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Semi-Invariant Submanifolds of a Kenmotsu Manifold with Generalized Almost r-Contact Structure

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Abstract

Semi-invariant submanifolds of a certain class of almost contact manifolds have been studied by Kobayashi, Shahid and many other geometers. In the present paper, submanifolds of the Kenmotsu manifold with generalized almost r-contact structure have been defined and studied. Certain interesting results have been obtained.

Keywords and Phrases : Kenmotsu manifold, Almost r-contact structure, Semi-invariant submanifolds, Parallel horizontal distributions.

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1. Introduction

Let $\tilde{M}$ be a (2n + r) dimensional Kenmotsu manifold with generalized almost r-contact structure $(\phi, \xi_p, \eta_p, \tilde{g})$ where ϕ is a tensor of type (1, 1), ξ_p are r-vector fields, η_p are r 1-forms and $\tilde{g}$ the associated Riemannian metric on $\tilde{M}$, satisfying

(a) $$\phi^2 = a^2 I + \sum_{p=1}^{r} \eta_p \otimes \xi_p,$$

(b) $$\eta_p(\xi_p) = \delta_{pq}, \qquad p, q = 1, 2, ..., r, \tag{1.1}$$

(c) $$\phi \xi_p = 0,$$

(d) $$\eta_p(\phi X) = 0$$

and

$$\tilde{g}(\phi X, \phi Y) + a^2 \tilde{g}(X, Y) + \sum_{p=1}^{r} \eta_p(X)\eta_p(Y) = 0, \tag{1.2}$$

$$\eta_p(X) = \tilde{g}(X, \xi_p), \tag{1.3}$$

$$(\tilde{\nabla}_X \phi) Y = -\sum_{p=1}^{r} \eta_p(Y)\phi X - \tilde{g}(X, \phi Y)\sum_{p=1}^{r} \xi_p, \tag{1.4}$$

$$\tilde{\nabla}_X \xi_p = X - \sum_{p=1}^{r} \eta_p(X)\xi_p, \tag{1.5}$$

where I is the identity tensor field and X, Y are vector fields in $\tilde{M}$ and $\tilde{\nabla}$ is the covariant differentiation operator on $\tilde{M}$ [6].

Now, let M be a Riemannian manifold isometrically immersed in $\tilde{M}$ such that ξ_p are tangent to $\tilde{M}$. We denote by g the Riemannian metric on M and ∇ the induced Levi-Civita connection on M with respect to g and $\nabla^{\perp}$ the linear connection induced by ∇ on the normal bundle $T^{\perp}M$. Then the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.6}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{1.7}$$

respectively, where h and A are called the second fundamental tensors satisfying

$$g(h(X, Y), N) = g(A_N X, Y).$$

2. Semi-invariant Submanifold

A submanifold M of $\tilde{M}$ is called semi-invariant if ξ_p is tangent to M and there exists a differentiable distribution

$$D : x \in M \rightarrow D_x \subset T_x M \text{ (being the tangent space of M at x)}$$

such that D is invariant distribution on M, that is

$$\phi D_x = D_x \qquad \text{for all } x \in M. \tag{2.1}$$

The orthogonal complementary distribution $D^{\perp}$ of D is anti-invariant, that is

$$\phi(D_X^{\perp}) \subset T_X^{\perp} M, \tag{2.2}$$

where $T_X^{\perp} M$ is the normal space at $x \in M$.

The distribution D (resp. $D^{\perp}$) can be defined by projection P(resp. Q) which satisfies the conditions

$$P^2 = P, \qquad Q^2 = Q, \qquad PQ = QP = 0. \tag{2.3}$$

The distribution D (resp. $D^{\perp}$) is called the horizontal (resp. vertical) distribution. Also the pair (D, $D^{\perp}$) is called ξ_p-horizontal (resp. ξ_p-vertical) if $(\xi_p)_x \in$ D(resp. $(\xi_p)_x \in D^{\perp}$) for all x in M.

It is clear that if M is ξ_p-horizontal (resp. ξ_p-vertical) then dim D = odd (resp. dim $D^{\perp}$ = even).

For a vector field X tangent to M, we put

$$X = PX + QX, \tag{2.4}$$

where PX and QX belong to D and $D^{\perp}$ respectively.

Also, for a vector field N normal to M, we put

$$\phi N = BN + CN, \tag{2.5}$$

where BN (resp. CN) denote the tangential (resp. normal) component of ϕN.

The horizontal distribution D is said to be parallel if

$$\nabla_X Y \in D \qquad \text{for any } X, Y \in D.$$

A semi-invariant submanifold M of $\widetilde{M}$ is said to be D-umbilic (resp. $D^{\perp}$ – umbilic) if

$$h(X, Y) = g(X, Y)\, L \tag{2.6}$$

holds for all X, Y in D(resp. X, Y in $D^\perp$), where L is some normal vector field. M is said to be D– totally geodesic (resp. $D^\perp$ – totally geodesic) if

$$h(X, Y) = 0 \qquad \text{for all} \qquad X, Y \in D \text{ (resp. } X, Y \in D^\perp).$$

Theorem (2.1). Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$. Then we have

$$P\nabla_X\phi PY - PA_{\phi QY}X = \phi P\nabla_X Y + g(\phi X, Y)\, P \sum_{p=1}^{r} \xi_p - \sum_{p=1}^{r} \eta_p(Y)\, \phi PX, \tag{2.7}$$

$$Q\nabla_X\phi PY - QA_{\phi QY}X = Bh(X, Y) + g(\phi X, Y)\, Q \sum_{p=1}^{r} \xi_p - \sum_{p=1}^{r} \eta_p(Y)\, \phi QX, \tag{2.8}$$

$$h(X, \phi PY) + \nabla^\perp_X \phi QY = \phi Q\nabla_X Y + Ch(X, Y). \tag{2.9}$$

Proof. We know that

$$(\tilde{\nabla}_X \phi)\, Y = g(\phi X, Y) \sum_{p=1}^{r} \xi_p - \sum_{p=1}^{r} \eta_p(Y)\, \phi X.$$

Using (2.4), we get

$$(\tilde{\nabla}_X \phi)\, Y = g(\phi X, Y)\, (P \sum_{p=1}^{r} \xi_p + Q \sum_{p=1}^{r} \xi_p) - \sum_{p=1}^{r} \eta_p(Y)\, (\phi PX + \phi QX). \tag{2.10}$$

We know that

$$(\tilde{\nabla}_X \phi)\, Y = \tilde{\nabla}_X \phi Y - \phi\, \tilde{\nabla}_X Y.$$

Using (1.6) and (2.4), the above equation takes the form

$$(\tilde{\nabla}_X \phi)\, Y = \tilde{\nabla}_X \phi PY + \tilde{\nabla}_X \phi QY - \phi\, \nabla_X Y - \phi h(X, Y).$$

By using the Gauss and Weingarten formulae and (2.5), we get

$$(\tilde{\nabla}_X \phi)\, Y = \nabla_X \phi PY + h(X, \phi PY) - A_{\phi QY}X + \nabla^\perp_X \phi QY - \phi P\nabla_X Y - \phi\, Q\nabla_X Y - Bh(X, Y) - Ch(X, Y).$$

$$= P\nabla_X \phi PY + Q\nabla_X \phi PY + h(X, \phi PY) - PA_{\phi QY}X - QA_{\phi QY}X$$

$$+\nabla^{\perp}_X \phi QY - \phi P\nabla_X Y - \phi Q\nabla_X Y - Bh(X, Y) - Ch(X, Y). \quad (2.11)$$

Comparing equations (2.10) and (2.11) and equating the horizontal, vertical and normal components, we get (2.7), (2.8) and (2.9) respectively.

Theorem (2.2). Let M be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$. If M is ξ_p-horizontal, then the distribution D is integrable if and only if

$$h(X, \phi Y) = h(\phi X, Y) \qquad \text{for all } X, Y \in D \quad (2.12)$$

and if M is ξ_p-vertical, then the distribution $D^{\perp}$ is integrable if and only if

$$A_{\phi X} Y - A_{\phi Y} X = \sum_{p=1}^{r} \eta_p(X)\phi Y - \sum_{p=1}^{r} \eta_p(Y)\phi X + 2g(\phi X, Y)\sum_{p=1}^{r} \xi_p . \quad (2.13)$$

Proof. Let M be ξ_p-horizontal, then (2.9) reduces to

$$h(X, \phi Y) = \phi Q\nabla_X Y + Ch(X, Y)$$

and hence, we have

$$h(X, \phi Y) - h(\phi X, Y) = \phi Q[X, Y].$$

Thus if M is ξ_p-horizontal, then $[X, Y] \in D$ i.e. $Q[X, Y] = 0$ if and only if

$$h(X, \phi Y) = h(\phi X, Y).$$

Let M be ξ_p-vertical, then (2.9) reduces to

$$\nabla^{\perp}_X \phi Y = \phi Q\nabla_X Y + Ch(X, Y) \quad \text{for all } X, Y \in D^{\perp}. \quad (2.14)$$

Using (1.4) and (2.4), we get

$$\tilde{\nabla}_X \phi Y = g(\phi X, Y)\sum_{p=1}^{r} \xi_p - \sum_{p=1}^{r} \eta_p(Y)\phi X + \phi P\nabla_X Y$$

$$+ \phi Q\nabla_X Y + Bh(X, Y) + Ch(X, Y). \quad (2.15)$$

Since M is ξ_p-vertical, then by Wiengarten formula, we have

$$\nabla^{\perp}_X \phi Y = \tilde{\nabla}_X \phi Y + A_{\phi Y} X.$$

Using (2.15), we have

$$\nabla^{\perp}_X \phi Y = g(\phi X, Y)\sum_{p=1}^{r} \xi_p - \sum_{p=1}^{r} \eta_p (Y)\, \phi X + \phi P \nabla_X Y$$

$$+ \phi Q \nabla_X Y + Bh(X, Y) + Ch(X, Y) + A_{\phi Y} X. \qquad (2.16)$$

From (2.14) and (2.16), we have

$$\phi P \nabla_X Y = \sum_{p=1}^{r} \eta_p (Y)\, \phi X - g(\phi X, Y)\sum_{p=1}^{r} \xi_p - Bh(X, Y) - A_{\phi Y} X.$$

Similarly

$$\phi P \nabla_Y X = \sum_{p=1}^{r} \eta_p (X)\, \phi Y - g(\phi Y, X)\sum_{p=1}^{r} \xi_p - Bh(Y, X) - A_{\phi X} Y.$$

Therefore, we have

$$\phi P \nabla_X Y - \phi P \nabla_Y X = \sum_{p=1}^{r} \eta_p (Y)\, \phi X - g(\phi X, Y)\sum_{p=1}^{r} \xi_p - Bh(X, Y) - A_{\phi Y} X$$

$$- \sum_{p=1}^{r} \eta_p (X)\, \phi Y + g(\phi Y, X)\sum_{p=1}^{r} \xi_p + Bh(Y, X) + A_{\phi X} Y,$$

$$\phi P[X, Y\} = \sum_{p=1}^{r} \eta_p (Y)\, \phi X - \sum_{p=1}^{r} \eta_p (X)\, \phi Y - 2g(\phi X, Y)\sum_{p=1}^{r} \xi_p + A_{\phi X} Y - A_{\phi Y} X.$$

Thus if M is ξ_p-vertical, we see that $[X, Y] \in D^{\perp}$ if and only if the equation (2.13) holds.

3. Parallel horizontal distributions

A non-zero normal vector field N is said to be D-parallel normal section if

$$\nabla^{\perp}_X N = 0 \qquad \text{for each } X \text{ in } D. \qquad (3.1)$$

Definition. M is said to be totally r-contact umbilical if there exists a normal vector H on M such that

$$h(X, Y) = g(\phi X, \phi Y)\, H + \sum_{p=1}^{r} \eta_p (X)\, h(Y, \xi_p) + \sum_{p=1}^{r} \eta_p (Y)\, h(X, \xi_p) \qquad (3.2)$$

for all vector fields X, Y tangent to M [4].

If $H = 0$, that is the fundamental form is given by

$$h(X, Y) = \sum_{p=1}^{r} \eta_p(X)\, h(Y, \xi_p) + \sum_{p=1}^{r} \eta_p(Y)\, h(X, \xi_p) \tag{3.3}$$

Then M is totally r-contact geodesic.

Theorem (3.1). If M is totally r-contact umbilical semi-invariant submanifold of a generalized Kenmotsu manifold $\widetilde{M}$ with parallel horizontal distribution, then M is totally r-contact geodesic.

Proof. Since M is semi-invariant submanifold of a generalized Kenmotsu manifold $\widetilde{M}$. From (2.7) and (2.8), we have

$$P\nabla_X \phi PY - PA_{\phi QY}X = \phi P\nabla_X Y + g(\phi X, Y)\, P \sum_{p=1}^{r} \xi_p - \sum_{p=1}^{r} \eta_p(Y)\, \phi PX,$$

$$Q\nabla_X \phi PY - QA_{\phi QY}X = Bh(X, Y) + g(\phi X, Y)\, Q \sum_{p=1}^{r} \xi_p - \sum_{p=1}^{r} \eta_p(Y)\, \phi QX.$$

Adding the above equations, we have

$$P\nabla_X \phi PY + Q\nabla_X \phi PY - (PA_{\phi QY}X + QA_{\phi QY}X)$$
$$= \phi P\nabla_X Y + Bh(X, Y) + g(\phi X, Y) \sum_{p=1}^{r} \xi_p - \sum_{p=1}^{r} \eta_p(Y)\, \phi X.$$

Using (2.4), we have

$$\nabla_X \phi PY - A_{\phi QY}X = \phi P\nabla_X Y + Bh(X, Y) + g(\phi X, Y) \sum_{p=1}^{r} \xi_p - \sum_{p=1}^{r} \eta_p(Y)\, \phi X. \tag{3.4}$$

Interchanging X and Y in (3.4), we have

$$\nabla_Y \phi PX - A_{\phi QX}Y = \phi P\nabla_Y X + Bh(Y, X) + g(\phi Y, X) \sum_{p=1}^{r} \xi_p - \sum_{p=1}^{r} \eta_p(X)\, \phi Y. \tag{3.5}$$

Adding (3.4) and (3.5), we get

$$\nabla_X \phi PY + \nabla_Y \phi PX - A_{\phi QY}X - A_{\phi QX}Y = \phi P\nabla_X Y + \phi P\nabla_Y X + 2\,Bh(X, Y) - \sum_{p=1}^{r} \eta_p(Y)\,\phi X - \sum_{p=1}^{r} \eta_p(X)\,\phi Y.$$

Operating 'g' on both sides of the above equation, we get

$$g(\nabla_X \phi PY + \nabla_Y \phi PX - A_{\phi QY}X - A_{\phi QX}Y, Z) = g(\phi P\nabla_X Y + \phi P\nabla_Y X + 2\,Bh(X, Y), Z) - \sum_{p=1}^{r} \eta_p(Y)\, g(\phi X, Z) - \sum_{p=1}^{r} \eta_p(X)\, g(\phi Y, Z).$$

Splitting the above equation and using (3.2), we get

$$g(\nabla_X \phi PY, Z) + g(\nabla_Y \phi PX, Z) - g(A_{\phi QY}X, Z) - g(A_{\phi QX}Y, Z)$$

$$= g(\phi P\nabla_X Y, Z) + g(\phi P\nabla_Y X, Z) + g\,[2B\{g(\phi X, \phi Y)\,H + \sum_{p=1}^{r} \eta_p(X)\,h(Y, \xi_p) + \sum_{p=1}^{r} \eta_p(Y)\,h(X, \xi_p)\}, Z] - \sum_{p=1}^{r} \eta_p(Y)\, g(\phi X, Z) - \sum_{p=1}^{r} \eta_p(X)\, g(\phi Y, Z).$$

$$g(\nabla_X \phi PY, Z) + g(\nabla_Y \phi PX, Z) - g(h(X, Z), \phi QY) - g(h(Y, Z), \phi QX)$$

$$= g(\phi P\nabla_X Y, Z) + g(\phi P\nabla_Y X, Z) + 2g(\phi X, \phi Y)\, g(BH, Z) + 2\sum_{p=1}^{r} \eta_p(X)\, g(Bh(Y, \xi_p), Z) + 2\sum_{p=1}^{r} \eta_p(Y)\, g(Bh(X, \xi_p), Z) - \sum_{p=1}^{r} \eta_p(Y)\, g(\phi X, Z) - \sum_{p=1}^{r} \eta_p(X)\, g(\phi Y, Z)$$

$$= g(\phi P\nabla_X Y, Z) + g(\phi P\nabla_Y X, Z) - 2a^2 g(X, Y)\, g(BH, Z) - 2\sum_{p=1}^{r} \eta_p(X)\,\eta_p(Y)\, g(BH, Z) + 2\sum_{p=1}^{r} \eta_p(X)\, g(h(Y, \xi_p), \phi Z)$$

$$+ 2 \sum_{p=1}^{r} \eta_p (Y) g(h(X, \xi_p), \phi Z) - \sum_{p=1}^{r} \eta_p (Y) g(\phi X, Z) - \sum_{p=1}^{r} \eta_p (X) g(\phi Y, Z).$$

Replacing Y by BH and Z by X and using (3.2), we get

$$g(\nabla_X \phi PBH, X) + g(\nabla_{BH} \phi PX, X) - g(X, X) g(H, \phi QBH) - g(BH, X) g(H, \phi QX)$$

$$= g(\phi P \nabla_X BH, X) + g(\phi P \nabla_{BH} X, X) - 2a^2 g(X, BH) g(BH, X)$$

$$- 2 \sum_{p=1}^{r} \eta_p (X) \eta_p (BH) g(BH, X) + 2 \sum_{p=1}^{r} \eta_p (X) g(h(BH, \xi_p), \phi X)$$

$$+ 2 \sum_{p=1}^{r} \eta_p (BH) g(h(X, \xi_p), \phi X) - \sum_{p=1}^{r} \eta_p (BH) g(\phi X, X)$$

$$- \sum_{p=1}^{r} \eta_p (X) g(\phi BH, X). \tag{3.6}$$

For X in D, we have

$$g(X, BH) = g(\phi X, BH) = 0.$$

Differentiating above covariantly along X, we find

$$g(\nabla_X \phi X, BH) + g(\phi X, \nabla_X BH) = 0.$$

Since, the horizontal distribution D is parallel, we have

$$g(\phi X, \nabla_X BH) = 0. \tag{3.7}$$

Using (3.7) in (3.6) and taking X in D as a unit vector, we get

$$g(\nabla_{BH} \phi PX, X) - g(H, \phi QBH) = g(\phi P \nabla_{BH} X, X),$$

$$g((\nabla_{BH} \phi P)X + \phi P \nabla_{BH} X, X) - g(H, \phi QBH) = g(\phi P \nabla_{BH} X, X),$$

$$g((\nabla_{BH} \phi P)X, X) + g(\phi P \nabla_{BH} X, X) - g(H, \phi QBH) = g(\phi P \nabla_{BH} X, X),$$

$$g((\nabla_{BH} \phi P)X, X) = g(H, \phi QBH) = - g(\phi H, QBH) = - g(BH, QBH). \tag{3.8}$$

Now, $g((\nabla_{BH} \phi P)X, X) = 0.$

From (3.8), we have

$$g(BH, QBH) = 0,$$

$$BH = 0.$$

Since $\phi H \in D^{\perp}$, we have $CH = 0$, hence $\phi H = 0$, thus $H = 0$.

Hence M is totally r-contact geodesic.

Remark. For a generalized Kenmotsu manifold, we have

$$\begin{aligned}\widetilde{\nabla}_X \xi_p &= \nabla_X \xi_p + h(X, \xi_p)\\ &= PX + QX - \sum_{p=1}^{r} \eta_p(X) - \sum_{p=1}^{r} \eta_p(X) P\xi_p.\end{aligned} \tag{3.9}$$

Equating the tangential and normal components, we have

$$\nabla_X \xi_p = PX - \sum_{p=1}^{r} \eta_p(X) P\xi_p. \tag{3.10}$$

$$h(X, \xi_p) = QX. \tag{3.11}$$

From (3.10) and (3.11), we can easily obtain

$$\nabla_X \xi_p = 0 \quad \text{for } X \text{ in } D^{\perp},$$

$$h(X, \xi_p) = 0 \quad \text{for } X \text{ in } D.$$

Also for X in D, we have

$$g(A_N \xi_p, X) = g(h(X, \xi_p), N) = 0$$

and so we have $A_N \xi_p \in D^{\perp}$.

Theorem (3.2). Let M be D-umbilic (resp. $D^{\perp}$-umbilic) semi-invariant submanifold of a generalized Kenmotsu manifold $\widetilde{M}$. If M is ξ_p-horizontal (resp. ξ_p-vertical), then M is D totally geodesic (resp. $D^{\perp}$-totally geodesic).

Proof. If M is D-umbilic semi-invariant submanifold of a generalized Kenmotsu manifold with ξ_p-horizontal, then we have

$$h(X, \xi_p) = g(X, \xi_p) L,$$

which means that $L = 0$,

from which, we get $h(X, \xi_p) = 0$.

Hence, M is D-totally geodesic.

Similarly, we can prove that if M is a $D^{\perp}$-umbilical semi-invariant submanifold with ξ_p-vertical, then M is $D^{\perp}$-totally geodesic.

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