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Deformation of a Sasakian Manifold

J. P. Srivastava and Sanjeev Kumar

Department Of Mathematics
University of Jammu, Jammu
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Abstract

A contact structure on a Brieskorn manifold is studied by K. Abe [1] and S. Saki and C. S. Hsu[2], K. Abe [1] proved that there exists Sasakian structures on Brieskorn manifold. We apply these deformation on the standard Sasakian spheres and show that Brieskorn manifold have almost contact metric manifold so that Brieskorn manifolds are invariant submanifolds of these deformed spheres.

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1. Introduction

Let (M, ϕ, ξ, η, g) be a Sasakian manifold which admits an infinitesimal automorphism μ. Under further conditions of μ, we will show that we can construct a deformation of a Sasakian structure with respect to μ.

When $\mu = \alpha.\xi$ for some real number α such that $1 + \alpha > 0$, then the deformation of a Sasakian manifold with respect to μ is called 0 transformation where 0 is a distribution defined by $\eta = 0$.

Let we denote it by (M, $\tilde{\phi}$, $\tilde{\xi}$, $\tilde{\eta}$, $\tilde{g}$) the Sasakian manifold deformed with respect to μ. We will find D is also defined by $\tilde{\eta} = 0$. There are other many deformation which are not D-deformation.

Let m be a (2n + 1) dimensional almost contact metric manifold. Let (M, ϕ, ξ, η, g) is called a Sasakian manifold when the following relation hold for the structure tensor field, η is a 1-form, ξ a vector field, ϕ is a (1, 1) tensor field and g is a Riemannian metric on M such that

$$\eta(\xi) = 1 \tag{1}$$

$$\tilde{\phi}(X) = -X + \eta(X)\,\xi, \tag{2}$$

$$d\eta\,(X, Y) = g\,(\phi(X), Y), \tag{3}$$

$$g\,(\xi, \xi) = 1, \tag{4}$$

$$N(X, Y) = 0, \tag{5}$$

where X and Y are arbitrary vector fields over M and N(X, Y) is a vector field defined by

$$N(X, Y) = [X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] - (X.\eta(Y) - Y.\eta(X))\,\xi. \tag{6}$$

This tensor field N is called Nijenhuis almost contact structure.

2. Theorem 2.1. Let (M, ϕ, ξ, η, g) be a Sasakian manifold and μ be a vector field over M which satisfies the next three conditions

$$£_{\mu}\, g = 0 \tag{7}$$

$$[\mu, \xi] = 0, \tag{8}$$

$$1 + \eta\,(\mu) > 0 \tag{9}$$

where $£_{\mu}$ is the Lie differentiation with respect to μ.

Now structure vector field denoted by $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ are defined by the relation

$$\tilde{\phi}(X) = \phi\,(X - \tilde{\eta}\,(X)\,\tilde{\xi}); \qquad \tilde{\eta} = (1 + \eta(\mu))^{-1}\,\eta, \qquad \tilde{\xi} = \xi + \mu$$

and $\quad \tilde{g}\,(X, Y) = (1 + \eta(\mu))^{-1}\, g\,(X - \tilde{\eta}\,(X)\,\tilde{\xi}, Y - \tilde{\eta}\,(Y)\,\tilde{\xi}) + \tilde{\eta}\,(X)\tilde{\eta}\,(Y)$

where X and Y are vector fields over M then, we have

Theorem 2.2. $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{\phi})$ is also a Sasakian manifold.

Proof. Let D be the distribution defined by $\eta = 0$. By the definition of $\tilde{\eta}$. We see also that distribution D is also defined by $\tilde{\eta} = 0$, we see that

$$\tilde{\eta}\,(\tilde{\xi}) = (1 + \eta(\mu))^{-1}.\ \eta(\xi + \mu)$$

$$= (1+\eta(\mu))^{-1}\,\eta(\xi) + (1+\eta(\mu))^{-1}.\,\eta(\mu)$$

$$= \eta(\xi) = 1$$

and

$$\tilde{g}(\tilde{\xi}, \tilde{\xi}) = g(\xi, \xi) = 1.$$

For the vector field X which belong to D, we have

$$\tilde{\eta}(X) = (1+\eta(\mu))^{-1}\,\eta(X) = 0,$$

$$\tilde{\eta}\phi(X) = 0 \text{ and}$$

$$\phi^2(X) = -X,$$

hence

$$\tilde{\phi}^2(X) = \tilde{\phi}\,(\phi\,(X - \tilde{\eta}(X)\,\tilde{\xi}))$$

$$= \tilde{\phi}\,(\phi X)$$

$$= \phi\,(\phi\,(X) - \tilde{\eta}(\phi(X))\,\tilde{\xi}$$

$$= \phi^2 X$$
$$= -X$$

because $\tilde{\phi}(\tilde{\xi}) = 0$.

The above equation holds for any vector field which belongs to D, we have

$$\tilde{\phi}^2(X) = -X + \tilde{\eta}(X)\,\tilde{\xi}$$

for any vector field X on M.

Next, we will prove that the torsion tensor field $\tilde{N}$ with respect to the almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ vanishes on M. As (ϕ, ξ, η) structure tensor fields of a Sasakian manifold, the tensor field N with respect to the almost contact structure (ϕ, ξ, η) vanishes. Sasaki and Hsu [2] has defined four torsion tensor fields as

$$N_1(X, Y) = [\xi, \phi X] - \phi\,[\xi, X] = 0 \quad (10)$$

$$N_2(X, Y) = -\eta\,([\phi X, Y]) - \eta([X, \phi Y]) + (\phi X)\,\eta(Y) - (\phi Y)\,\eta(X) = 0 \quad (11)$$

$$N_3(X, Y) = \xi.\eta(X) - \eta([\xi, X]) = 0 \tag{12}$$

For the vector fields X and Y which belong D, we get from (11)

$$N_2(X, Y) = -\eta\ ([\phi X, Y]) - \eta([X, \phi Y]) = 0$$

because $\phi(X) = \tilde{\phi}\ (X)$ and $\tilde{\eta}(X) = 0$, for the vector field X and Y which belong to D. Then, we have

$$\begin{aligned}\tilde{N}\ (X, Y) &= [X, Y] + \tilde{\phi}\ ([\phi X, \tilde{Y}]) + \tilde{\phi}\ ([X, \tilde{\phi}\ (Y)]) - [\phi(X), \phi(Y)] \\ &= N(X, Y) - \phi\ (\tilde{\eta}\ [\phi X, Y])\ \tilde{\xi} + \tilde{\eta}\ ([X, \phi Y]) \\ &= N(X, Y) + (1 + \eta(\mu))^{-1}.N_2(X, Y)\ \phi(\tilde{\xi}\ ) \\ &= 0\end{aligned} \tag{13}$$

where X and Y belong to D.

Since $£_\xi\phi = 0$ and $£_\xi\eta = 0$, clearly, it is easy to show that $£_{\tilde{\xi}}\ \phi = 0$ and $£_{\tilde{\xi}}\ \eta = 0$. Hence

$$£_{\tilde{\xi}}\ \eta = (1 + \eta(\mu))^{-1}.\ £_{\tilde{\xi}}\ \eta = 0$$

and

$$\begin{aligned}(£_{\tilde{\xi}}\ \tilde{\phi})(X) &= £_{\tilde{\xi}}\ ((\phi(X - \tilde{\eta}\ (X)\tilde{\xi}\ )) - \tilde{\phi}([\xi, X]) \\ &= \phi\ [\ \tilde{\xi}, X - \tilde{\eta}(X)\ \tilde{\xi}\ ] - \phi\ ([\xi, X] - \tilde{\eta}([\xi, X])\tilde{\xi} \\ &= 0\end{aligned}$$

From these two equations, we find for a vector field X on M, we have

$$\tilde{N}\ (\tilde{\xi}, X) = [\tilde{\xi}, X] + \tilde{\phi}\ [\tilde{\xi}, \phi X] = 0 \tag{14}$$

Thus, we have

Theorem 2.3. $\tilde{N}\ (\tilde{\xi}, X) = 0.$

Thus we have (13) and (14) shows that the torsion tensor field $\tilde{N}$ with respect to $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}\ )$ vanishes on M.

Since (ϕ, ξ, η, g) is structure tensor field of a Sasakian manifold. The next equation hold for any X and Y vector field

$$2g(\phi X, Y) = 2d\eta\,(X, Y) = X.\eta(Y) - Y.\eta(X) - \eta([X, Y]).$$

Hence for the vector fields X and Y belonging to D. We have

$$\begin{aligned}2d\,\tilde{\eta}\,(X, Y) &= X\tilde{\eta}\,(Y) - Y.\tilde{\eta}\,(X) - \tilde{\eta}\,([X, Y])\\ &= -\,\tilde{\eta}\,([X, Y])\\ &= -\,(1 + \eta(\mu))^{-1}.\eta\,([X, Y])\\ &= 2(1 + \eta(\mu))^{-1}\,g\,(\phi X, Y)\\ &= 2(1 + \eta(\mu))^{-1}\,g\,(\tilde{\phi}\,X, Y)\end{aligned}$$

$$2d\,\tilde{\eta}\,(X, Y) = 2\,\tilde{g}\,(\tilde{\phi}\,X, Y) \tag{15}$$

because of the definition of $\tilde{\eta}$ and $\tilde{g}$ we have $\phi(X) = \tilde{\phi}\,(X)$ and $\eta(X) = \tilde{\eta}\,(X) = 0$.

Further more, since $£_{\tilde{\xi}}\,\tilde{\eta} = 0$, the equation

$$2d\,\tilde{\eta}\,(\tilde{\xi}, X) = -\,\tilde{\eta}\,([\tilde{\xi}, X]) = 0 \tag{16}$$

hold for any vector field X belonging to D. By (15) and (16), it was proved that the relation (1), (2), (3), (4) and (5) for $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ hold good.

So we can now introduced the new Sasakian structure $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on M.

Thus $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is called a Sasakian manifold deformed with μ. This can be explain by an example :

Example 2.4. Let $(S^{2n+1}, \phi, \xi, \eta, g)$ be the unit sphere with the standard Sasakian structure and be imbedded in Euclidean space E^{2n+2} with $(x^1, y^1, \ldots\ldots, x^{n+1}, y^{n+1})$.

At the point P of S^{2n+1}, we have

$$\xi_\beta = \sum_{j=1}^{n+1}\left(x^j \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial x^j}\right)$$

$$\eta_P = \sum_{j=1}^{n+1} \left(x^j \,(x^j\, dy^j - y^j\, dx^j\,) \right)$$

where $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^j}$ are vector fields over E^{2n+2} respectively and $(x^1, y^1, \ldots\ldots\ldots,$ $x^{n+1}, y^{n+1})$ is the coordinate of P.

When we introduce a complex structure J on E^{2n+2} as $Z^j = x^j + i\, y^j$ for j = 1, ..., n + 1, then ϕ is restriction of J on D which is orthogonal complement of R.ξ in the tangent space at each point on S^{2n+1} and 0 on R.ξ. The Riemannian metric g on S^{2n+1} is induced by the Riemannina metric of E^{2n+2}. When we put

$$\mu = \sum_{j=1}^{n+1} r_j \,(x^j\, \partial y^j - y^j\, \partial x^j)$$

where $(r_1, r_2, \ldots., r_{n+1})$ is a (n + 1) tuple at real numbers such that

$$1 + \sum_{j=1}^{n+1} r_j \,((x^j)^2 + (y^j)^2\,) > 0 \text{ on } S^{2n+1},$$

then μ satisfies the condition of theorem [1] and the new trajectories of $\tilde{\xi}$ with the initial condition P is given by

$$x^i\,(t) = x^i \cos\,(1 + r_i)\, t - y^i \sin\,(1 + r_i\,)t$$

$$y^j\,(t) = x^j \sin\,(1 + r_i)\, t + y^j \cos\,(1 + r_i\,)t$$

for j = 1, ..., n + 1.

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