{n(n-2)}\right] \\ &\quad + \frac{n-1}{n(n-2)}\{g(\bar{Y}, Z)\bar{X} - g(\bar{X}, Z)\bar{Y} - 2g(\bar{X}, Y)\bar{Z}\}. \end{aligned}$$

$$\text{or, } \hat{T}(X, Y)Z = T(X, Y)Z + l[g(X, Z)Y - g(Y, Z)X] + m\{g(\bar{Y}, Z)\bar{X} - g(\bar{X}, Z)\bar{Y} - 2g(\bar{X}, Y)\bar{Z}\},$$

where  $l = 1 - \frac{n-1}{n(n-2)}$  and  $m = \frac{n-1}{n(n-2)}$ .



### References

1. Tarafdar, M., Bhattacharyya, A. and Debnath, D. : A type of pseudo-projective  $\varphi$ -recurrent trans-Sasakian manifold, *Analele Stiintifice ale Universitatii "Al.I. Cuza" Iasi, Matematica*, 2006, f.2., 417-422.
2. Bhattacharyya, A. : Quasi-conformally flat contact metric manifolds, *Analele Stiintifice ale Universitatii "Al.I. Cuza" Iasi, Matematica*, 2006, f.1., 71-80.
3. Matsumoto, K. : On Lorentzian Para-contact manifold, *Bull of Yamagata Univ., Nat. Sci.* 12(1989), 151-156.
4. Mishra, R.S. : Structure on a differentiable manifold and their application, Chandrama Prakashan, Allahabad, 1984.
5. Yano, K. : On semi-symmetric metric connection, *Rev. Roum. Math. Pures et. Appl.*, 15 (1970), 1579-81.
6. Bhattacharyya, A. : On a type of conharmonically flat LP sasakian manifold, *Analele Stiintifice ale Universitatii "Al.I. Cuza" Iasi, Matematica*, 2001, f.1., 183-188.
7. Jaiswal, V.K., Ojha, R. H. and Prasad, B. : A semi-symmetric metric  $\phi$ -connection in an LSP-Sasakian manifold, *J. Nat. Acad. Math.*, 15 (2001), 73-78.
8. Yano, K. and Imai, T. : On semi-symmetric metric  $\phi$ -connection in a Sasakian manifold, *Kodai Math. Sem. Rep.*, 28 (1977), 1-7.
9. Tarafdar, M. and Mayra, A. : On nearly Kahler manifold with conservative, concircular and H-projective curvature tensor, *Acta Ciencia*, Vol.XXIIM, No. 4 (1996), 491-493.