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On Some Measures, Analogues to Carlson's Dirichlet Measure, Averages, Generalized Erde'lyi-Kober Integral Operators and Applications

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Abstract

In the present paper, we define two measures identical to Carlson's Dirichlet measure and then, with them, find two integral averages and again, construct two generalized fractional integral operators involving these averages those are analogues to generalized Erde'lyi-Kober integral operators. Then, we make their applications to evaluate the transformation formulae of elementary functions on analytic continuation theory. Also, we discuss some of their particular cases.

Key Words : Carlson's Dirichlet measure, new measures analogues to Carlson's Dirichlet measure, averages, generalized Erde'lyi-Kober integral operators.

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1. Introduction

Carlson [1-3] has defined a Dirichlet measure $d\mu_b(u)$ on the standard simplex E in R^{k-1}, $k \geq 2$, such that [4]

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1 - 1} \ldots\ldots u_{k-1}^{b_{k-1} - 1} (1 - u_1 - \ldots - u_{k-1})^{b_k - 1} du_1 \ldots du_{k-1} \quad (1.1)$$

where $b = (b_1, \ldots, b_k) \in C^k$, $0 \leq u_1 \leq 1, \ldots, 0 \leq u_{k-1} \leq 1$, $u_k = 1 - u_1 - \ldots - u_{k-1}$; $B(b) = \{\Gamma(b_1) \ldots \Gamma(b_k)\}/\{\Gamma(b_1 + \ldots + b_k)\}$, where $\mathrm{Re}(b_i) > 0$, $\forall\, i = 1, \ldots, k$.

Due to (1.1), a Dirichlet average F of the function f measurable on the convex set Ω with respect to the Dirichlet measure $d\mu_b(u)$ on the standard simplex E in R^{k-1}, $k \geq 2$, is given by

$$F(b; z) = \int_E f(u. z)\, d\mu_b(u) \tag{1.2}$$

where $z = (z_1, ..., z_k) \in \Omega^k \subset C^k$ and $(u.z) = \sum_{i=1}^{k} u_i z_i$ is the convex combination of $z_1, ..., z_k$ to the $u_1, .., u_k$ and all $u_1, ..., u_k \in R_k$.

Also for $k = 1$, $$F(b; z) = f(z). \tag{1.3}$$

Now, introducing new non-zero real parameters $z_1, ..., z_k$, we define a new measure $d\mu_{b,z}(u)$ on the standard simplex E in R^{k-1}, $k \geq 2$, in the form

$$d\mu_{b,z}(u) = \frac{z_k^{b_k}}{B(b)} u_1^{b_1 - 1} \cdots u_{k-1}^{b_{k-1} - 1} \left(1 - \frac{u_1}{z_1} - \cdots - \frac{u_{k-1}}{z_{k-1}}\right)^{b_k - 1} du_1 \ldots du_{k-1} \tag{1.4}$$

where B(b), and b are given in (1.1) and $u_1, ..., u_k$ are such that

$$0 \leq u_1 \leq z_1, \ldots, 0 \leq u_{k-1} \leq z_{k-1}, \quad \frac{u_k}{z_k} = 1 - \frac{u_1}{z_1} - \cdots - \frac{u_{k-1}}{z_{k-1}}.$$

Again, for non-zero real numbers $z_1, ..., z_k$ and $u_1, ..., u_k$, we define second new measure $d\mu_{*b,z}(u)$ on the standard simplex E in R^{k-1}, $k \geq 2$, in the form

$$d\mu_{*b,z}(u) = \frac{z_k^{-b_k}}{B(b)} u_1^{-b_1 - 1} \cdots u_{k-1}^{-b_{k-1} - 1} \left(1 - \frac{z_1}{u_1} - \cdots - \frac{z_{k-1}}{u_{k-1}}\right)^{b_k - 1} du_1 \ldots du_{k-1} \tag{1.5}$$

where B(b) and b are given in (1.1) and $z_1 \leq u_1 < \infty, ..., z_{k-1} \leq u_{k-1} < \infty$, $\frac{z_k}{u_k} = 1 - \frac{z_1}{u_1} - \cdots - \frac{z_{k-1}}{u_{k-1}}$.

Further, for obtaining the generalizations of the elementary functions, we define an integral average F {f}(b; z) of an arbitrary function f measurable on the standard simplex E in R^{k-1}, $k \geq 2$, with respect to the measure $d\mu_{b,z}(u)$, in the form

$$F\{f\}(b; z) = \int_E f(u_1 + ... + u_k)\, d\mu_{b,z}(u) \tag{1.6}$$

where, all conditions given in (1.4) are satisfied.

Also, for k = 1, it is given by

$$F\{f\}(b; z) = z^b f(z). \tag{1.7}$$

The second integral average $F^*\{f\}(b; z)$ of an arbitrary function f measurable on the standard simplex E in R^{k-1}, $k \geq 2$, with respect to the measure $d\mu_{*b,z}(u)$, in the form

$$F^*\{f\}(b; z) = \int_E f(u_1^{-1} + ... + u_k^{-1})\, d\mu_{*b,z}(u) \tag{1.8}$$

where all conditions given in (1.5) are satisfied. Also, for k = 1, it is given by

$$F^*\{f\}(b; z) = z^{-b} f(z). \tag{1.9}$$

Now, to get more generalizations of both these integral averages in terms of the analytic functions, we define following generalized fractional integral operators associated with these integral averages as kernels and an arbitrary multivariable function which are analogous to the generalized Erde'lyi-Kober operators such that :

If $c = (c_1, ..., c_k)$, $b = (b_1, .., b_k)$, $z_i \in (0, x_i)$, where $x_i > 0$, Re $(c_i) > 0$, Re$(b_i) > 0$, $\forall\, i = 1. ...., k$,

then, for all the conditions given in (1.4), (1.6) and (1.7), the generalized fractional integral operator, associated with the function $F\{f\}(b; z)$ and an arbitrary multivariable function $\Phi(z_1, ..., z_k)$, is defined by

$$I^{b,c,f}\{\Phi\}(x_1, ..., x_k) = \frac{x_1^{-b_1-c_1}}{\Gamma(c_1)} \cdots \frac{x_k^{-b_k-c_k}}{\Gamma(c_k)} \int_0^{x_1} \cdots \int_0^{x_k} (x_1 - z_1)^{c_1-1} \cdots (x_k - z_k)^{c_k-1} F\{f\}(b; z)\, \Phi(z_1, ..., z_k)\, dz_1 \dots dz_k. \tag{1.10}$$

The second generalized fractional integral operator, associated with the function $F^*\{f\}(b; z)$ and an arbitrary multivariable function $\Phi(z_1, ..., z_k)$, for $c = (c_1, ..., c_k)$, $b = (b_1, ..., b_k)$, $x_i \leq z_i < \infty$, $x_i > 0$, Re $(c_i) > 0$, Re $(b_i) > 0$, $\forall\, i = 1, ..., k$ and for all the conditions given in (1.5), (1.8) and (1.9), is defined by

$$J^{b,c,f}\{\Phi\}(x_1, ..., x_k) = \frac{x_1^{b_1}}{\Gamma(c_1)} \cdots \frac{x_1^{b_k}}{\Gamma(c_k)} \int_{x_1}^{\infty} \cdots \int_{x_k}^{\infty} (z_1 - x_1)^{c_1 - 1} \cdots\cdots$$

$$(z_k - x_k)^{c_k - 1} x_1^{-c_1} \cdots\cdots x_k^{-c_k} F^*\{f\}(b; z)\, \Phi(z_1, ..., z_k)\, dz_1 \ldots dz_k. \tag{1.11}$$

Particularly, for k = 2 and f = 1, from (1.10) and (1.11), we get

$$I^{b,c,1}\{\Phi\}(x, y) = \frac{x^{-b-c}}{\Gamma(c)} \frac{y^{-b'-c'}}{\Gamma(c')} \int_0^x \int_0^y (x - z)^{c-1} (y - t)^{c'-1} z^b t^{b'} \Phi(z, t)\, dz\, dt. \tag{1.12}$$

and

$$J^{b,c,1}\{\Phi\}(x, y) = \frac{x^b}{\Gamma(c)} \frac{y^{b'}}{\Gamma(c')} \int_x^{\infty} \int_y^{\infty} (z - x)^{c-1} (t - y)^{c'-1} z^{-b-c} t^{b'-c'} \Phi(z, t)\, dz\, dt. \tag{1.13}$$

respectively, those are double Erde'lyi-Kober operators.

We consider the multivariable general class of polynomials defined by [9]

$$S_L^{h_1,\ldots,h_m}(x_1, \ldots, x_m) = \sum_{r_1,\ldots,r_m=0}^{h_1 r_1 + \ldots + h_m r_m \le L} (-L)_{h_1 r_1 + \ldots + h_m r_m} A[L; r_1, .., r_m] . \frac{x_1^{r_1}}{r_1!} \cdots\cdots \frac{x_m^{r_m}}{r_m!} \tag{1.14}$$

where $h_1, \ldots h_m$ are arbitrary positive integers and the coefficients $A[L; r_1, \ldots, r_m]$, $(L, r_j \in N_0 = \{0, 1, 2, \ldots\}, j = 1, \ldots, m)$ are arbitrary constants real or complex.

For m = 1, of the polynomials defined by (1.14) would corresponds to the polynomials due to Srivastava [9]

$$S_L^h = \sum_{r=0}^{[L/h]} (-L)_{hr} A_{L,r} \frac{x^r}{r!} \tag{1.15}$$

where h is arbitrary positive integer and the coefficients, $A_{L,r}$ $(L, r \in N_0 = \{0, 1, 2, \ldots\})$ are arbitrary constants real or complex.

The Pochhammer symbol is defined by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}. \tag{1.16}$$

Here, in our investigations, we evaluate the transformation formulae of the polynomials defined in (1.14) due to our operators defined by (1.10) and (1.11), finally we discuss some of their particular cases.

2. Integral Transformation Formulae

In this section, we evaluate, following integral transform formulae for the special function $f(x) = e^{-\alpha x}$, α real or complex :

If x has a transformation $x = u_1 + \ldots + u_k$, under the domain $0 \le u_1 \le z_1, \ldots,$ $0 \le u_{k-1} \le z_{k-1}$, $\frac{u_k}{z_k} = 1 - \frac{u_1}{z_1} - \ldots - \frac{u_{k-1}}{z_{k-1}}$, where $z_i > 0 \; \forall \; i = 1, .., k$, then for $c = (c_1, \ldots, c_k)$, $b = (b_1, .., b_k)$, $z_i \in (0, x_i)$, such that $x_i > 0$, $\text{Re}(c_i) > 0$, $\text{Re}(b_i) > 0$, $\forall i = 1, ..k$, and for the special function $f(x) = e^{-\alpha x}$, α real or complex. Then following transformation formula holds

$$\int_0^{x_1} \ldots \int_0^{x_k} z_1^{\sigma_1 - 1} \ldots (x_1 - z_1)^{c_1 - 1} \ldots z_k^{\sigma_k - 1} (x_k - z_k)^{c_k - 1} F\{f\}(b; z)\, dz_1 \ldots dz_k =$$

$$\frac{\Gamma(c_1)\ldots\Gamma(c_k)\, x_1^{\sigma_1 + b_1 + c_1 - 1} \ldots x_k^{\sigma_k + b_k + c_k - 1}}{B(b)} G^{0,0:(1,2);\ldots;(1,2)}_{0,1:[2,2];\ldots;[2,2]}$$

$$\left[\begin{matrix} [\ldots\ldots..]:[1-b_1:1],[1-\sigma_1-b_1:1];\ldots[1-b_k:1],[1-\sigma_k-b_k:1]; \\ [1-b_1-\ldots-b_k:1,\ldots,1]:[0:1],[1-\sigma_1-b_1-c_1:1];\ldots;[0,1],[1-\sigma_k-b_k-c_k:1]; \end{matrix} \alpha x_1, .., \alpha x_k\right] \tag{2.1}$$

provided that $\text{Re}(\sigma_i) > \text{Re}(-b_i)$ and $|\arg(\alpha x_i)| < \pi/2$, $\forall i = 1, 2, \ldots, k$, the multivariable G-function corresponds to the H-function due to Srivastava and Panda [12].

Proof. In the left hand side of (2.1), define the average function $F\{f\}(b; z)$, with the help of (1.6) and (1.7), for the function $f(x) = e^{-\alpha x}$, α real or complex, then apply the Mellin Barnes Formula for the exponential function due to Mathai and Saxena [6]

$$e^{-x} = \frac{1}{2\pi\omega}\int_{-\omega\infty}^{\omega\infty} \Gamma(-\xi)\, x^{\xi}\, d\xi, \quad |x| < \infty, \quad \omega = \sqrt{(-1)}, \tag{2.2}$$

in it and then set $u_i = z_i v_i$, $\forall\, i = 1, 2, ..., k$ and again make the transformations with the help of the conditions given in (1.4), and thus change the order of integrations, we find that

$$\frac{1}{B(b)}\frac{1}{(2\pi\omega)^k}\int_{-\omega\infty}^{\omega\infty} \cdots \int_{-\omega\infty}^{\omega\infty} \Gamma(-\xi_1) \cdots\cdots \Gamma(-\xi_k)\,(\alpha)^{\xi_1} \cdots (\alpha)^{\xi_k}$$

$$\int_0^{x_1} \cdots \int_0^{x_k} z_1^{\sigma_1+b_1+\xi_1-1}(x_1-z_1)^{c_1-1} \cdots z_k^{\sigma_k+b_k+\xi_k-1}(x_k-z_k)^{c_k-1}\, dz_1 \ldots dz_k$$

$$\int \cdots \int v_1^{b_1+\xi_1-1} \cdots v_{k-1}^{b_{k-1}+\xi_{k-1}-1}(1-v_1-\cdots-v_{k-1})^{b_k+\xi_k-1} dv_1 \ldots dv_{k-1} d\xi_1 \ldots d\xi_k \tag{2.3}$$

Now, use the Dirichlet integral formula and the formula of the Beta function occurring in the inner integrals of (2.3), we get

$$\Gamma(c_1)\, x_1^{b_1+c_1+\sigma_1-1} \cdots \Gamma(c_k)\, x_k^{b_k+c_k+\sigma_k-1}\, \frac{1}{B(b)\,(2\pi\omega)^k}$$

$$\int_{-\omega\infty}^{\omega\infty} \cdots \int_{-\omega\infty}^{\omega\infty} \frac{1}{\Gamma(b_1+\ldots+b_k+\xi_1+\ldots+\xi_k)}\, \frac{\Gamma(-\xi_1)\,\Gamma(b_1+\xi_1)\,\Gamma(\sigma_1+b_1+\xi_1)}{\Gamma(\sigma_1+b_1+c_1+\xi_1)}$$

$$\cdots\cdots \frac{\Gamma(-\xi_k)\,\Gamma(b_k+\xi_k)\,\Gamma(\sigma_k+b_k+\xi_k)}{\Gamma(\sigma_k+b_k+c_k+\xi_k)}\,(\alpha x_1)^{\xi_1} \cdots (\alpha x_k)^{\xi_k}\, d\xi_1 \ldots d\xi_k. \tag{2.4}$$

Finally, define the multivariable G-function corresponding to the multivariable H-function of Srivastava and Panda [12], we obtain the right hand side of (2.1).

If x has the transformation $x = u_1^{-1} + \cdots + u_k^{-1}$, under the domain $z_1 \le u_1 < \infty$, ..., $z_{k-1} \le u_{k-1} < \infty$, $\frac{z_k}{u_k} = 1 - \frac{z_1}{u_1} - \cdots - \frac{z_{k-1}}{u_{k-1}}$, where u_i and $z_i > 0\ \forall\, i = 1, .., k$, then for $c = (c_1, ..., c_k)$, $b = (b_1, .., b_k)$, $z_i \in (x_i, \infty)$, such that $x_i > 0$, $\mathrm{Re}(c_i) > 0$, $\mathrm{Re}(b_i) > 0$, $\forall i = 1, ..., k$, and for the special function $f(x) = e^{-\alpha x}$, α real or complex, the second transformation formula holds

$$\int_{x_1}^{\infty} \cdots \int_{x_k}^{\infty} z_1^{\sigma_1 - c_1 - 1} (z_1 - x_1)^{c_1 - 1} \cdots z_k^{\sigma_k - c_k - 1} (z_k - x_k)^{c_k - 1} F^*\{f\}(b; z)\, dz_1 \ldots dz_k =$$

$$\frac{\Gamma(c_1)\ldots\Gamma(c_k)\, x_1^{\sigma_1 - b_1 - 1} \cdots x_k^{\sigma_k - b_k - 1}}{B(b)} G^{0,0:(1,2);\ldots;(1,2)}_{0,1:[2,2];\ldots;[2,2]}$$

$$\left[\begin{matrix} [\ldots\ldots\ldots]:[1-b_1:1],[\sigma_1 - b_1:1];\ldots[1-b_k:1],[\sigma_k - b_k:1]; \\ [1-b_1-\ldots-b_k:1,\ldots,1]:[0:1],[\sigma_1 - b_1 - c_1:1];\ldots;[0,1],[\sigma_k - b_k - c_k:1]; \end{matrix} \frac{\alpha}{x_1},\ldots,\frac{\alpha}{x_k}\right] \tag{2.5}$$

provided that $\mathrm{Re}(\sigma_i) < 1 + \mathrm{Re}(b_i)$, and $|\arg(\alpha/x_i)| < \pi/2$, $\forall\, i = 1, 2, \ldots, k$.

Proof. In left hand side of (2.5), define $F^*\{f\}(b; z)$, with the help of (1.8) and (1.9), for the function $f(x) = e^{-\alpha x}$, a real or complex, then apply the Mellin Barnes Formula for the exponential function due to Mathai and Saxena [6], given in (2.2), and then set $z_i = u_i v_i$, $\forall\, i = 1, 2, \ldots, k$, and make the transformations with the help of the conditions given in (1.5), and thus change the order of integrations, we find that

$$\frac{1}{B(b)} \frac{1}{(2\pi\omega)^k} \int_{-\omega\infty}^{\omega\infty} \cdots \int_{-\omega\infty}^{\omega\infty} \Gamma(-\xi_1) \cdots \Gamma(-\xi_k)\, (\alpha)^{\xi_1} \cdots (\alpha)^{\xi_k}$$

$$\int_{x_1}^{\infty} \cdots \int_{x_k}^{\infty} z_1^{\sigma_1 - b_1 - c_1 - \xi_1 - 1} (z_1 - x_1)^{c_1 - 1} \cdots z_k^{\sigma_k - b_k - c_k - \xi_k - 1} (z_k - x_k)^{c_k - 1} dz_1 \ldots dz_k$$

$$\int \cdots \int v_1^{b_1 + \xi_1 - 1} \cdots v_{k-1}^{b_{k-1} + \xi_{k-1} - 1} (1 - v_1 - \cdots - v_{k-1})^{b_k + \xi_k - 1} dv_1 \ldots dv_{k-1} d\xi_1 \ldots d\xi_k \tag{2.6}$$

Now, in (2.6) do the same processes as analyzed after (2.3), we evaluate the right hand side of (2.5).

3. Applications

Theorem A. If $\mathrm{Re}(b_i) > 0$, $\mathrm{Re}(c_i) > 0$, $\mathrm{Re}(\sigma_i) > \mathrm{Re}(-b_i)$ and $|\arg(\alpha x_i)| < \pi/2$, $\forall\, i = 1, 2, \ldots, k$, and $I^{b,c,f}\{.\}$ be the generalized fractional integral operator involving the kernel $F\{f\}(b; z)$, where the function $f(x) = e^{-\alpha x}$, α real or complex and $z_i \in (0, x_i) > 0$, $\forall\, i = 1, 2, \ldots, k$, defined by (1.10), then for arbitrary positive integers $h_1, \ldots, h_k$, $L \in N_0 = \{0, 1, 2, \ldots\}$ and the complex numbers μ_i and β_i, where μ_i and $\beta_i \neq 0$, $\forall\, i = 1, 2, \ldots, k$, the image

$I^{b,c,f}\{z_1^{\sigma_1 - 1} \ldots z_k^{\sigma_k - 1} S_L^{h_1,\ldots,h_k}(\beta_1 z_1^{\mu_1} \ldots \beta_k z_k^{\mu_k})\}(x_1, \ldots, x_k)$ of the function $z_1^{\sigma_1 - 1} \ldots z_k^{\sigma_k - 1} S_L^{h_1,\ldots,h_k}(\beta_1 z_1^{\mu_1} \ldots \beta_k z_k^{\mu_k})$ exists and there holds the formula

$$I^{b,c,f}\{z_1^{\sigma_1 - 1} \ldots z_k^{\sigma_k - 1} S_L^{h_1,\ldots,h_k}(\beta_1 z_1^{\mu_1} \ldots \beta_k z_k^{\mu_k})\}(x_1, \ldots, x_k) =$$

$$\frac{x_1^{\sigma_1 - 1} \ldots x_k^{\sigma_k - 1}}{B(b)} \sum_{r_1,\ldots,r_k = 0}^{h_1 r_1 + \ldots + h_k r_k \le L} (-L)_{h_1 r_1 + \ldots + h_k r_k} A[L; r_1, \ldots, r_k]$$
$$\frac{(\beta_1 x_1^{\mu_1})^{r_1}}{r_1!} \cdots \frac{(\beta_k x_k^{\mu_k})^{r_k}}{r_k!} G^{0,0:(1,2);\ldots;(1,2)}_{0,1:[2,2];\ldots;[2,2]}$$
$$\left[\begin{matrix} [\ldots\ldots]:[1-b_1:1],[1-\sigma_1-b_1-\mu_1 r_1:1];\ldots[1-b_k:1],[1-\sigma_k-b_k-\mu_k r_k:1]; \\ [1-b_1-\ldots-b_k:1,\ldots,1]:[0:1],[1-\sigma_1-b_1-c_1-\mu_1 r_1:1];\ldots;[0,1],[1-\sigma_k-b_k-c_k-\mu_k r_k:1]; \\ \alpha x_1,\ldots,\alpha x_k \end{matrix} \right] \tag{3.1}$$

where the general coefficients $A[L; r_1, \ldots, r_k]$, $(L, r_i \in N_0 = \{0, 1, 2, \ldots\}\ \forall\, i = 1, 2, \ldots k)$ are arbitrary real or complex.

Theorem B. If $\mathrm{Re}(b_i) > 0$, $\mathrm{Re}(c_i) > 0$, $\mathrm{Re}(\sigma_i) < 1 + \mathrm{Re}(b_i)$ and $|\arg(\alpha/x_i)| < \pi/2$, $\forall\, i = 1, 2, \ldots, k$, and $J^{b,c,f}\{.\}$ be the generalized fractional integral operator involving the kernel $F^*\{f\}(b; z)$, where the function $f(x) = e^{-\alpha x}$, α real or complex and $z_i \in (x_i, \infty)$, $x_i > 0$, $\forall\, i = 1, 2, \ldots, k$, defined by (1.11), then for arbitrary positive integers $h_1, \ldots, h_k$, $L \in N_0 = \{0, 1, 2, \ldots\}$ and the complex numbers μ_i and β_i, where μ_i and $\beta_i \neq 0$, $\forall\, i = 1, 2, \ldots, k$, the image $J^{b,c,f}\{z_1^{\sigma_1 - 1} \ldots z_k^{\sigma_k - 1} S_L^{h_1,\ldots,h_k}(\beta_1 z_1^{\mu_1} \ldots \beta_k z_k^{\mu_k})\}(x_1, \ldots, x_k)$ of the function $z_1^{\sigma_1 - 1} \ldots z_k^{\sigma_k - 1} S_L^{h_1,\ldots,h_k}(\beta_1 z_1^{\mu_1} \ldots \beta_k z_k^{\mu_k})$ exists and there holds the formula

$$J^{b,c,f}\{z_1^{\sigma_1 - 1} \ldots z_k^{\sigma_k - 1} S_L^{h_1,\ldots,h_k}(\beta_1 z_1^{\mu_1} \ldots \beta_k z_k^{\mu_k})\}(x_1, \ldots, x_k) =$$
$$\frac{x_1^{\sigma_1 - 1} \ldots x_k^{\sigma_k - 1}}{B(b)} \sum_{r_1,\ldots,r_k = 0}^{h_1 r_1 + \ldots + h_k r_k \le L} (-L)_{h_1 r_1 + \ldots + h_k r_k} A[L; r_1, \ldots, r_k]$$
$$\frac{(\beta_1 x_1^{\mu_1})^{r_1}}{r_1!} \cdots \frac{(\beta_k x_k^{\mu_k})^{r_k}}{r_k!} G^{0,0:(1,2);\ldots;(1,2)}_{0,1:[2,2];\ldots;[2,2]}$$

$$\left[\begin{array}{c}[\ldots\ldots..]:[1-b_1:1],[1-\sigma_1-b_1+\mu_1 r_1:1];\ldots[1-b_k:1],[\sigma_k-b_k+\mu_k r_k:1];\\ [1-b_1-\ldots-b_k:1,\ldots,1]:[0:1],[\sigma_1-b_1-c_1+\mu_1 r_1:1];\ldots;[0,1],[\sigma_k-b_k-c_k+\mu_k r_k:1];\\ \frac{\alpha}{x_1},\ldots\ldots,\frac{\alpha}{x_k}\end{array}\right] \tag{3.2}$$

where the coefficients $A[L; r_1, \ldots, r_k]$, $(L, r_i \in N_0 = \{0, 1, 2, \ldots\}\ \forall\, i = 1, 2, \ldots k)$ are arbitrary real or complex.

Corollary A. For $k = 2$, from theorem A, we find the transformation formula in the series form involving two variables G-function due to Srivastava and Joshi [13] (See, Srivastava, Gupta and Goyal [10, p. 7])

$$I^{b,b',c,c',f}\{z^{\sigma-1}\, t^{\sigma'-1}\, S_L^{h,h'}(\beta z^{\mu}\ldots\ldots \beta' t^{\mu'})\}(x, y) =$$

$$\frac{x^{\sigma-1}\, y^{\sigma'-1}\,\Gamma(b+b')}{\Gamma(b)\,\Gamma(b')}\sum_{r,s=0}^{hr+h's\le L}(-L)_{hr+h's}\, A[L; r, s]\,\frac{(\beta x^{\mu})^r}{r_1!}\,\frac{(\beta' y^{\mu'})^s}{s!}$$

$$G^{0,0:(1,2);\ldots;(1,2)}_{0,1:[2,2];\ldots;[2,2]}\left[\begin{array}{l}(\ldots\ldots..):(1-b,1-\sigma-b-\mu r);(1-b',1-\sigma'-b'-\mu' s);\\ (1-b-b'):(0,1-\sigma-b-c-\mu r);(0,1-\sigma'-b'-c'-\mu' r);\end{array}\alpha x,\alpha y\right] \tag{3.3}$$

where $\mathrm{Re}(b) > 0$, $\mathrm{Re}(b') > 0$, $\mathrm{Re}(c) > 0$, $\mathrm{Re}(c') > 0$, $\mathrm{Re}(\sigma) > \mathrm{Re}(-b)$, $\mathrm{Re}(\sigma') > \mathrm{Re}(-b')$, $|\arg(\alpha x)| < \pi/2$ and $|\arg(\alpha y)| < \pi/2$, h and h' are arbitrary positive integers, $L \in N_0 = \{0, 1, 2, \ldots\}$ and the complex numbers $\mu, \mu', \beta, \beta' \neq 0$. The coefficients $A[L; r, s]$, $(L, r. s \in N_0 = \{0, 1, 2, \ldots\})$ are arbitrary constants real or complex and the function $f(x) = e^{-\alpha x}$, α real or complex.

Corollary B. For $k = 2$, from theorem B, we find the transformation formula in the series form involving two variables G-function due to Srivastava and Joshi [13] (See, Srivastava, Gupta and Goyal [10, p. 7]), such that

$$J^{b,b',c,c',f}\{z^{\sigma-1}\, t^{\sigma'-1}\, S_L^{h,h'}(\beta z^{\mu}\ldots\ldots \beta' t^{\mu'})\}(x, y) =$$

$$\frac{x^{\sigma-1}\, y^{\sigma'-1}\,\Gamma(b+b')}{\Gamma(b)\,\Gamma(b')}\sum_{r,s=0}^{hr+h's\le L}(-L)_{hr+h's}\, A[L; r, s]\,\frac{(\beta x^{\mu})^r}{r_1!}\,\frac{(\beta' y^{\mu'})^s}{s!}$$

$$G^{0,0:1,2;1,2}_{0,1:2,2;2,2}\left[\begin{array}{l}(\ldots\ldots..):(1-b,\sigma-b+\mu r);(1-b',\sigma'-b'+\mu' s);\\ (1-b-b'):(0,\sigma-b-c+\mu r);(0,\sigma'-b'-c'+\mu' s);\end{array}\alpha/x,\alpha/y\right] \tag{3.4}$$

where Re(b) > 0, Re (b′) > 0, Re(c) > 0, Re(c′) > 0, Re(σ) < 1 + Re (b), Re(σ′) < 1 + Re(b′), $|\arg(\alpha/x)| < \pi/2$ and $|\arg(\alpha/y)| < \pi/2$, h and h′ are arbitrary positive integers, $L \in N_0 = \{0, 1, 2, ...\}$ and the complex numbers μ, μ′, β, β′ ≠ 0. The coefficients A[L; r, s], $(L, r. s \in N_0 = \{0, 1, 2, ...\})$ are arbitrary constants real or complex and the function $f(x) = e^{-\alpha x}$, α real or complex.

Since the computational representation of two variables G-function is presented by Mathai and Saxena [5], therefore, from (3.3) and (3.4), we may compute various results of continuation theory.

4. Particular Cases

Setting $A[L; r, s] = \dfrac{\prod_{j=1}^{E}(1-e_j-L)_{hr+h's}\prod_{j=1}^{G}(g_j)_{hr}\prod_{j=1}^{G'}(g'_j)_{h's}}{\prod_{j=1}^{P}(1-p_j-L)_{hr+h's}\prod_{j=1}^{Q}(q_j)_{hr}\prod_{j=1}^{Q'}(q'_j)_{h's}}$ in the equations (3.3) and (3.4) respectively, we get following transformation formulae of the two variables Srivastava and Daoust function [8] in the form

$$I^{b,b',c,c',f}\{z^{\sigma-1}t^{\sigma'-1}F^{E+1:G,G'}_{P:Q,Q'}\left[\begin{matrix}[(1-e-L):h,h'],[-L:h,h']:[(g):h];[(g'):h'];\\ [(1-p-L):h,h']:[(q):h];[(q'):h'];\end{matrix}\beta z^{\mu},\beta' t^{\mu'}\right]\}(x,y)=$$

$$\frac{x^{\sigma-1}y^{\sigma'-1}\Gamma(b+b')}{\Gamma(b)\Gamma(b')}\sum_{r,s=0}^{hr+h's\le L}\frac{(-L)_{hr+h's}\prod_{j=1}^{E}(1-e_j-L)_{hr+h's}\prod_{j=1}^{G}(g_j)_{hr}\prod_{j=1}^{G'}(g'_j)_{h's}}{\prod_{j=1}^{P}(1-p_j-L)_{hr+h's}\prod_{j=1}^{Q}(q_j)_{hr}\prod_{j=1}^{Q'}(q'_j)_{h's}}$$

$$\frac{(\beta x^{\mu})^r}{r_1!}\frac{(\beta' y^{\mu'})^s}{s!}G^{0,0:1,2;1,2}_{0,1:2,2;2,2}\left[\begin{matrix}(\dots\dots\dots):(1-b,1-\sigma-b-\mu r);(1-b',1-\sigma'-b'-\mu' s);\\ (1-b-b'):(0,1-\sigma-b-c-\mu r);(0,1-\sigma'-b'-c'-\mu' s);\end{matrix}\alpha x,\alpha y\right] \tag{4.1}$$

where all conditions of (3.3) are included.

$$J^{b,b',c,c',f}\{z^{\sigma-1}t^{\sigma'-1}F^{E+1:G,G'}_{P:Q,Q'}\left[\begin{matrix}[(1-e-L):h,h'],[-L:h,h']:[(g):h];[(g'):h'];\\ [(1-p-L):h,h']:[(q):h];[(q'):h'];\end{matrix}\beta z^{\mu},\beta' t^{\mu'}\right]\}(x,y)=$$

$$\frac{x^{\sigma-1}\,y^{\sigma'-1}\,\Gamma(b+b')}{\Gamma(b)\,\Gamma(b')}\sum_{r,s=0}^{hr+h's\le L}\frac{(-L)_{hr+h's}\prod_{j=1}^{E}(1-e_j-L)_{hr+h's}\prod_{j=1}^{G}(g_j)_{hr}\prod_{j=1}^{G'}(g'_j)_{h's}}{\prod_{j=1}^{P}(1-p_j-L)_{hr+h's}\prod_{j=1}^{Q}(q_j)_{hr}\prod_{j=1}^{Q'}(q'_j)_{h's}}$$

$$\frac{(\beta\,x^{\mu})^{r}}{r_1!}\,\frac{(\beta'\,y^{\mu'})^{s}}{s!}\,G^{0,0:1,2;1,2}_{0,1:2,2;2,2}$$

$$\left[\begin{matrix}(\ldots\ldots\ldots):(1-b,\sigma-b+\mu r);(1-b',\sigma'-b'+\mu' s);\\(1-b-b'):(0,\sigma-b-c+\mu r);(0,\sigma'-b'-c'+\mu' s);\end{matrix}\,\alpha/x,\alpha/y\right] \tag{4.2}$$

where all conditions of (3.4) are satisfied, respectively.

Due to Mellin Barnes integral formula, we have the relation between following G-function and the Kampé de Fériet function due to Srivastava and Panda [13]

$$G^{0,0:1,2;1,2}_{0,1:2,2;2,2}\left[\begin{matrix}(\ldots\ldots\ldots):(1-b,1-\sigma-b-\mu r);(1-b',1-\sigma'-b'-\mu' s);\\(1-b-b'):(0,1-\sigma-b-c-\mu r);(0,1-\sigma'-b'-c'-\mu' s);\end{matrix}\,\alpha x,\alpha y\right]=$$

$$\frac{\Gamma(b)\,\Gamma(b')}{\Gamma(b+b')}\,\frac{\Gamma(\sigma+b+\mu r)\,\Gamma(\sigma'+b'+\mu' s)}{\Gamma(\sigma+b+c+\mu r)\,\Gamma(\sigma'+b'+c'+\mu' s)}\,F^{0:2;2}_{1:1;1}$$

$$\left[\begin{matrix}(\ldots\ldots\ldots):(b,\sigma+b+\mu r);(b',\sigma'+b'+\mu' s);\\(b+b'):(\sigma+b+c+\mu r);(\sigma'+b'+c'+\mu' s);\end{matrix}\,-\alpha x,-\alpha y\right]. \tag{4.3}$$

Thus, from (4.3), we get the approximation formula of G-function in the limiting form

$$\lim_{\alpha\to 0}\;G^{0,0:1,2;1,2}_{0,1:2,2;2,2}\left[\begin{matrix}(\ldots\ldots\ldots):(1-b,1-\sigma-b-\mu r);(1-b',1-\sigma'-b'-\mu' s);\\(1-b-b'):(0,1-\sigma-b-c-\mu r);(0,1-\sigma'-b'-c'-\mu' s);\end{matrix}\,\alpha x,\alpha y\right]$$

$$=\frac{\Gamma(b)\,\Gamma(b')}{\Gamma(b+b')}\,\frac{\Gamma(\sigma+b+\mu r)\,\Gamma(\sigma'+b'+\mu' s)}{\Gamma(\sigma+b+c+\mu r)\,\Gamma(\sigma'+b'+c'+\mu' s)} \tag{4.4}$$

provided that Re (b) > 0, Re (b′) > 0, Re (σ) > Re(− b) and Re (σ') > Re(− b′) and setting μ and μ' are positive real $\forall$ r, s = 0, 1, 2, ...

Now, make an appeal to (1.12), (4.1) and (4.4), we get the transformation formula

$$\lim_{\alpha\to 0}\; I^{b,b',c,c',f}\{z^{\sigma-1}\,t^{\sigma'-1}\,F^{E+1:G,G'}_{P:Q,Q'}$$

$$\left[\begin{matrix}[(1-e-L): h, h'], [-L: h, h'] : [(g): h]; [(g'): h'];\\ [(1-p-L): h, h']: [(q): h]; [(q'): h'];\end{matrix} \beta z^{\mu}, \beta' t^{\mu'}\right]\}(x, y) =$$

$$I^{b, b', c, c', 1}\{z^{\sigma-1} t^{\sigma'-1} F^{E+1: G, G'}_{P: Q, Q'}$$

$$\left[\begin{matrix}[(1-e-L): h, h'], [-L: h, h'] : [(g): h]; [(g'): h'];\\ [(1-p-L): h, h']: [(q): h]; [(q'): h'];\end{matrix} \beta z^{\mu}, \beta' t^{\mu'}\right]\}(x, y) =$$

$$x^{\sigma-1} y^{\sigma'-1} \sum_{r, s=0}^{hr+h's \le L} \frac{(-L)_{hr+h's} \prod_{j=1}^{E} (1-e_j-L)_{hr+h's} \prod_{j=1}^{G} (g_j)_{hr} \prod_{j=1}^{G'} (g'_j)_{h's}}{\prod_{j=1}^{P} (1-p_j-L)_{hr+h's} \prod_{j=1}^{Q} (q_j)_{hr} \prod_{j=1}^{Q'} (q'_j)_{h's}}$$

$$\times \frac{\Gamma(\sigma+b+\mu r)\,\Gamma(\sigma'+b'+\mu' s)}{\Gamma(\sigma+b+c+\mu r)\,\Gamma(\sigma'+b'+c'+\mu' s)} \frac{(\beta x^{\mu})^r}{r_1!} \frac{(\beta' y^{\mu'})^s}{s!}$$

$$= x^{\sigma-1} y^{\sigma'-1} \frac{\Gamma(\sigma+b)\,\Gamma(\sigma'+b')}{\Gamma(\sigma+b+c)\,\Gamma(\sigma'+b'+c')} F^{E+1: G+1, G'+1}_{P: Q+1, Q'+1}$$

$$\left[\begin{matrix}[(1-e-L): h, h'], [-L: h, h'] : [(g): h], [\sigma+b: \mu]; [(g'): h'], [\sigma'+b': \mu'];\\ [(1-p-L): h, h']: [(q): h], [\sigma+c+b: \mu]; [(q'): h'], [\sigma'+b'+c': \mu'];\end{matrix} \beta z^{\mu}, \beta' t^{\mu'}\right] \tag{4.5}$$

All conditions of (4.4) are included.

Similarly, the following G-function may be computed in the limiting form

$$\lim_{\alpha \to 0} G^{0,0:1,2;1,2}_{0,1:2,2;2,2}\left[\begin{matrix}(\dots\dots\dots): (1-b, \sigma-b+\mu r); (1-b', \sigma'-b'+\mu' s);\\ (1-b-b'): (0, \sigma-b-c+\mu r); (0, \sigma'-b'-c'-\mu' s);\end{matrix} \alpha/x,\ \alpha/y\right]$$

$$= (-1)^{c+c'} \frac{\Gamma(b)\,\Gamma(b')}{\Gamma(b+b')} \frac{\Gamma(\sigma-b-c+\mu r)\,\Gamma(\sigma'-b'-c+\mu' s)}{\Gamma(\sigma-b+\mu r)\,\Gamma(\sigma'-b'+\mu' s)} \tag{4.6}$$

provided that Re $(b) > 0$, Re $(b') > 0$, Re $(c) > 0$, Re$(c') > 0$, Re $(\sigma) >$ Re$(b + c)$ and Re $(\sigma') >$ Re$(b' + c')$ taking μ and μ' are positive real $\forall\, r, s = 0, 1, 2, \dots$.

Therefore, from (1.13), (4.2) and (4.6), we get another transformation formula

$$\lim_{\alpha \to 0} J^{b, b', c, c', f}\{z^{\sigma-1} t^{\sigma'-1} F^{E+1: G, G'}_{P: Q, Q'}$$

$$\left[\begin{matrix}[(1-e-L):h,h'],[-L:h,h']:[(g):h];[(g'):h']; \\ [(1-p-L):h,h']:[(q):h];[(q'):h'];\end{matrix}\beta z^{\mu},\beta' t^{\mu'}\right]\}(x,y)=$$

$$J^{b,b',c,c',1}\{z^{\sigma-1}t^{\sigma'-1}F^{E+1:G,G'}_{P:Q,Q'}$$

$$\left[\begin{matrix}[(1-e-L):h,h'],[-L:h,h']:[(g):h];[(g'):h']; \\ [(1-p-L):h,h']:[(q):h];[(q'):h'];\end{matrix}\beta z^{\mu},\beta' t^{\mu'}\right]\}(x,y)=$$

$$x^{\sigma-1}y^{\sigma'-1}(-1)^{c+c'}\sum_{r,s=0}^{hr+h's\le L}\frac{(-L)_{hr+h's}\prod_{j=1}^{E}(1-e_j-L)_{hr+h's}\prod_{j=1}^{G}(g_j)_{hr}\prod_{j=1}^{G'}(g'_j)_{h's}}{\prod_{j=1}^{P}(1-p_j-L)_{hr+h's}\prod_{j=1}^{Q}(q_j)_{hr}\prod_{j=1}^{Q'}(q'_j)_{h's}}$$

$$\times\frac{\Gamma(\sigma-b-c+\mu r)\,\Gamma(\sigma'-b'-c'+\mu' s)}{\Gamma(\sigma-b+\mu r)\,\Gamma(\sigma'-b'+\mu' s)}\frac{(\beta x^{\mu})^r}{r_1!}\frac{(\beta' y^{\mu'})^s}{s!}$$

$$=x^{\sigma-1}y^{\sigma'-1}(-1)^{c+c'}\frac{\Gamma(\sigma-b-c)\,\Gamma(\sigma'-b'-c')}{\Gamma(\sigma-b)\,\Gamma(\sigma'-b')}F^{E+1:G+1,G'+1}_{P:Q+1,Q'+1}$$

$$\left[\begin{matrix}[(1-e-L):h,h'],[-L:h,h']:[(g):h],[\sigma-b-c:\mu];[(g'):h'],[\sigma'-b'-c':\mu']; \\ [(1-p-L):h,h']:[(q):h],[\sigma-b:\mu];[(q'):h'],[\sigma'-b':\mu'];\end{matrix}\beta z^{\mu},\beta' t^{\mu'}\right] \tag{4.7}$$

All conditions of (4.6) are included.

Similarly, on specializing the coefficients A [L; r, s], (L, r.s $\in N_0 = \{0, 1, 2, ...\}$ in (3.3) and (3.4) various other results for orthogonal and non-orthogonal functions of two variables occuring in many physical sciences may be obtained. Again, on making some specializations in the general coefficients A[L; $r_1,..., r_k$], (L, $r_i \in N_0 = \{0, 1, 2,...\}$, $\forall$ i = 1, 2, .., k) appearing in (3.1) and (3.2) various analytic results for orthogonal and non-orthogonal functions of several variables occuring in many physical sciences may be evaluated as particular cases.

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