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Some Properties of Curvature Tensors on Quarter-Symmetric Metric Connection in a P-Sasakian Manifold

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Abstract

The object of the present paper is to study properties of concircular and conharmonic curvature tensor of a quarter symmetric metric connection in a P-Sasakian manifold.

Key Words : P-Sasakian manifold, concircular and conharmonic curvature tensor, connection.

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1. Introduction

An n-dimensional differentiable manifold M is called an almost para-contact manifold if it admits an almost para-contact structure (F, ξ, η) consisting of a (1, 1) tensor field F, a vector field ξ and a 1-form η satisfying

$$\overline{\overline{X}} = X - \eta(X)\,\xi, \tag{1.1}$$

$$\overline{X} = F(X) \tag{1.2}$$

$$\eta(\xi) = 1. \tag{1.3}$$

Let g be the compatible Riemannian metric with (F, ξ, η) that is

$$g(FX, FY) = g(X, Y) - \eta(X)\,\eta(Y) \tag{1.4}$$

or equivalently

$$g(X, FY) = g(FX, Y) \tag{1.5}$$

and

$$g(X, \xi) = \eta(X) \quad \text{for all } X, Y \in TM \tag{1.6}$$

Then M becomes almost para-contact Riemannian manifold equipped with an almost para-contact Riemannian structure (F, ξ, η, γ). An almost para-contact Riemannian manifold is called a P-Sasakian manifold if it satisfies

$$(\nabla_X F)\, Y = -\, g(X, Y)\, \xi - \eta(Y)\, X + 2\eta(X)\, \eta(Y)\, \xi, \qquad X, Y \in TM \tag{1.7}$$

where ∇ denote the covariant differentiation with respect to g. It follows that

$$(\nabla_X\, \xi) = \overline{X}, \tag{1.8}$$

$$(\nabla_X\, \eta) = (\nabla_Y\, \eta)\, X = g(X, \overline{Y}), \qquad X \in TM. \tag{1.9}$$

In an n-dimensional P-Sasakian manifold M, the curvature tensor R, the Ricci tensor S and the Ricci operator Q, satisfy

$$R(X, Y)\, \xi = \eta(X)\, Y - \eta(Y)\, X \tag{1.10}$$

$$R(\xi, X)\, Y = \eta(Y)\, X - g(X, Y)\, \xi \tag{1.11}$$

$$R(\xi, X)\, \xi = X - \eta(X)\, \xi \tag{1.12}$$

$$S(X, \xi) = -\,(n-1)\, \eta(X) \tag{1.13}$$

$$Q\xi = -\,(n-1)\, \xi \tag{1.14}$$

$$\eta\, (R(X, Y)\, U) = g(X, U)\, \eta(Y) - g(Y, U)\, \eta(X) \tag{1.15}$$

$$\eta\, (R(X, Y)\, \xi) = 0 \tag{1.16}$$

$$\eta\, (R\, (\xi, X)\, Y) = \eta(X)\, \eta(Y) - g(X, Y). \tag{1.17}$$

An almost para-contact Riemannian manifold M is said to be η-Einstein [2], if the Ricci operator Q satisfying

$$Q(X) = aX + b\, \eta(X)\, \xi$$

where a and b are smooth function on the manifold. In particular if b = 0, the manifold is an Einstein manifold.

Let (M, g) be an n-dimensional Riemannian manifold. Then the concircular curvature tensor V and the conharmonic curvature tensor L are given by

$$V(X, Y, Z) = R(X, Y, Z) - \frac{r}{n\,(n-1)}\,[g(Y, Z)\,X - g(X, Z)\,Y] \qquad (1.18)$$

$$L(X, Y, Z) = R(X, Y, Z) - \frac{1}{(n-2)}\,[Ric\,(Y, Z)\,X - Ric\,(X, Z)\,Y$$

$$+\, g\,(Y, Z)\,QX - g(X, Z)\,QY] \qquad (1.19)$$

for all X, Y $\in$ TM respectively, where r is the scalar curvature of M.

A linear connection $\tilde{\nabla}$ in a Riemannian manifold M is said to be a quarter symmetric connection if its torsion tensor T satisfies

$$T(X, Y) = \eta(Y)\,F(X) - \eta(X)\,F(Y) \qquad (1.20)$$

where η is a 1-form and F is a (1, 1) tensor field [4].

A linear connection $\tilde{\nabla}$ satisfying (1.20) and (1.21) is called a quarter symmetric metric connection [3].

2. Curvature Tensor

We consider a linear connection and be a Riemannian connection such that [6]

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X, Y) \qquad (2.1)$$

where U is a tensor of type (1, 2), and

$$T(X, Y) = \eta(Y)\,\overline{X} - \eta(X)\,\overline{Y} = U(X, Y) - U(Y, X). \qquad (2.2)$$

If a connection $\tilde{\nabla}$ is metric connection, i.e.

$$(\tilde{\nabla}_X\, g)\,(Y, Z) = 0 \qquad (2.3)$$

holds.

From (2.2), we have

$$'U(X, Y, Z) + {}'U(X, Z, Y) = 0 \qquad (2.4)$$

where $'U(X, Y, Z) = g(U(X, Y), Z)$.

Since

$$(\tilde{\nabla}_X g)(Y, Z) = 0 \Rightarrow g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}_X Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$\Rightarrow g(U(X, Y), Z) + g(Y, U(X, Z)) = 0$$

$$\Rightarrow 'U(X, Y, Z) + 'U(X, Z, Y) = 0$$

and also

$$U(X, Y) = \frac{1}{2}[T(X, Y) + T'(X, Y) + T'(Y, X)] \tag{2.5}$$

where $$g(T'(Y, X), Z) = g(T(Z, X), Y) \quad \text{see [3]} \tag{2.6}$$

Assume that the torsion tensor $T(X, Y)$ of the linear connection is of the form

$$T(X, Y) = \eta(Y)\,\overline{X} - \eta(X)\,\overline{Y}. \tag{2.7}$$

From (2.6) and (2.7), we have

$$T(X, Y) = \eta(X)\,Y - 'F(X, Y) \tag{2.8}$$

where $'F(X, Y) = g(\overline{X}, Y)$, η is a 1-form and ξ is the associated vector field.

From (2.5), (2.7) and (2.8), we get

$$U(X, Y) = \eta(Y)\,\overline{X} - 'F(X, Y)\,\xi. \tag{2.9}$$

From (2.1) and (2.9), we get

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\,\overline{X} - 'F(X, Y)\,\xi. \tag{2.10}$$

Hence a quarter symmetric metric connection $\tilde{\nabla}$ in a P-Sasakian manifold is given by (2.10).

If R and $\tilde{R}$ be the curvature tensors of the connection ∇ and $\tilde{\nabla}$ respectively. Then we have

$$R(X, Y)\,Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \tag{2.11}$$

$$\tilde{R}(X, Y) Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z. \tag{2.12}$$

Using (2.10) and (2.11) in (2.12), we have

$$\tilde{R}(X, Y) Z = R(X, Y) Z + 3\ 'F(X, Z)Y - 3\ 'F(Y, Z) X + [(\nabla_X F)(Y) - (\nabla_Y F)(X)]\ \eta(Z) - [(\nabla_X\ 'F)(Y, Z) - (\nabla_Y\ 'F(X, Z))]\ \xi. \tag{2.13}$$

From (1.7) and (2.13), we have

$$\tilde{R}(X, Y) Z = R(X, Y) Z + 3\ 'F(X, Z)\bar{Y} - 3\ 'F(Y, Z)\bar{X} + [\eta(X)Y - \eta(Y)X]\ \eta(Z) - [\eta(X)\, g(Y, Z) - \eta(Y)\, g(X, Z)]\ \xi. \tag{2.14}$$

From (2.14), we have

$$'\tilde{R}(X, Y, Z, U) = 'R(X, Y, Z, U) + 3\ 'F(X, Z)\ 'F(Y, U) - 3\ 'F(Y, Z)\ 'F(X, U) + \eta(X)\,\eta(Z)\, g(Y, U) - \eta(Y)\,\eta(Z)\, g(X, U) - [\eta(X)\, g(Y, Z) - \eta(Y)\, g(X, Z)]\ \eta(U). \tag{2.15}$$

A relation between the curvature tensor of M with respect to the quarter-symmetric connection $\tilde{\nabla}$ and the Riemannian connection, ∇ is given by the equation (2.15).

Putting $X = U = e_i$ in (2.15) where $\{e_i\}$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \le i \le n$, we get

$$\tilde{S}(Y, Z) = S(Y, Z) + \eta(Y)\,\eta(Z) - n\,\eta(Y)\,\eta(Z) - g(Y, Z) + \eta(Y)\,\eta(Z)$$

$$\tilde{S}(Y, Z) = S(Y, Z) - (n-2)\,\eta(Y)\,\eta(Z) - g(Y, Z) \tag{2.16}$$

contracting (2.16) with respect to Z, we get

$$\tilde{r} = r - 2(n-1) \tag{2.17}$$

where $\tilde{r}$ and r are the scalar curvatures of the connection $\tilde{\nabla}$ and ∇ respectively.

From (2.14) and (2.15), we have

$$\tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y = 0$$

$$'\tilde{R}(X, Y, Z, U) + '\tilde{R}(X, Y, U, Z) = 0$$

and
$$'\tilde{R}(X, Y, Z, U) - '\tilde{R}(Z, U, X, Y) = 0$$

putting $U = \xi$ in (2.15), we have

$$'\tilde{R}(X, Y, Z, \xi) = 2\,'R(X, Y, Z, \xi).$$

Putting $Y = Z = e_i$ in (2.15) and taking summation over i, we have

$$\tilde{S}(X, \xi) = 2S(X, \xi).$$

Thus we can state the following theorem :

Theorem 1. For a P-Sasakian manifold M with quarter symmetric metric connection $\tilde{\nabla}$, we have

(a) $\tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y = 0$

(b) $'\tilde{R}(X, Y, Z, U) + '\tilde{R}(X, Y, U, Z) = 0$

(c) $'\tilde{R}(X, Y, Z, U) - '\tilde{R}(Z, U, X, Y) = 0$

(d) $'\tilde{R}(X, Y, Z, \xi) = 2\,'R(X, Y, Z, \xi)$

(e) $\tilde{S}(X, \xi) = 2S(X, \xi)$.

The concircular curvature tensor $'V$ of type (0, 4) of M^n with respect to Riemannian connection is given by

$$'V(X, Y, Z, U) = 'R(X, Y, Z, U) - \frac{r}{n(n-1)}[g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)]. \tag{2.19}$$

Analogous to this definition, we define concircular curvature tensor of M^n with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ by

$$'\tilde{V}(X, Y, Z, U) = '\tilde{R}(X, Y, Z, U) - \frac{\tilde{r}}{n(n-1)}[g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)]. \tag{2.20}$$

From (2.15), (2.17), (2.19) and (2.20), we have

$$
\begin{aligned}
{}'\tilde{V}(X, Y, Z, U) = {}'V(X, Y, Z, U) + 3\,{}'F(X, Z)\,{}'F(Y, U) - 3\,{}'F(Y, Z)\,{}'F(X, U) \\
+ \eta(X)\,\eta(Z)\,g(Y, U) - \eta(Y)\,\eta(Z)\,g(X, U) - [\eta(X)\,g(Y, Z) \\
- \eta(Y)\,g(X, Z)]\,\eta(U) + \frac{2}{n}[g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)].
\end{aligned} \tag{2.21}
$$

From (2.21), we get

$$'\tilde{V}(X, Y, Z, U) + {}'\tilde{V}(Y, Z, X, U) + {}'\tilde{V}(Z, X, Y, U) = 0.$$

Putting $U = \xi$ in (2.21), we have

$$'\tilde{V}(X, Y, Z, \xi) = \frac{r - 2(n-1)}{n(n-1)}[g(X, Z)\,\eta(Y) - g(Y, Z)\,\eta(X)].$$

Hence we can state the following theorem :

Theorem 2. In a P-Sasakian manifold the concircular curvature $\tilde{V}$ of a quarter-symmetric metric connection $\tilde{\nabla}$ satisfying

(i) $$
\begin{aligned}
{}'\tilde{V}(X, Y, Z, U) = {}'V(X, Y, Z, U) + 3\,{}'F(X, Z)\,{}'F(Y, U) - 3\,{}'F(Y, Z)\,{}'F(X, U) \\
+ \eta(X)\,\eta(Z)\,g(Y, U) - \eta(Y)\,\eta(Z)\,g(X, U) - [\eta(X)\,g(Y, Z) \\
- \eta(Y)\,g(X, Z)]\,\eta(U) + \frac{2}{n}[g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)]
\end{aligned}
$$

(ii) $$'\tilde{V}(X, Y, Z, U) + {}'\tilde{V}(Y, Z, X, U) + {}'\tilde{V}(Z, X, Y, U) = 0$$

(iii) $$'\tilde{V}(X, Y, Z, \xi) = \frac{r - 2(n-1)}{n(n-1)}[g(X, Z)\,\eta(Y) - g(Y, Z)\,\eta(X)].$$

The conharmonic curvature tensor $'L$ of type (0, 4) of M^n with respect to the Riemannian connection is given by

$$
\begin{aligned}
{}'L(X, Y, Z, U) = {}'R(X, Y, Z, U) - \frac{1}{n-2}[S(Y, Z)\,g(X, U) - S(X, Z)\,g(Y, U) \\
+ g(Y, Z)\,S(X, U) - g(X, Z)\,S(Y, U)].
\end{aligned} \tag{2.22}
$$

Analogous to this definition, we define conharmonic curvature tensor of M^n with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ by

$$'\tilde{L}(X, Y, Z, U) = '\tilde{R}(X, Y, Z, U) - \frac{1}{n-2}[\tilde{S}(Y, Z)\, g(X, U) - \tilde{S}(X, Z)\, g(Y, U)$$

$$+ g(Y, Z)\,\tilde{S}(X, U) - g(X, Z)\,\tilde{S}(Y, U)]. \qquad (2.23)$$

From (2.15), (2.16), (2.17), (2.22) and (2.23), we have

$$'\tilde{L}(X, Y, Z, U) = 'L(X, Y, Z, U) + 3\, 'F(X, Z)\, 'F(Y, U) - 3\, 'F(Y, Z)\, 'F(X, U)$$

$$- \frac{2}{n-2}[g(X, Z)\, g(Y, U) - g(Y, Z)\, g(X, U)]. \qquad (2.24)$$

From (2.24), we get

$$'\tilde{L}(X, Y, Z, U) + '\tilde{L}(Y, Z, X, U) + '\tilde{L}(Z, X, Y, U) = 0.$$

Putting $U = \xi$ in (2.24), we have

$$'\tilde{L}(X, Y, Z, \xi) = \frac{1}{n-2}[\{3g(Y, Z) - S(Y, Z)\}\,\eta(X) - \{3g(X, Z) - S(X, Z)\}\,\eta(Y)].$$

Hence we can state the following theorem :

Theorem 3. In a P-Sasakian manifold the conharmonic curvature $\tilde{L}$ of a quarter-symmetric metric connection $\tilde{\nabla}$ satisfying

(i) $$'\tilde{L}(X, Y, Z, U) = 'L(X, Y, Z, U) + 3\, 'F(X, Z)\, 'F(Y, U) - 3\, 'F(Y, Z)\, 'F(X, U)$$

$$- \frac{2}{n-2}[g(X, Z)\, g(Y, U) - g(Y, Z)\, g(X, U)]$$

(ii) $$'\tilde{L}(X, Y, Z, U) + '\tilde{L}(Y, Z, X, U) + '\tilde{L}(Z, X, Y, U) = 0.$$

(iii) $$'\tilde{L}(X, Y, Z, \xi) = \frac{1}{n-2}[\{3g(Y, Z) - S(Y, Z)\}\,\eta(X)$$

$$- \{3g(X, Z) - S(X, Z)\}\,\eta(Y)].$$

3. Ricci-parallel P-Sasakian manifold with respect to quarter-symmetric metric connection

A p-Sasakian manifold is Ricci-parallel if [5]

$$S(Y, Z) = -(n-1)\, g(Y, Z). \tag{3.1}$$

Using (3.1) in (2.16), we get

$$\tilde{S}(Y, Z) = -ng(Y, Z) - (n-2)\,\eta(Y)\,\eta(Z). \tag{3.2}$$

Hence we can state the following theorem :

Theorem 4. In a Ricci-parallel p-Sasakian manifold with respect to quarter-symmetric metric connection $\tilde{\nabla}$, we have

$$\tilde{S}(Y, Z) = -ng(Y, Z) - (n-2)\,\eta(Y)\,\eta(Z).$$

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