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Affine Motion in a Finsler Space with Non-Symmetric Connections

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Abstract

Takano [5] studied the affine motion and its properties in a recurrent Finsler space. Further, Pandey and Tiwari [3] developed the existence of affine motion in a $R-\oplus$ recurrent Finsler space equipped with non-symmetric connections. The object of the present paper is to study the infinitesimal affine motion $\overline{x^i} = x^i + v^i(x)\, dt$ in a Finsler space with non-symmetric connection. Some useful investigation have been obtained in the paper.

Keywords : Affine motion, Non-symmetric connection, Covariant derivative.

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1. Introduction

Consider an n-dimensional Finsler space Fn, [4] having 2n-line elements $(x, \dot{x})$ equipped with non-symmetric connection $\Gamma^i_{jk}(x, \dot{x})$ (i, j etc. = 1, 2, 3,..., n). The non-symmetric connection Γ^i_{jk} is based on non-symmetric fundamental metric tensor $g_{ij}(x, \dot{x}) \neq g_{ji}(x, \dot{x})$. Also the non-symmetric connection Γ^i_{jk} is homogeneous of degree zero in its directional arguments $\dot{x}$. Here, the non-symmetric connection Γ^i_{jk} can be written as :

$$\Gamma^i_{jk} = M^i_{jk}(x, \dot{x}) + \frac{1}{2} N^i_{jk}(x, \dot{x}), [1] \tag{1.1}$$

where M^i_{jk} and $\frac{1}{2} N^i_{jk}$ denote symmetric and skew-symmetric parts of Γ^i_{jk} respectively.

The covariant derivative of a tensor field T^i_j (x, $\dot{x}$) with respect to x^k are defined in two ways : [2]

$$T^i_{j \underset{+}{|} k} = \partial_k T^i_j - (\dot{\partial}_m T^i_j)\,\Gamma^m_{pk}\,\dot{x}^p + T^m_j\,\Gamma^i_{mk} - T^i_m\,\Gamma^m_{jk} \tag{1.2}$$

$$T^i_{j \underset{-}{|} k} = \partial_k T^i_j - (\dot{\partial}_m T^i_j)\,\hat{\Gamma}^m_{pk}\,\dot{x}^p + T^m_j\,\hat{\Gamma}^i_{mk} - T^i_m\,\Gamma^m_{jk}, \tag{1.3}$$

where $\hat{\Gamma}^i_{jk} = \Gamma^i_{kj}$.

The covariant derivative defined by (1.2) is called ⊕-covariant derivative while defined by (1.3) is called Θ-covariant derivative.

The commutation formula involving ⊕-covariant derivative for a tensor T^i_j is given by

$$T^i_{j \underset{+}{|} hk} - T^i_{j \underset{+}{|} kh} = (\dot{\partial}_m T^i_j)\,R^m_{hk} + T^m_j\,R^i_{mhk} - T^i_m\,R^m_{jhk} + T^i_{j \underset{+}{|} m}\,N^m_{kj},\ [3] \tag{1.4}$$

where

$$R^i_{hjk} = \partial_k \Gamma^i_{hj} - \partial_j \Gamma^i_{hk} + (\dot{\partial}_m \Gamma^i_{hj})\,\Gamma^m_{sk}\,\dot{x}^s - (\dot{\partial}_m \Gamma^i_{hk})\,\Gamma^m_{sj}\,\dot{x}^s + \Gamma^p_{hj}\,\Gamma^i_{pk} - \Gamma^p_{hk}\,\Gamma^i_{pj}, \tag{1.5}$$

is called relative curvature tensor field and satisfies the following identities : [3]

(a) $R^i_{hjk} = - R^i_{hkj}$

(b) $R^i_{hjl} = R_{hj}$. (1.6)

Let us consider an infinitesimal point transformation

$$x^{-i} = x^i + v^i(x)\,dt, \quad [4] \tag{1.7}$$

where $v^i(x)$ be a vector field and dt an infinitesimal point constant.

In view of transformation (1.7) and covariant derivative defined by (1.4) and (1.5), the Lie derivative of T^i_j (x, $\dot{x}$) and Γ^i_{jk} (x, $\dot{x}$) are defined as : [3]

$$L_V T^i_j = T^i_{j \underset{+}{|} h}\, v^h - T^h_j\, V^i_{\underset{-}{|} h} + T^i_h\, V^h_{\underset{-}{|} j} + \dot{\partial}_h T^i_j\, V^h_{\underset{-}{|} s}\, \dot{x}^s \tag{1.8}$$

$$L_V \Gamma^i_{jk} = V^i_{\underset{-}{|}j\underset{+}{|}k} + R^i_{jkh} V^h + \dot{\partial}_r \Gamma^i_{jk} V^r_{\underset{-}{|}s} \dot{x}^s. \tag{1.9}$$

The Lie-derivation of the curvature tensor R^i_{jkh} may be given by

$$L_V R^i_{hjk} = R^i_{hjk\underset{+}{|}s} V^s + R^i_{sjk} V^s_{\underset{-}{|}h} + R^i_{hsk} V^s_{\underset{-}{|}j} + R^i_{hjk} V^s_{\underset{-}{|}k}$$

$$- R^i_{hjk} V^s_{\underset{-}{|}s} + \dot{\partial}_r R^i_{hjk} V^r_{\underset{-}{|}s} \dot{x}^s. \tag{1.10}$$

With the help of (1.9), (1.10) and covariant derivatives defined in (1.2) and (1.3), we have following two commutation formula

$$\left(L_V T^i_{jk}\right)_{\underset{+}{|}s} - L_V (T^i_{jk\underset{+}{|}s}) = T^i_{mk} (L_V \Gamma^m_{js}) + \Gamma^i_{jm} (L_V \Gamma^m_{ks}) - T^m_{jk} (L_V \Gamma^i_{ms})$$

$$+ \dot{\partial}_m T^i_{jk} (L_V \Gamma^m_{rs}) \dot{x}^r. \tag{1.11}$$

$$\left(L_V \Gamma^i_{jh}\right)_{\underset{+}{|}k} - \left(L_V \Gamma^i_{kh}\right)_{\underset{+}{|}j} = L_V R^i_{hjk} + \dot{\partial}_r \Gamma^i_{jh} (L_V \Gamma^r_{ks}) \dot{x}^s - \dot{\partial}_r \Gamma^i_{hk} (L_V \Gamma^r_{js}) \dot{x}^s. \tag{1.12}$$

We shall use following results : [2]

(a) $\dot{x}^i_{\underset{+}{|}k} = 0.$

(b) $\dot{x}^i_{\underset{-}{|}k} = 0.$ (1.13)

2. Discussion

In order that the transformation (1.7) considered at each point of the space is infinitesimal affine motion if and only if

$$L_V \Gamma^i_{jk} = 0, \; [6]. \tag{2.1}$$

Introducing the result (2.1) in commutation formula (1.12), we find

$$L_V R^i_{hjk} = 0. \tag{2.2}$$

Thus, we have

Theorem (2.1). In a Finsler space F_n with non-symmetric connection admitting affine motion, the curvature tensor R^h_{hjk} becomes a Lie-invariant.

Now, applying the condition (2.1) in relation (1.11), we get

$$\left(L_V T^i_{jk}\right)_{\overset{+}{|s}} - L_V \left(T^i_{jk\overset{+}{|s}}\right) = 0. \tag{2.3}$$

Therefore, we have

Theorem (2.2). In a Finsler space F_n with non-symmetric connection admitting affine motion, the tensor field T^i_j satisfies the differential equation (2.3).

Let us suppose that vector V^i be parallel in F_n admitting an affine motion.

Then, we have

$$V^i_{\overset{-}{|j}} = 0. \tag{2.4}$$

From equations (1.10), (2.2) and (2.4), we get following useful result

$$R^i_{jhk\overset{+}{|s}} V^s = 0. \tag{2.5}$$

Theorem (2.3). If the vector V^i be parallel in a Finsler space with non-symmetric connection admitting an affine motion, then conditions (2.5) holds.

Applying condition (2.1) in (1.9), we get

$$V^i_{\overset{-}{|j}\overset{+}{|k}} + R^i_{jkh} V^h + \dot{\partial}_r \Gamma^i_{jk} V^r_{\overset{-}{|s}} \dot{x}^s = 0. \tag{2.6}$$

Suppose the vector V^{is} is parallel in F_n, then relation (2.4) will hold and consequently the equation (2.6) yields

$$R^i_{jkh} V^h = 0. \tag{2.7}$$

Thus, we state

Theorem (2.4). In a Finsler space F_n with non-symmetric connection admitting affine motion, the relation (2.7) holds if the vector V^i is parallel.

3. Special Form

Consider the existence of affine motion in following form

$$\bar{x}^i = x^i + v(x)\,dt, \qquad V^i_{\overset{-}{|}j} = \rho\, \delta^i_j, \tag{3.1}$$

Here $\rho(x, \dot{x})$ is an arbitrary non-scalar function and it is homogeneous of degree zero in its directional arguments $\dot{x}^i$.

Introducing the relation (3.1) and (2.3) in (1.10), we find

$$R^i_{hjk\overset{+}{|}s} V^s + R^i_{sjk}(\rho\, \delta^s_h) + R^i_{hsk}(\rho\, \delta^s_k) + R^i_{hjs}(\rho\, \delta^s_j)$$
$$- R^i_{hjk}(\rho\, \delta^i_s) + \dot{\partial}_r R^i_{hjk}(\rho\, \delta^r_s)\, \dot{x}^s = 0$$

or

$$R^i_{hjk\overset{+}{|}s} V^s + 2\rho\, R^i_{hjk} + \rho\, \dot{\partial}_s R^i_{hjk}\, \dot{x}^s = 0. \tag{3.2}$$

Due to homogeneity property of R^i_{hjk}, we have

$$\dot{\partial}_s R^i_{hjk}\, \dot{x}^s = 0. \tag{3.3}$$

Using (3.3) in (3.2), we get

$$2\rho\, R^i_{hjk} + R^i_{hjk\overset{+}{|}s} V^s = 0. \tag{3.4}$$

Contracting the equation (3.4) for the indices i and k and transvecting by $\dot{x}^h$, $\dot{x}^j$ successively, we get

$$2\rho\, R_{hj}\, \dot{x}^h\, \dot{x}^j + R_{hj\overset{+}{|}s}\, \dot{x}^h\, \dot{x}^j\, V^s = 0. \tag{3.5}$$

Using (1.13)(a) in (3.5), we get

$$2\rho\, R_{hj}\, \dot{x}^h\, \dot{x}^j + \left(R_{hj}\, \dot{x}^h\, \dot{x}^j\right)_{\underset{I\ s}{+}}\, \dot{x}^h\, \dot{x}^j\, V^s = 0. \tag{3.6}$$

Assume a non-null scalar function $S(x, \dot{x})$ defined as

$$S\,(x, \dot{x}) \overset{\text{def}}{=} R_{hj}\, \dot{x}^h\, \dot{x}^j\,. \tag{3.7}$$

In view of relation (3.7), (3.6) reduces to

$$2\rho\, S + S_{\underset{I\ s}{+}}\, V^s = 0. \tag{3.8}$$

Thus we conclude following :

Theorem (3.1). In a Finsler space F_n with non-symmetric connection admitting the affine motion of form (3.1), the non-null scalar function $S\,(x, \dot{x})$ defined by (3.7) satisfies the differential equation (3.8).

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