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Semi Invariant Submanifolds of a Para Kenmotsu Manifold with Constant ϕ Holomorphic Sectional Curvature

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Abstract

In this paper we have studied submanifolds para Kenmotsu manifold to be semi-invariant submanifold. In particular case when it is a para Kenmotsu space form of constant ϕ holomorphic sectional curvature.

Keywords : Semi-invariant submanifold, Para Kenmotsu manifold, ϕ–holomorphic sectional curvature.

1. Introduction

The semi-invariant submanifolds of Kenmotsu manifold have been introduced and studied Kenmotsu and Kobayashi. Dube have defined para Kenmotsu manifold and studied different curvature tensors on para Kenmotsu manifold. He also studied semi-invariant submanifolds of an almost para r-contact manifold. Bhatt and Dube have studied semi-invariant submanifolds of r-Kenmotsu and para Kenmotsu manifold. Bhatt and others investigated non-invariant hypersurfaces of Kenmotsu manifold and Dube on curvatures on para Kenmotsu manifold.

Let $\overline{M}$ be an $(2m + 1)$ dimensional almost para contact metric manifold with the almost para contact metric structure (ϕ, ξ, η, g), where a tensor ϕ of type $(1, 1)$, a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the following

$$\phi^2 X = X - \eta(X)\,\xi, \qquad g(X, \xi) = \eta(X) \tag{1.1a}$$

$$\eta(\xi) = 1, \qquad \phi(\xi) = 0 \qquad \eta \circ \phi = 0 \tag{1.1b}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y) \tag{1.2}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the tangent bundle of $\overline{M}$.

An almost para contact metric structure (ϕ, ξ, η, g) on $\overline{M}$ is called para Kenmotsu if and only if

$$(\overline{\nabla}_X \phi)\, Y = -\, g\,(\phi X, Y)\,\xi - \eta(Y)\,\phi X \tag{1.3}$$

$$\overline{\nabla}_X \xi = X - \eta\,(X)\,\xi. \tag{1.4}$$

For all $X, Y \in T\overline{M}$, where $\overline{\nabla}$ denotes Levi Civita connection on $\overline{M}$.

Now, let M be a submanifold immersed in $\overline{M}$ and suppose that structure vector field ξ of $\overline{M}$ is tangent to M. We denote by TM and $T^{\perp}M$, the tangent bundle and normal bundle of M, then the submanifold M of M is called a semi invariant submanifold if it is endowed with a pair of distribution $(D, D^{\perp})$ satisfying the following condition :

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$ where $\{\xi\}$ denotes the distribution spanned.

(ii) The distribution D is invariant by ϕ that is $\phi(D_x) = D_x$, for each $x \in M$.

(iii) The distribution $D^{\perp}$ is anti invariant by ϕ that is $\phi\,(D_x^{\perp}) \subset T_x\, M^{\perp}$, for each $x \in M$.

In this paper we discuss some results concerned with the maximal invariant subspace $D_x^{\perp}$ of T_x M where M is the submanifold of para Kenmotsu manifold $\overline{M}$. We suppose that M is the submanifold of para Kenmotsu manifold $\overline{M}$ such that the structure vector field ξ of $\overline{M}$ is tangent to M. We by H the orthogonal complementary distribution of $\{\xi\}$ in M, that is $T_xM = H_x \oplus \xi_x$, for each $x \in M$. Now Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1.5}$$

$$\overline{\nabla}_X V = -\,A_v\, X + \nabla_x^{\perp} V \tag{1.6}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$, where ∇ is the Riemannian connection determined by induced metric g on M, $\nabla^{\perp}$ the metric connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_V X, Y) = g(h(X, Y), V). \tag{1.7}$$

Let $\overline{R}$ (resp. R) be the curvature tensor of $\overline{M}$ (resp. M) then equations of Gauss and Codazis are given by

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(Y, W), h(X, Z) \tag{1.8}$$

$$(R(X, Y) Z)^{\perp} = \nabla_X^{\perp} h(Y, Z) - (\nabla_X^{\perp} h)(X, Z) \tag{1.9}$$

where Z, W are vector fields tangent to M the left hand side of (1.9) denotes the normal component of (R(X, Y) Z) and

$$\nabla_X h(Y, Z) = \nabla_X^{\perp} h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2. Para Kenmotsu manifold with constant ϕ holomorphic sectional curvature

Let $\overline{R}$ be the curvature tensor of the connection $\overline{\nabla}$, then a para Kenmotsu manifold $\overline{M}$ is of constant ϕ holomorphic sectional curvature, C if

$$\overline{R}(X, Y) Z = \left(\frac{C-3}{4}\right)[g(Y, Z) X - g(X, Z) Y] + \left(\frac{C+1}{4}\right)[\eta(X)\eta(Z) Y$$
$$- \eta(Y)\eta(Z) X + \eta(Y) g(X, Z)\xi - \eta(X) g(Y, Z)\xi + g(X, \phi Z)\phi Y$$
$$- g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z] \tag{2.1}$$

where X, Y, Z $\in T\overline{M}(C)$. Using Gauss equation as

$$g(R(X, Y) Z, W) = g(\overline{R}(X, Y) Z, W) + g(h(X, W), h(Y, Z)) - g(h(Y, W), h(X, Z))$$

From equation (2.1), we get

$$g(R(X, Y) Z, W) = \left(\frac{C-3}{4}\right)[g(Y, Z) X - g(X, Z) Y] + \left(\frac{C+1}{4}\right)[\eta(X)\eta(Z) Y$$
$$- \eta(Y)\eta(Z) X + \eta(Y) g(X, Z)\xi - \eta(X) g(Y, Z)\xi + g(X, \phi Z)\phi Y$$

$$- g(Y, \phi Z)\, \phi X + 2g(X, \phi Y)\, \phi Z] + g(h(X, W), h(Y, Z))$$

$$- g(h(Y, W), h(X, Z)) \tag{2.2}$$

where X, Y, Z, W ∈ TM.

The sectional curvature of para Kenmotsu space form C is

$\overline{R} - (X, Y) = - \overline{R}(X, Y, X, Y)$ where X, Y are the orthonormal vectors

$$= \left(\frac{C-3}{4}\right) - \left(\frac{C+1}{4}\right) [\eta^2(X) + \eta^2(Y) - 3g(Y, \phi X)]$$

$$+ g(h(X, X), h(Y, Y)) - \|h(X, Y)\|^2 \tag{2.3}$$

and the holomorphic sectional curvature H of para Kenmotsu manifold

$$H(X) = \overline{R}_{\overline{M}}(X, \phi X)$$

$$= \left(\frac{C-3}{4}\right) - \left(\frac{C+1}{4}\right) [7\eta^2(X) - 3 - 3\eta^4(X)]$$

$$+ g(h(X, X), h(\phi X, \phi X)) - | h(X, \phi X) |^2. \tag{2.4}$$

From (1.3), we have

$$(\overline{\nabla}_X \phi)(Y, Z) = - g((\nabla_X \phi) Y, Z) = \overline{R}(X, Y, Z, \xi)$$

and hence

$$\overline{R}(X, Y, Z, \xi) = \eta(R(X, Y, Z)) = g(X, Z)\, \eta(Y) - g(Y, Z)\, \eta(X). \tag{2.5}$$

This gives

$$R(X, Y)\, \xi = \eta(X)\, Y - \eta(Y)\, X.$$

On contracting (1.9), we get Ricci tensor as

$$\mathrm{Ric}(Y, Z) = \left\{\frac{(C+1)(n+1)}{4}\right\} g(\phi Y, \phi Z) - (n-1)\, g(Y, Z)$$

and $$r = \left(\frac{n-1}{4}\right) [(C+1)(n+1) - 4n]$$

replacing Y by ξ in (2.5), we get

$$\overline{R}(X, \xi, Z, \xi) = g(X, Z)\, \eta(\xi) - g(\xi, Z)\, \eta(X) = g(X, \xi) - \eta(X)\, \eta(Z).$$

Corollary 2.1 : A para Kenmotsu manifold $\overline{M}$ of constant ϕ holomorphic curvature can not be flat.

Proof. The corollary is obvious.

3. Submanifolds of Para Kenmotsu Manifold

Let M be a submanifold of a para Kenmotsu manifold. Since ξ is tangent to M then we have from Gauss formula

$$\overline{\nabla}_X \xi = \nabla_X \xi + h(X, \xi)$$

this due to (1.4) gives

$$\overline{\nabla}_X \xi = X - \eta(X)\, \xi$$

and $h(X, \xi) = 0$.

Lemma 3.1 : Let M be a submanifold tangent to ξ of a para Kenmotsu manifold $\overline{M}$ and D_X the maximal invariant subspace of H_X. Suppose Dim D_X = constant. Then the invariant distribution D is integrable.

Proof. Now we have

$$\begin{aligned} g([X, Y], \xi) &= g(\overline{\nabla}_X Y - \overline{\nabla}_Y X, \xi) \\ &= g(X, \overline{\nabla}_Y \xi) - g(Y, \overline{\nabla}_X \xi) \\ &= g(X, Y - \eta(Y)\, \xi) - Y, X - \eta(X)\, \xi) \\ &= \eta(X)\, \eta(Y) - \eta(Y)\, \eta(X) \\ &= 0 \end{aligned} \tag{3.1}$$

for any $X, Y \in D$.

If we take unit vector $X \in D$ and $Y = \phi X$, then from (3.1), we get

$$g([X, Y], \xi) = 0$$

hence invariant distribution is integrable.

Theorem 3.1 : Let M be a submanifold tangent to ξ-isometrically immersed in a para Kenmotsu manifold $\overline{M}$ and D_X the maximal invariant subspace of H_X. Suppose Dim D_X = constant. Then the distribution $D \oplus \{\xi\}$ is completely integrable if and only if the second fundamental form h of M satisfies

$$h(X, \phi Y) = h(\phi X, Y) \text{ for } X, Y \in D.$$

Proof. For X, Y $\in$ D, we have

$$\begin{aligned}\phi([X, Y]) &= \phi(\nabla_X Y - \nabla_Y X)\\ &= \phi(\overline{\nabla}_X Y - h(X, Y) - \overline{\nabla}_X Y - h(X, Y))\\ &= \overline{\nabla}_X \phi Y + \eta(Y)\, \phi X + g(X, \phi Y)\, \xi - h(X, \phi Y) + \phi h(X, Y)\\ &\quad -\overline{\nabla}_Y \phi X - \eta(X)\, \phi Y - g(Y, \phi X)\, \xi + h\,(\phi X, Y) - \phi h\,(Y, Z)\\ &= h(\phi X, Y) - h(X, \phi Y) + \phi h\,(X, Y) - \phi h(Y, X)\\ &= h(\phi X, Y) - h(X, \phi Y)\end{aligned}$$

where ϕ [X, Y] shows the component of $\nabla_X Y$ from the orthogonal complementary distribution of $D \oplus \{\xi\}$ in M. Since h is symmetrical morphism of vector bundles and hence from above we have

$$h(\phi X, Y) - h(X, \phi Y) = \phi\,([X, Y])$$

thus, we get $[X, Y] \in D \oplus \{\xi\}$ if and only if $\phi([X, Y]) = 0$, that is

$$h(\phi X, Y) = h(X, \phi Y)$$

conversely using (1.2), (1.3), (1.4) along with Gauss and Weingarten equations, we get $[X, Y] \in D \oplus \{\xi\}$ for each $X \in D$, which proves our assertions.

Theorem 3.2 : Let M be a submanifold tangent to ξ of a para Kenmotsu manifold $\overline{M}$ and $D_X^{\perp}$ the maximal anti-invariant subspace of H_X. Suppose Dim $D_X^{\perp}$ = constant. Then the anti-invariant distribution $D^{\perp}$ is always integrable.

Proof. For Z, W $\in D^{\perp}$, we have

$$\phi([Z, W]) = \phi(\nabla_Z W - \nabla_W Z)$$

$$= \overline{\nabla}_Z \phi W - (\overline{\nabla}_Z \phi) W - \phi h(Z, W) - (\overline{\nabla}_W \phi Z) - (\overline{\nabla}_W \phi) Z - \phi h(W, Z)$$

$$= - A_{\phi Z} W + \nabla_Z^{\perp} \phi W + \eta(W) \phi Z + g(Z, \phi W) \xi - \phi h (Z, W)$$

$$= (A_{\phi W} Z - A_{\phi Z} W) + \nabla_Z^{\perp} \phi W - \nabla_W^{\perp} \phi Z .$$

Since $A_{\phi W}Z - A_{\phi Z} W$ is tangential to M and $\nabla_Z^{\perp} \phi W - \nabla_W^{\perp} \phi Z$ is normal to M then it follows that $[Z, W] \in D^{\perp}$, if and only if

$$A_{\phi W}Z = A_{\phi Z} W \quad \text{for any } Z, W \in D^{\perp} \tag{3.2}$$

and

$$g([Z, W], \xi) = 0. \tag{3.3}$$

conversely using (1.2), (1.3), (1.4) and (1.6), we have

$$g(A_{\phi Z} W, X) = g (h(W, X), \phi Z) = g(\overline{\nabla}_X W, \phi Z)$$

$$= - g(\phi \overline{\nabla}_X W, Z) = - g(\overline{\nabla}_X \phi W, Z)$$

$$= g(A_{\phi W} X, Z) = g(A_{\phi X} W, X)$$

for any $Z, W \in D^{\perp}$ and $X \in TM$. That is $A_{\phi Z} W = A_{\phi W}Z$ holds.

Hence $D^{\perp}$ is always integrable.

Finally using (1.4) and taking account that $D^{\perp}$ is anti-invariant distribution, we have

$$g([Z, W], \xi) = g(\overline{\nabla}_Z W - \overline{\nabla}_W Z, \xi)$$

$$= g (Z, \overline{\nabla}_W \xi) - g (W, \overline{\nabla}_Z \xi)$$

$$= g(\phi Z, \phi W) - g(\phi W, \phi Z)$$

$$= 0, \quad \text{for any } Z, W \in D^{\perp}.$$

Therefore (3.2) and (3.3) holds. This completes the proof of the theorem.

4. Semi-invariant Submanifolds of a Kenmotsu Manifold with constant ϕ Holomorphic Sectional Curvature C

Here we consider a semi-invariant submanifold as the particular case when para Kenmotsu manifold is a para Kenmotsu space form $\overline{M}(C)$ of constant ϕ holomorphic sectional curvature C.

Theorem 4.1 : Let M be a submanifold tangent to ξ of para Kenmotsu space form $\overline{M}(C)$ with $C \neq -1$ then M is a semi-invariant if and only if the maximal invariant subspaces $D_X = H_X \cap \phi(H_X)$, $x \in M$ define a non-trivial differentiable distribution D on M such that

$$\overline{R}(D, D; D^{\perp}, D) = 0 \tag{4.1}$$

where $D^{\perp}$ denotes the orthogonal complementary distribution to $D \oplus \{\xi\}$ in M and $\overline{R}$ denotes the curvature tensor of Kenmotsu space form $\overline{M}(C)$.

Proof. The curvature tensor of a para Kenmotsu space form $\overline{M}(C)$ is given by (2.1) for any vector field X, Y, Z tangent to $\overline{M}(C)$. If M is a semi-invariant submanifold of $\overline{M}(C)$ from (2.1), we get $\overline{R}(X, Y, Z, W) = g(\overline{R}(X, Y) Z, W) = 0$ for any $X, Y \in D$ and $Z, W \in D^{\perp}$ hence (4.1) holds.

Conversely suppose that the maximal invariant subspace D of H defines a non-trivial differentiable distribution such that (4.1) holds, then using (2.1) and (4.1) we get

$$\overline{R}(X, \phi X, Z, W) = -\frac{(C+1)}{2} g(X, X)\, g(\phi Z, W) = 0, \text{ for all } X \in D \text{ and } Z, W \in D^{\perp}$$

since $C \neq -1$, we get $g(\phi Z, W) = 0$ for all $Z, W \in D^{\perp}$. So, we have

$$\phi D^{\perp} \perp D^{\perp}. \tag{4.2}$$

Since D is an invariant distribution, using (1.2), we have

$$g(X, \phi Z) = -g(\phi X, Z) = 0 \text{ for any } X \in D \text{ and } Z \in D^{\perp}.$$

We get

$$\phi D^{\perp} \perp D. \tag{4.3}$$

Finally $g(\xi, \phi Z) = 0$ for any $Z \in D^{\perp}$ and taking account of (4.2) and (4.3) we get $\phi D^{\perp} \subset TM^{\perp}$. That is $D^{\perp}$ is an anti-invariant distribution. Thus M is a semi-invariant submanifold. Hence the theorem is proved.

Theorem 4.2 : Let M be a submanifold tangent to ξ of para Kenmotsu space form $\overline{M}(C)$ with $C \neq 0$ then M is a semi-invariant if and only if the maximal anti-invariant subspaces $D_X^{\perp} \subset H_X$, $x \in M$ define a non-trivial differentiable distribution $D^{\perp}$ on M such that

$$\overline{R}\,(D, \phi D; \mu, D) = 0 \tag{4.4}$$

where $D^{\perp}$ denotes the orthogonal complementary distribution to $D \oplus \{\xi\}$ in M and μ denotes the distribution for which μ_X is orthogonal complement of $\phi D^{\perp}$ in $T_X M^{\perp}$.

Proof. Let M be a semi-invariant submanifold of $\overline{M}(C)$ then using the definition and (1.2) we obtain from (2.1) $\overline{R}(X, \phi Y; L, Z) = 0$ for any $X, Y, Z \in D$ and $L \in \mu$, that is (4.4) is satisfied.

Conversely, if the maximal anti-invariant subspaces $D_X^{\perp} \subset H_X$; $x \in M$ define a non-trivial differentiable distribution $D^{\perp}$ on M such that (4.4) holds then using (1.2) and (2.1), we get

$$\overline{R}(X, \phi Y; L, Z) = Cg(X, X)\, g(\phi X, L) = 0 \tag{4.5}$$

for any $X \in D$ and $L \in \mu$.

Since $X \notin D$ then from (4.5) it follows

$$\phi D \perp \mu \tag{4.6}$$

also using (1.2) and considering $D^{\perp}$ is an anti-invariant distribution, we get

$$\phi D \perp \xi \quad \text{and} \quad \phi D \perp \phi D^{\perp}. \tag{4.7}$$

Since by hypothesis, we have $D \perp \{\xi\}$ and from (1.1) and (1.2), we have

$$\phi\, D \perp \{\xi\} \tag{4.8}$$

then (4.6), (4.7) and (4.8) shows that $\phi D \subset TM$ and $\phi D = D$, that is D is a invariant distribution and M is semi-invariant submanifold. This proves our assertions.

References

1. Kenmotsu, K. : A class of almost contact Riemannian manifold, Tohoku Math. J., 24, (1972), 93-103.

2. Bhatt, L. and Dube, K. K. : Semi-invariant submanifolds of a r-Kenmotsu manifold, Acta Ciencia Indica, 31 (1), (2003), 167-172.

3. Dube, K. K. : Study of curvatures on para Kenmotsu manifold, The Nepali Math. Sci. Report, 15, (1996), 83-88.

4. Dube, K. K. : Semi-invariant submanifolds of an almost para r-contact manifolds, Acta Ciencia Indica, 28 (3), (2002), 431-434.

5. Kobayashi, M. : Semi-invariant submanifolds of a certain class of almost contact manifolds, Tensor, N. S., 43, (1986), 28-36.