J. T. S
Vol. 3 (2009), pp.79-104
https://doi.org/10.56424/jts.v3i01.9974

On n-Recurrent HGF-Structure Metric Manifold

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(Received : 28 December, 2008)

Abstract

Various geometers viz, Takano, K. [10], Pandey and Pant [7, 8], Mishra, Pandey and Sharma [3], Mishra and Pandey [2, 4] and Dasila [5] studied different recurrent, birecurrent and n-recurrent differentiable manifolds. In this paper, we have extensively studied n-recurrent HGF-structure metric manifold taking different curvature tensors.

Keywords : Recurrent manifold, metric manifold, birecurrent manifold.

Mathematics Subject Classification 2000 : 53C05, 53C15.

1. Introduction

Let us consider a differentiable manifold M_n of differentiability class C^∞. Let there be in M_n a vector-valued linear function F of the class C^∞, satisfying the algebraic equation

$$\overline{\overline{X}} = -a^2 X, \text{ for arbitrary vector field } X, \tag{1.1)(a}$$

where $\overline{X} = FX$ and 'a' is a complex number. (1.1)(b)

Then {F} is said to give to M_n a hyperbolic differentiable structure, briefly known as HGF-structure, defined by the equation (1.1)(a) and the manifold M_n is called HGF-manifold. The equations (1.1)(a) gives different structures for different values of 'a'.

If $a \neq 0$, it is hyperbolic π-structure, if $a = \pm 1$, it is an almost complex or a hyperbolic almost product structure, if $a = \pm i$, it is an almost product or a hyperbolic almost complex structure and if $a = 0$, it is an almost tangent or hyperbolic almost tangent structure.

Let the HGF-structure be endowed with a Hermite tensor g, such that

$$g(\overline{X}, \overline{Y}) - a^2 g(X, Y) = 0. \tag{1.2}$$

The {F, g} is said to give to M_n, a hyperbolic differentiable metric structure and the manifold M_n is called a hyperbolic differentiable metric structure manifold.

Let us put

$$'F(X, Y) = g(\overline{X}, Y). \tag{1.3}$$

Then the following equations holds :

$$'F(X, Y) = -\, 'F(Y, X), \tag{1.4)(a}$$

i.e. 'F is skew-symmetric in X and Y.

$$'F(\overline{X}, \overline{Y}) = a^2\, 'F(X, Y), \tag{1.4)(b}$$

$$'F(\overline{X}, Y) = -\, 'F(X, \overline{Y}). \tag{1.4)(c}$$

A bilinear function B in HGF-metric manifold is said to be pure in two slots, if

$$B(\overline{X}, \overline{Y}) + a^2 B(X, Y) = 0. \tag{1.5}$$

It is said to be hybrid in two slots, if

$$B(\overline{X}, \overline{Y}) - a^2 B(X, Y) = 0. \tag{1.6}$$

It shows that 'F is hybrid in X and Y.

Let D be a connexion and X, Y and Z be C^{∞} vector fields, then the function K, defined by

$$K(X, Y, Z) \overset{\text{def}}{=} D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z, \tag{1.7}$$

is called the curvature tensor of connexion D.

Let us put

$$'K(X, Y, Z, T) \overset{\text{def}}{=} g(K(X, Y, Z), T), \tag{1.8}$$

Then 'K is a real-valued 4-linear function, called associated curvature tensor or Riemann Christoffel curvature tensor of the first kind which satisfies the following properties :

(i) It is skew-symmetric in first two slots :

$$'K(X, Y, Z, T) = -\, 'K(Y, X, Z, T). \qquad (1.9)$$

(ii) It is skew-symmetric in last two slots :

$$'K(X, Y, Z, T) = -\, 'K(X, Y, T, Z). \qquad (1.10)$$

(iii) It is skew-symmetric in two pairs of slots :

$$'K(X, Y, Z, T) = 'K(Z, T, X, Y). \qquad (1.11)$$

(iv) It satisfies Bianchi's first identities :

$$'K(X, Y, Z, T) + 'K(Y, Z, X, T) + 'K(Z, X, Y, T) = 0. \qquad (1.12)$$

(v) It also satisfies Bianchi's second identities :

$$(D_X\, 'K)(Y, Z, T, U) + (D_Y\, 'K)(Z, X, T, U) + (D_Z\, 'K)(X, Y, T, U) = 0. \qquad (1.13)$$

The tensor defined by

$$\mathrm{Ric}\,(Y, Z) \overset{\mathrm{def}}{=} (C_1^1 K)(Y, Z), \qquad (1.14)$$

is called the Ricci tensor, C_1^1 being the contraction operator. It is a symmetric tensor of the type (0, 2) :

$$\mathrm{Ric}\,(Y, Z) = \mathrm{Ric}\,(Z, Y), \qquad (1.15)$$

The linear map r, defined by

$$g\,(r(X), Y) = g(X, r(Y)) \qquad (1.16)$$

is called Ricci map. The scalar R, defined by

$$R = (C_1^1\, r) \qquad (1.17)$$

is called the scalar curvature of M_n at any point P.

Agreement (1.1). The discussion which follow will hold for manifolds other than almost tangent or hyperbolic almost tangent manifolds.

In the HGF-metric structure manifold, Pseudo projective curvature tensor W^*, Pseudo conformal curvature tensor C^*, Pseudo conharmonic curvature tensor L^*, Pseudo concircular curvature tensor V^*, Pseudo H-projective curvature tensor P^*, Pseudo H-conharmonic curvature tensor S^* and Pseudo Bochner curvature tensor B^* are given by

$$W^*(X, Y, Z) = K(X, Y, Z) - \frac{1}{a^2 (n-1)} [a^2 \operatorname{Ric}(Y, Z) X - a^2 \operatorname{Ric}(X, Z) Y], \tag{1.18}$$

$$C^*(X, Y, Z) = K(X, Y, Z) + \frac{a^2}{(n+2)} [a^2 \operatorname{Ric}(X, Z) Y - a^2 \operatorname{Ric}(Y, Z) X + g(X, Z) r(Y) - g(Y, Z) r(X)] - \frac{a^2 R}{(n-1)(n-2)} [g(X, Z)Y - g(Y, Z)X], \tag{1.19}$$

$$L^*(X, Y, Z) = K(X, Y, Z) + \frac{a^2}{(n-2)} [a^2 \operatorname{Ric}(X, Z) Y - a^2 \operatorname{Ric}(Y, Z) X + g(X, Z) r(Y) - g(Y, Z) r(X)], \tag{1.20}$$

$$V^*(X, Y, Z) = K(X, Y, Z) + \frac{a^2 R}{n(n-1)} [g(X, Z) Y - g(Y, Z) X], \tag{1.21}$$

$$P^*(X, Y, Z) = K(X, Y, Z) + \frac{1}{a^2 (n+2)} [a^2 \operatorname{Ric}(Y, Z) X - a^2 \operatorname{Ric}(X, Z) Y - \operatorname{Ric}(\overline{Y}, Z) \overline{X} + \operatorname{Ric}(\overline{X}, Z)\overline{Y} - 2 \operatorname{Ric}(X, \overline{Y}) \overline{Z}], \tag{1.22}$$

$$S^*(X, Y, Z) = K(X, Y, Z) - \frac{a^2}{(n+4)} [a^2 \operatorname{Ric}(Y, Z) X - a^2 \operatorname{Ric}(X, Z) Y - g(X, Z) r(Y) + g(Y, Z) r(X) + \operatorname{Ric}(X, \overline{Z}) \overline{Y} - \operatorname{Ric}(Y, \overline{Z})\overline{X} - 2 \operatorname{Ric}(\overline{X}, Y) \overline{Z} + g(X, \overline{Z}) r(\overline{Y}) - g(Y, \overline{Z}) r(\overline{X}) - 2g(\overline{X}, Y) r(\overline{Z})]. \tag{1.23}$$

$$B^*(X, Y, Z) = K(X, Y, Z) - \frac{a^2}{(n+4)} [a^2 \operatorname{Ric}(Y, Z) X - a^2 \operatorname{Ric}(X, Z) Y - g(X, Z) r(Y) + g(Y, Z) r(X) + \operatorname{Ric}(X, \overline{Z}) \overline{Y} - \operatorname{Ric}(Y, \overline{Z}) \overline{X} - 2 \operatorname{Ric}(\overline{X}, Y) \overline{Z} + g(X, \overline{Z}) r(\overline{Y}) - g(Y, \overline{Z}) r(\overline{X}) - 2g(\overline{X}, Y) r(\overline{Z})]$$

$$+\frac{a^2 R}{(n+2)(n+4)}[g(Y, Z)X - g(X, Z)Y + g(X, \overline{Z})\overline{Y} - g(Y, \overline{Z})\overline{X} - 2g(\overline{X}, Y)\overline{Z}]. \tag{1.24}$$

The associated tensors of $'W^*$, $'C^*$, $'L^*$, $'V^*$, $'P^*$, $'S^*$ and $'B^*$ are given by

$$'W^* \overset{\text{def}}{=} g(W^*(X, Y, Z), T), \tag{1.25(a)}$$

$$'C^* \overset{\text{def}}{=} g(C^*(X, Y, Z), T), \tag{1.25(b)}$$

$$'L^* \overset{\text{def}}{=} g(L^*(X, Y, Z), T), \tag{1.25(c)}$$

$$'V^* \overset{\text{def}}{=} g(V^*(X, Y, Z), T), \tag{1.25(d)}$$

$$'P^* \overset{\text{def}}{=} g(P^*(X, Y, Z), T), \tag{1.25(e)}$$

$$'S^* \overset{\text{def}}{=} g(S^*(X, Y, Z), T), \tag{1.25(f)}$$

$$'B^* \overset{\text{def}}{=} g(B^*(X, Y, Z), T). \tag{1.25(g)}$$

Consequently,

$$'W^*(X, Y, Z, T) = 'K(X, Y, Z, T) - \frac{1}{a^2(n-1)}[a^2 \operatorname{Ric}(Y, Z)g(X,T) - a^2 \operatorname{Ric}(X, Z)g(Y, T)], \tag{1.26}$$

$$\begin{aligned}'C^*(X, Y, Z, T) = {}&'K(X, Y, Z, T) + \frac{a^2}{(n+2)}[a^2 \operatorname{Ric}(X, Z)g(Y, T) \\ &- a^2 \operatorname{Ric}(Y, Z)g(X, T) + g(X, Z)\operatorname{Ric}(Y, T) - g(Y, Z)\operatorname{Ric}(X, T)] \\ &- \frac{a^2 R}{(n-1)(n-2)}[g(X, Z)g(Y, T) - g(Y, Z)g(X, T)],\end{aligned} \tag{1.27}$$

$$\begin{aligned}'L^*(X, Y, Z, T) = {}&'K(X, Y, Z, T) + \frac{a^2}{(n-2)}[a^2 \operatorname{Ric}(X, Z)g(Y, T) \\ &- a^2 \operatorname{Ric}(Y, Z)g(X, T) + g(X, Z)\operatorname{Ric}(Y, T) - g(Y, Z)\operatorname{Ric}(X, T)],\end{aligned} \tag{1.28}$$

$$'V^*(X, Y, Z, T) = 'K(X, Y, Z, T) + \frac{a^2 R}{n(n-1)} [g(X, Z)\, g(Y, T) - g(Y, Z)\, g(X, T)], \tag{1.29}$$

$$'P^*(X, Y, Z, T) = 'K(X, Y, Z, T) + \frac{1}{a^2 (n+2)} [a^2 \operatorname{Ric}(Y, Z) g(X, T) - a^2 \operatorname{Ric}(X, Z)\, g(Y, T) - \operatorname{Ric}(\overline{Y}, Z)\, g(\overline{X}, T) + \operatorname{Ric}(\overline{X}, Z) g(\overline{Y}, T) - 2 \operatorname{Ric}(X, \overline{Y})\, g(\overline{Z}, T)], \tag{1.30}$$

$$'S^*(X, Y, Z, T) = 'K(X, Y, Z, T) - \frac{a^2}{(n+4)} [a^2 \operatorname{Ric}(Y, Z) g(X, T) - a^2 \operatorname{Ric}(X, Z) g(Y, T) + \operatorname{Ric}(X, \overline{Z})\, g(\overline{Y}, T) - \operatorname{Ric}(Y, \overline{Z})\, g(\overline{X}, T) - 2 \operatorname{Ric}(\overline{X}, Y)\, g(\overline{Z}, T) - g(X, Z) \operatorname{Ric}(Y, T) + g(Y, Z) \operatorname{Ric}(X, T) + g(X, \overline{Z}) \operatorname{Ric}(\overline{Y}, T) - g(Y, \overline{Z}) \operatorname{Ric}(\overline{X}, T) - 2g(\overline{X}, Y) \operatorname{Ric}(\overline{Z}, T)], \tag{1.31}$$

$$'B^*(X, Y, Z, T) = 'K(X, Y, Z, T) - \frac{a^2}{(n+4)} [a^2 \operatorname{Ric}(Y, Z) g(X, T) - a^2 \operatorname{Ric}(X, Z) g(Y, T) - g(X, Z) \operatorname{Ric}(Y, T) + g(Y, Z) \operatorname{Ric}(X, T) + \operatorname{Ric}(X, \overline{Z})\, g(\overline{Y}, T) - \operatorname{Ric}(Y, \overline{Z})\, g(\overline{X}, T) - 2 \operatorname{Ric}(\overline{X}, Y)\, g(\overline{Z}, T) + g(X, \overline{Z}) \operatorname{Ric}(\overline{Y}, T) - g(Y, \overline{Z}) \operatorname{Ric}(\overline{X}, T) - 2g(\overline{X}, Y) \operatorname{Ric}(\overline{Z}, T)] + \frac{a^2 R}{(n+2)(n+4)} [g(Y, Z) g(X, T) - g(X, Z) g(Y, T) + g(X, \overline{Z})\, g(\overline{Y}, T) - g(Y, \overline{Z})\, g(\overline{X}, T) - 2g(\overline{X}, Y)\, g(\overline{Z}, T)]. \tag{1.32}$$

Let $'P$ be any of the curvature tensors of $'W^*$, $'C^*$, $'L^*$, $'V^*$, $'P^*$, $'S^*$ and $'B^*$. Then the HGF-structure metric manifold is said to be (4)-recurrent in $'P$, if

$$a^2 (\nabla\, 'P)(X, Y, Z, T, U) - 'P(X, Y, Z, (\nabla F)(\overline{T}, U)) = a^2 B_1(U)\, 'P(X, Y, Z, T),$$

where $B_1(U)$ is a C^∞ function, called recurrence parameter, and (4)-birecurrent in $'P$, if

$$a^2 (\nabla\nabla\, 'P)(X, Y, Z, T, U, V) - (\nabla' P)(X, Y, Z, (\nabla F)(\overline{T}, U), V)$$
$$- (\nabla' P)(X, Y, Z, (\nabla F)(\overline{T}, V), U) + a^2\, 'P(X, Y, Z, (\nabla\nabla F)(\overline{T}; U, V))$$
$$= a^2 B_2(U, V)\, 'P(X, Y, Z, T),$$

where $B_2(U, V)$ is a C^∞ function, called birecurrence parameter.

The HGF-structure metric manifold is said to be (14)-recurrent in 'P, if

$$a^2 (\nabla\, 'P)(X, Y, Z, \overline{T}, U) - 'P((\nabla F)(\overline{X}, U), Y, Z, \overline{T}) + a^2\, 'P(X, Y, Z, (\nabla F)(T, U))$$
$$= a^2 B_1(U)\, 'P(X, Y, Z, \overline{T}),$$

or equivalently

$$a^2 (\nabla\, 'P)(\overline{X}, Y, Z, T, U) - 'P(\overline{X}, Y, Z, (\nabla F)(\overline{T}, U)) + a^2\, 'P((\nabla F)(X, U), Y, Z, T)$$
$$= a^2 B_1(U)\, 'P(\overline{X}, Y, Z, T),$$

and (14)-birecurrent in 'P, if

$$a^2 (\nabla\nabla\, 'P)(X, Y, Z, \overline{T}, U, V) - (\nabla' P)((\nabla F)(\overline{X}, U), Y, Z, \overline{T}, V)$$
$$- (\nabla\, 'P)((\nabla F)(\overline{X}, V), Y, Z, \overline{T}, U) - P'((\nabla\nabla F)(\overline{X}, U, V), Y, Z, T)$$
$$+ a^2 (\nabla' P)(X, Y, Z, (\nabla F)(T, U), V) + a^2 (\nabla' P)(X, Y, Z, (\nabla F)(T, V), U)$$
$$+ a^2\, 'P(X, Y, Z, (\nabla\nabla F)(T, U, V)) - P'((\nabla F)(\overline{X}, U), Y, Z, (\nabla F)(T, V))$$
$$- P'((\nabla F)(\overline{X}, V), Y, Z, (\nabla F)(T, U)) = a^2 B_2(U, V)\, 'P(X, Y, Z, \overline{T}),$$

or equivalently

$$a^2 (\nabla\nabla\, 'P)(\overline{X}, Y, Z, T, U, V) - (\nabla' P)(\overline{X}, Y, Z, (\nabla F)(\overline{T}, U), V)$$
$$- (\nabla' P)(\overline{X}, Y, Z, (\nabla F)(\overline{T}, V), U) - 'P(\overline{X}, Y, Z, (\nabla\nabla F)(\overline{T}, U, V))$$
$$+ a^2 (\nabla' P)((\nabla F)(X, U), Y, Z, T, V) + a^2 (\nabla' P)((\nabla F)(X, V), Y, Z, T, U)$$
$$+ a^2\, 'P((\nabla\nabla F)(X, U, V), Y, Z, T) - 'P((\nabla F)(X, U), Y, Z, (\nabla F)(\overline{T}, V))$$
$$- 'P((\nabla F)(X, V), Y, Z, (\nabla F)(\overline{T}, U)) = a^2 B_2(U, V)\, 'P(\overline{X}, Y, Z, T).$$

The HGF-structure metric manifold is said to be (124)-recurrent in 'P, if

$$a^2 (\nabla\, 'P)(X, \overline{Y}, Z, \overline{T}, U) - 'P((\nabla F)(\overline{X}, U), \overline{Y}, Z, \overline{T}) + a^2\, 'P(X, (\nabla F)(Y, U), Z, \overline{T}) +$$

$$a^2\, 'P(X, \overline{Y}, Z, (\nabla F)(T, U)) = a^2 B_1(U)\, 'P(X, \overline{Y}, Z, \overline{T}),$$

or equivalently

$$a^2 (\nabla\, 'P)(\overline{X}, Y, Z, \overline{T}, U) - 'P(\overline{X}, (\nabla F)(\overline{Y}, U), Z, \overline{T}) + a^2\, 'P((\nabla F)(X, U), Y, Z, \overline{T}) +$$

$$a^2\, 'P(\overline{X}, Y, Z, (\nabla F)(T, U)) = a^2 B_1(U)\, 'P(\overline{X}, Y, Z, \overline{T}).$$

Similarly we can obtain other equivalent condition applying structure on different vectors.

and (124)-birecurrent in 'P, if

$$\begin{aligned}
& a^2 (\nabla\nabla\, 'P)(X, \overline{Y}, Z, \overline{T}, U, V) - (\nabla' P)((\nabla F)(\overline{X}, U), \overline{Y}, Z, \overline{T}, V) \\
& - (\nabla\, 'P)((\nabla F)(\overline{X}, V), \overline{Y}, Z, \overline{T}, U) - P'((\nabla\nabla F)(\overline{X}, U, V), \overline{Y}, Z, \overline{T}) \\
& + a^2 (\nabla' P)(X, (\nabla F)(Y, U), Z, \overline{T}, V) + a^2 (\nabla' P)(X, (\nabla F)(Y, V), Z, \overline{T}, U) \\
& + a^2\, 'P(X, (\nabla\nabla F)(Y, U, V), Z, \overline{T}) + a^2 (\nabla\, 'P)(X, \overline{Y}, Z, (\nabla F)(T, U), V) \\
& + a^2 (\nabla\, 'P)(X, \overline{Y}, Z, (\nabla F)(T, V), U) + a^2\, 'P(X, \overline{Y}, Z, (\nabla\nabla F)(T, U, V)) \\
& - P'((\nabla F)(\overline{X}, U), (\nabla F)(Y, V), Z, \overline{T}) - P'((\nabla F)(\overline{X}, V), (\nabla F)(Y, U), Z, \overline{T}) \\
& - P'((\nabla F)(\overline{X}, U), \overline{Y}, Z, (\nabla F)(T, V)) - P'((\nabla F)(\overline{X}, V), \overline{Y}, Z, (\nabla F)(T, U)) \\
& + a^2\, 'P(X, (\nabla F)(Y, U), Z, (\nabla F)(T, V)) + a^2\, 'P(X, (\nabla F)(Y, V), Z, (\nabla F)(T, U)) \\
& = a^2\, 'B_2(U, V)\, 'P(X, \overline{Y}, Z, \overline{T}),
\end{aligned}$$

or equivalently

$$\begin{aligned}
& a^2 (\nabla\nabla\, 'P)(\overline{X}, \overline{Y}, Z, T, U, V) - (\nabla' P)(\overline{X}, \overline{Y}, Z, (\nabla F)(\overline{T}, U), V) \\
& - (\nabla' P)(\overline{X}, \overline{Y}, Z, (\nabla F)(\overline{T}, V), U) - P'(\overline{X}, \overline{Y}, Z, (\nabla\nabla F)(\overline{T}, U, V)) \\
& + a^2 (\nabla' P)((\nabla F)(X, U), \overline{Y}, Z, T, V) + a^2 (\nabla' P)((\nabla F)(X, V), \overline{Y}, Z, T, U)
\end{aligned}$$

$$+ a^2\,'P((\nabla\nabla F)(X, U, V), \overline{Y}, Z, T) + a^2(\nabla\,'P)(\overline{X}, (\nabla F)(Y, U), Z, T, V)$$

$$+ a^2(\nabla\,'P)(\overline{X}, (\nabla F)(Y, V), Z, T, U) + a^2\,'P(\overline{X}, (\nabla\nabla F)(Y, U, V), Z, T)$$

$$-\,'P((\nabla F)(X, U), \overline{Y}, Z, (\nabla F)(\overline{T}, V)) -\,'P((\nabla F)(X, V), \overline{Y}, Z, (\nabla F)(\overline{T}, U))$$

$$-\,'P(\overline{X}, (\nabla F)(Y, V), Z, (\nabla F)(\overline{T}, U)) -\,'P(\overline{X}, (\nabla F)(Y, U), Z, (\nabla F)(\overline{T}, V))$$

$$+ a^2\,'P((\nabla F)(X, U), (\nabla F)(Y, V), Z, T) + a^2\,'P((\nabla F)(X, V), (\nabla F)(Y,U), Z, T)$$

$$= a^2\,'B_2(U, V)\,'P(\overline{X}, \overline{Y}, Z, T).$$

Similarly we can obtain other equivalent condition applying structure on different vectors.

The HGF-structure metric manifold is said to be (1234)-recurrent in 'P, if

$$a^2 (\nabla\,'P)(X, \overline{Y}, \overline{Z}, \overline{T}, U) -\,'P((\nabla F)(\overline{X}, U), \overline{Y}, \overline{Z}, \overline{T}) + a^2\,'P(X, (\nabla F)(Y, U), \overline{Z}, \overline{T})$$

$$+ a^2\,'P(X, \overline{Y}, (\nabla F)(Z, U), \overline{T}) + a^2\,'P(X, \overline{Y}, \overline{Z}, (\nabla F)(T, U)) = a^2 B_1(U)\,'P(X, \overline{Y}, \overline{Z}, \overline{T}),$$

or equivalently

$$a^2 (\nabla\,'P)(\overline{X}, Y, \overline{Z}, \overline{T}, U) -\,'P(\overline{X}, (\nabla F)(\overline{Y}, U), \overline{Z}, \overline{T}) + a^2\,'P(\overline{X}, Y, \overline{Z}, (\nabla F)(T, U))$$

$$+ a^2\,'P((\nabla F)(X, U), Y, \overline{Z}, \overline{T}) + a^2\,'P(\overline{X}, Y, (\nabla F)(Z, U), \overline{T}) = a^2 B_1(U)\,'P(\overline{X}, Y, \overline{Z}, \overline{T}).$$

Similarly we can obtain other equivalent condition applying structure on different vectors.

and (1234)-birecurrent in 'P, if

$$a^2 (\nabla\nabla\,'P)(X, \overline{Y}, \overline{Z}, \overline{T}, U, V) - (\nabla'P)((\nabla F)(\overline{X}, U), \overline{Y}, \overline{Z}, \overline{T}, V)$$

$$- (\nabla\,'P)((\nabla F)(\overline{X}, V), \overline{Y}, \overline{Z}, \overline{T}, U) - P'((\nabla\nabla F)(\overline{X}, U, V), \overline{Y}, \overline{Z}, \overline{T})$$

$$+ a^2 (\nabla'P)(X, (\nabla F)(Y, U), \overline{Z}, \overline{T}, V) + a^2 (\nabla'P)(X, (\nabla F)(Y, V), \overline{Z}, \overline{T}, U)$$

$$+ a^2\,'P(X, (\nabla\nabla F)(Y, U, V), \overline{Z}, \overline{T}) + a^2(\nabla\,'P)(X, \overline{Y}, (\nabla F)(Z, U), \overline{T}, V)$$

$$+ a^2(\nabla\,'P)(X, \overline{Y}, (\nabla F)(Z, V), \overline{T}, U) + a^2\,'P(X, \overline{Y}, (\nabla\nabla F)(Z, U, V), \overline{T})$$

$$+ a^2(\nabla\,'P)(X, \overline{Y}, \overline{Z}, (\nabla F)(T, U), V) + a^2(\nabla\,'P)(X, \overline{Y}, \overline{Z}, (\nabla F)(T, V), U)$$

$$+ a^{2}\,{}'P(X, \bar{Y}, \bar{Z},(\nabla\nabla F)(T, U, V)) - {}'P((\nabla F)(\bar{X}, U), (\nabla F)(Y, V),\bar{Z}, \bar{T})$$

$$- {}'P((\nabla F)(\bar{X}, V), (\nabla F)(Y, U),\bar{Z}, \bar{T}) + a^{2}\,{}'P(X, \bar{Y}, \bar{Z},(\nabla\nabla F)(T, U, V))$$

$$- {}'P((\nabla F)(\bar{X}, U), (\nabla F)(Y, V),\bar{Z}, \bar{T}) - {}'P((\nabla F)(\bar{X}, V), (\nabla F)(Y, U),\bar{Z}, \bar{T})$$

$$- {}'P((\nabla F)(\bar{X}, V),\bar{Y}, \bar{Z},(\nabla F)(T, U)) + a^{2}\,{}'P(X,(\nabla F)(Y, U), (\nabla F)(Z, V), \bar{T})$$

$$+ a^{2}\,{}'P(X,(\nabla F)(Y, V), (\nabla F)(Z, U), \bar{T}) + a^{2}\,{}'P(X, (\nabla F)(Y, U), \bar{Z},(\nabla F)(T,V))$$

$$+ a^{2}\,{}'P(X, (\nabla F)(Y, V), \bar{Z},(\nabla F)(T,U)) + a^{2}\,{}'P(X,\bar{Y}, (\nabla F)(Z, U),(\nabla F)(T,V))$$

$$+ a^{2}\,{}'P(X,\bar{Y}, (\nabla F)(Z, V),(\nabla F)(T, U)) = a^{2}\,{}'B_{2}(U, V)\,{}'P(X, \bar{Y}, \bar{Z}, \bar{T}),$$

or equivalently

$$a^{2} (\nabla\nabla\,{}'P)(\bar{X}, Y, \bar{Z}, \bar{T}, U, V) - (\nabla{}'P)(\bar{X}, (\nabla F)(\bar{Y}, U), \bar{Z}, \bar{T}, V)$$

$$- (\nabla\,{}'P)(\bar{X}, (\nabla F)(\bar{Y}, V), \bar{Z}, \bar{T}, U) - P'(\bar{X},(\nabla\nabla F)(\bar{Y}, U,V), \bar{Z}, \bar{T})$$

$$+ a^{2} (\nabla{}'P)((\nabla F)(X, U),Y, \bar{Z}, \bar{T}, V) + a^{2} (\nabla{}'P)((\nabla F)(X, V),Y, \bar{Z}, \bar{T}, U)$$

$$+ a^{2}\,{}'P((\nabla\nabla F)(X, U, V), Y, \bar{Z}, \bar{T}) + a^{2}(\nabla\,{}'P)(\bar{X}, Y, (\nabla F)(Z, U), \bar{T}, V)$$

$$+ a^{2}(\nabla\,{}'P)(\bar{X}, Y, (\nabla F)(Z, V), \bar{T}, U) + a^{2}\,{}'P(\bar{X}, Y, (\nabla\nabla F)(Z, U, V),\bar{T})$$

$$+ a^{2}(\nabla\,{}'P)(\bar{X}, Y, \bar{Z}, (\nabla F)(T, U), V) + a^{2}(\nabla\,{}'P)(\bar{X}, Y, \bar{Z}, (\nabla F)(T, V), U)$$

$$+ a^{2}\,{}'P(\bar{X}, Y, \bar{Z},(\nabla\nabla F)(T, U, V)) - {}'P((\nabla F)(X, U), (\nabla F)(\bar{Y}, V),\bar{Z}, \bar{T})$$

$$- {}'P((\nabla F)(X, V), (\nabla F)(\bar{Y}, U),\bar{Z}, \bar{T}) - {}'P(\bar{X},(\nabla F)(\bar{Y}, U),(\nabla F)(Z, V),\bar{T})$$

$$- {}'P(\bar{X},(\nabla F)(\bar{Y}, V),(\nabla F)(Z, U),\bar{T}) - {}'P(\bar{X},(\nabla F)(\bar{Y}, U), \bar{Z},(\nabla F)(T, V))$$

$$- {}'P(\bar{X},(\nabla F)(\bar{Y}, V), \bar{Z},(\nabla F)(T, U)) + a^{2}\,{}'P((\nabla F)(X, U), Y,(\nabla F)(Z, V), \bar{T})$$

$$+ a^{2}\,{}'P((\nabla F)(X, V), Y,(\nabla F)(Z, U), \bar{T}) + a^{2}\,{}'P((\nabla F)(X, U), Y, \bar{Z},(\nabla F)(T,V))$$

$$+ a^{2}\,{}'P((\nabla F)(X, V), Y, \bar{Z},(\nabla F)(T, U)) + a^{2}\,{}'P(\bar{X},Y, (\nabla F)(Z, U),(\nabla F)(T,V))$$

$$+ a^{2}\,{}'P(\bar{X},Y, (\nabla F)(Z, V),(\nabla F)(T, U)) = a^{2}\,{}'B_{2}(U, V)\,{}'P(\bar{X}, Y, \bar{Z}, \bar{T}).$$

Similarly, we can obtain other equivalent condition applying structure on different vectors.

2. n-Recurrence and symmetry of different kinds

Definition (2.1). The HGF-structure metric manifold is said to be (4) n-recurrent in 'P, if

$$a^2 (\nabla_n \nabla_{n-1} \dots \nabla_2 \nabla_1 \,'P)(X, Y, Z, T, U_1, U_2, \dots, U_{n-1}, U_n)$$
$$-(\nabla_{n-1} \nabla_{n-2} \dots \nabla_2 \nabla_1 \,'P)(X, Y, Z, (\nabla_n F)(\bar{T}, U_n), U_1, U_2, \dots, U_{n-1})$$
$$-(\nabla_n \nabla_{n-2} \dots \nabla_2 \nabla_1 \,'P)(X, Y, Z, (\nabla_{n-1} F)(\bar{T}, U_{n-1}), U_1, U_2, \dots, U_{n-2}, U_n)$$
$$\dots\dots\dots$$
$$-(\nabla_n \nabla_{n-1} \dots \nabla_3 \nabla_2 \,'P)(X, Y, Z, (\nabla_1 F)(\bar{T}, U_1), U_2, U_3, \dots, U_n)$$
$$-(\nabla_{n-2} \nabla_{n-3} \dots \nabla_3 \nabla_2 \nabla_1 \,'P)(X, Y, Z, (\nabla_n \nabla_{n-1} F)(\bar{T}, U_{n-1}, U_n), U_1, U_2, \dots, U_{n-2})$$
$$\dots\dots\dots$$
$$-(\nabla_n \nabla_{n-1} \dots \nabla_4 \nabla_3 \,'P)(X, Y, Z, (\nabla_2 \nabla_1 F)(\bar{T}, U_1, U_2), U_3, \dots, U_n)$$
$$-(\nabla_{n-3} \nabla_{n-4} \dots \nabla_2 \nabla_1 \,'P)(X, Y, Z, (\nabla_n \nabla_{n-1} \nabla_{n-2} F)(\bar{T}, U_{n-2}, U_{n-1}, U_n), U_1, U_2, \dots, U_{n-3})$$
$$\dots\dots\dots$$
$$-(\nabla_n \nabla_{n-1} \dots \nabla_4 \,'P)(X, Y, Z, (\nabla_3 \nabla_2 \nabla_1 F)(\bar{T}, U_1, U_2, U_3), U_4, U_5 \dots, U_n)$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$-(\nabla_n \nabla_{n-1} \,'P)(X, Y, Z, (\nabla_{n-2} \dots \nabla_1 F)(\bar{T}, U_1, U_2, \dots U_{n-2}), U_{n-1}, U_n)$$
$$\dots\dots\dots$$
$$-(\nabla_2 \nabla_1 \,'P)(X, Y, Z, (\nabla_n \nabla_{n-1} \dots \nabla_3 F)(\bar{T}, U_3, \dots U_n), U_1, U_2)$$
$$-(\nabla_n \,'P)(X, Y, Z, (\nabla_{n-1} \dots \nabla_2 \nabla_1 F)(\bar{T}, U_1, U_2, \dots U_{n-1}), U_n)$$
$$\dots\dots\dots$$
$$-(\nabla_1 \,'P)(X, Y, Z, (\nabla_n \nabla_{n-1} \dots \nabla_3 \nabla_2 F)(\bar{T}, U_2, U_3, \dots U_{n-1}, U_n), U_1)$$
$$-\,'P(X, Y, Z, (\nabla_n \nabla_{n-1} \dots \nabla_2 \nabla_1 F)(\bar{T}, U_1, U_2, \dots, U_n))$$
$$= a^2 B_n (U_1, U_2, \dots, U_n) \,'P(X, Y, Z, T). \quad (2.1)$$

Where $B_n (U_1, U_2, \dots, U_n)$ is a non-vanishing C^∞ function.

Definition (2.2). The HGF-structure metric manifold is said to be (14) n-recurrent in 'P, if

$$a^2 (\nabla_n \nabla_{n-1} \ldots\ldots \nabla_2 \nabla_1 \,'P)(X, Y, Z, \bar{T}, U_1, U_2, \ldots\ldots, U_{n-1}, U_n)$$

$$-(\nabla_{n-1} \nabla_{n-2} \ldots\ldots \nabla_2 \nabla_1 \,'P)((\nabla_n F)(\bar{X}, U_n), Y, Z, \bar{T}, U_1, U_2, \ldots\ldots, U_{n-1})$$

$$+a^2(\nabla_{n-1} \nabla_{n-2} \ldots\ldots \nabla_2 \nabla_1 \,'P)(X, Y, Z, (\nabla_n F)(T, U_n), U_1, U_2, \ldots\ldots, U_{n-1})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_2 \,'P)((\nabla_1 F)(\bar{X}, U_1), Y, Z, \bar{T}, U_2, U_3, \ldots\ldots, U_n)$$

$$+a^2(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_3 \nabla_2 \,'P)(X, Y, Z, (\nabla_1 F)(T, U_1), U_2, U_3, \ldots\ldots, U_n)$$

$$-(\nabla_{n-2} \ldots\ldots \nabla_1 \,'P)((\nabla_n F)(\bar{X}, U_n), Y, Z, (\nabla_{n-1} F)(T, U_{n-1}), U_1, \ldots\ldots, U_{n-2})$$

$$-(\nabla_{n-2} \ldots\ldots \nabla_1 \,'P)((\nabla_{n-1} F)(\bar{X}, U_{n-1}), Y, Z, (\nabla_n F)(T, U_n), U_1, \ldots\ldots, U_{n-2})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_3 \,'P)((\nabla_2 F)(\bar{X}, U_2), Y, Z, (\nabla_1 F)(T, U_1), U_3, \ldots\ldots, U_n)$$

$$-(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_3 \,'P)((\nabla_1 F)(\bar{X}, U_1), Y, Z, (\nabla_2 F)(T, U_2), U_3, \ldots\ldots, U_n)$$

$$-(\nabla_{n-2} \ldots\ldots \nabla_2 \nabla_1 \,'P)((\nabla_n \nabla_{n-1} F)(\bar{X}, U_{n-1}, U_n), Y, Z, \bar{T}, U_1, U_2, \ldots\ldots, U_{n-2})$$

$$+a^2(\nabla_{n-2} \ldots\ldots \nabla_2 \nabla_1 \,'P)(X, Y, Z, (\nabla_n \nabla_{n-1} F)(\bar{T}, U_{n-1}, U_n), U_1, \ldots\ldots, U_{n-2})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_3 \,'P)((\nabla_2 \nabla_1 F)(\bar{X}, U_1, U_2), Y, Z, T, U_3, \ldots\ldots, U_n)$$

$$+a^2(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_3 \,'P)(X, Y, Z, (\nabla_2 \nabla_1 F)(T, U_1, U_2), U_3, \ldots\ldots, U_n)$$

$$-(\nabla_{n-3} \ldots\ldots \nabla_n \,'P)((\nabla_n F)(\bar{X}, U_n), Y, Z, (\nabla_{n-1} \nabla_{n-2} F)(T, U_{n-1}, U_n), U_1, \ldots\ldots, U_n)$$

$$-(\nabla_{n-3} \ldots\ldots \nabla_n \,'P)((\nabla_{n-1} \nabla_{n-2} F)(\bar{X}, U_{n-2}, U_{n-1}), Y, Z, (\nabla_n F), (\bar{T}, U_n), U_1, \ldots\ldots, U_{n-3})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_4 \,'P)((\nabla_1 F)(\bar{X}, U_1), Y, Z, (\nabla_3 \nabla_2 F)(T, U_2, U_3), U_4, \ldots\ldots, U_n)$$

$$-(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_4 \,'P)((\nabla_3 \nabla_2 F)(\bar{X}, U_2, U_3), Y, Z, (\nabla_1 F)(T, U_1), U_4, \ldots\ldots, U_n)$$

$$-(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_4 \,'P)((\nabla_3 \nabla_2 \nabla_1 F)(\bar{X}, U_1, U_2, U_3), Y, Z, \bar{T}, U_4, \ldots\ldots, U_n)$$

$$+a^2(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_4 \,'P)(X, Y, Z, (\nabla_3 \nabla_2 \nabla_1 F)(T, U_1, U_2, U_3), U_4, \ldots\ldots, U_n)$$

$$-(\nabla_{n-4} \ldots\ldots \nabla_1 \,'P)((\nabla_n \nabla_{n-1} F)(\bar{X}, U_{n-1}, U_n), Y, Z, (\nabla_{n-2} \nabla_{n-3} F), (\bar{T}, U_{n-3}, U_{n-2}), U_1, \ldots\ldots, U_{n-4})$$

$$-(\nabla_{n-4} \ldots\ldots \nabla_1 \,'P)((\nabla_{n-2} \nabla_{n-3} F)(\bar{X}, U_{n-3}, U_{n-2}), Y, Z, (\nabla_n \nabla_{n-1} F), (\bar{T}, U_{n-1}, U_n), U_1, \ldots\ldots, U_{n-4})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_5 \,'P)((\nabla_4 \nabla_3 F)(\bar{X}, U_3, U_4), Y, Z, (\nabla_2 \nabla_1 F), (T, U_1, U_2), U_5, \ldots\ldots, U_n)$$

$$-(\nabla_n \nabla_{n-1} \ldots\ldots \nabla_5 \,'P)((\nabla_2 \nabla_1 F)(\bar{X}, U_1, U_2), Y, Z, (\nabla_4 \nabla_3 F), (T, U_3, U_4), U_5, \ldots\ldots, U_{n-1}, U_n)$$

$$-(\nabla_{n-4}\ldots\ldots\nabla_1\,'P)((\nabla_n F)(\overline{X},U_n),Y,Z,(\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F)(T,U_{n-3},U_{n-2},U_{n-1}),U_1,\ldots\ldots,U_{n-4})$$

$$-(\nabla_{n-4}\ldots\ldots\nabla_1\,'P)((\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F)(\overline{X},U_{n-3},U_{n-2},U_{n-1}),Y,Z,(\nabla_n F)(T,U_n),U_1,\ldots\ldots,U_{n-4})$$

....

$$-(\nabla_n\ldots\ldots\ldots\ldots\nabla_5\,'P)((\nabla_1 F)(\overline{X},U_1),Y,Z,(\nabla_4\nabla_3\nabla_2 F)(T,U_2,U_3,U_4),U_5,\ldots\ldots\ldots\ldots,U_n)$$

$$-(\nabla_n\ldots\ldots\ldots\ldots\nabla_5\,'P)((\nabla_4\nabla_3\nabla_2 F)(\overline{X},U_2,U_3,U_4),Y,Z,(\nabla_1 F)(T,U_1),U_5,\ldots\ldots\ldots\ldots,U_n)$$

$$-(\nabla_{n-4}\ldots\ldots\ldots\nabla_1\,'P)((\nabla_n\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F)(\overline{X},U_{n-3},U_{n-2},U_{n-1},U_n),Y,Z,\overline{T},U_1,\ldots\ldots\ldots,U_{n-4})$$

$$+a^2(\nabla_{n-4}\ldots\ldots\ldots\nabla_1\,'P)(X,Y,Z,(\nabla_n\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F)(\overline{T},U_{n-3},U_{n-2},U_{n-1},U_n),U_1,\ldots\ldots\ldots,U_{n-4})$$

....

$$-(\nabla_n\ldots\ldots\ldots\ldots\ldots\ldots\nabla_5\,'P)((\nabla_4\nabla_3\nabla_2\nabla_1 F)(\overline{X},U_1,U_2,U_3,U_4),Y,Z,\overline{T},U_5,\ldots\ldots\ldots\ldots\ldots,U_n)$$

$$+a^2(\nabla_n\ldots\ldots\ldots\ldots\ldots\ldots\nabla_5\,'P)(X,Y,Z,(\nabla_4\nabla_3\nabla_2\nabla_1 F)(T,U_1,U_2,U_3,U_4),U_5,\ldots\ldots\ldots\ldots\ldots,U_n)$$

....

....

$$-(\nabla_n\,'P)((\nabla_{n-1}F)(\overline{X},U_{n-1}),Y,Z,(\nabla_{n-2}\ldots\ldots\ldots\ldots\ldots\nabla_1 F)(T,U_1,\ldots\ldots\ldots\ldots\ldots,U_{n-2}),U_n)$$

$$-(\nabla_n\,'P)((\nabla_{n-2}\ldots\ldots\ldots\ldots\ldots\nabla_1 F)(\overline{X},U_1,\ldots\ldots\ldots\ldots\ldots,U_{n-2}),Y,Z,(\nabla_{n-1}F)(T,U_{n-1}),U_n)$$

....

$$-(\nabla_1\,'P)((\nabla_2 F)(\overline{X},U_2),Y,Z,(\nabla_n\ldots\ldots\ldots\ldots\ldots\nabla_3 F)(T,U_3,\ldots\ldots\ldots\ldots\ldots,U_n),U_1)$$

$$-(\nabla_1\,'P)((\nabla_n\ldots\ldots\ldots\ldots\ldots\nabla_3 F)(\overline{X},U_3,\ldots\ldots\ldots\ldots\ldots,U_n),Y,Z,(\nabla_2 F)(T,U_2),U_1)$$

$$-\,'P((\nabla_n\nabla_{n-1}\ldots\ldots\ldots\ldots\ldots\nabla_1 F)(\overline{X},U_1,\ldots\ldots\ldots\ldots\ldots,U_{n-1},U_n),Y,Z,\overline{T})$$

$$+a^2\,'P(X,Y,Z,(\nabla_n\nabla_{n-1}\nabla_{n-2}\ldots\ldots\ldots\ldots\ldots\nabla_1 F)(T,U_1,U_2,\ldots\ldots\ldots\ldots\ldots,U_n)$$

$$=a^2 B_n(U_1,U_2,\ldots\ldots\ldots\ldots\ldots,U_n)\,'P(X,Y,Z,\overline{T}). \tag{2.2}$$

Barring X and T in the equation (2.1) we can obtain other form of the definition.

Similarly, we can define (12), (13), (23), (34), (24) n-recurrent HGF-metric manifold.

Definition (2.3). The HGF-structure metric manifold is said to be (124) n-recurrent in 'P, if

$$a^2(\nabla_n\nabla_{n-1}\ldots\ldots\ldots\ldots\ldots\nabla_2\nabla_1\,'P)(X,\overline{Y},Z,\overline{T},U_1,U_2,\ldots\ldots\ldots\ldots\ldots,U_{n-1},U_n)$$

$$-(\nabla_{n-1}\nabla_{n-2}\ldots\ldots\ldots\ldots\nabla_2\nabla_1\,'P)((\nabla_n F)(\overline{X},U_n),Y,Z,\overline{T},U_1,U_2,\ldots\ldots\ldots\ldots,U_{n-1})$$

$$+ a^2(\nabla_{n-1}\nabla_{n-2} \ldots\ldots \nabla_2\nabla_1 \,{}'P)(X,(\nabla_n F)(Y,U_n),Z,\overline{T},U_1,U_2, \ldots\ldots, U_{n-1})$$

$$+ a^2(\nabla_{n-1}\nabla_{n-2} \ldots\ldots \nabla_2\nabla_1 \,{}'P)(X,\overline{Y},Z,(\nabla_n F)(T,U_n),U_1,U_2, \ldots\ldots, U_{n-1})$$

$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- (\nabla_n\nabla_{n-1} \ldots\ldots \nabla_2 \,{}'P)((\nabla_1 F)(\overline{X},U_1),\overline{Y}, Z,\overline{T},U_2, U_3, \ldots\ldots, U_n)$$

$$+ a^2(\nabla_n\nabla_{n-1} \ldots\ldots \nabla_3\nabla_2 \,{}'P)(X,(\nabla_1 F)(Y,U_1),Z,\overline{T},U_2, U_3, \ldots\ldots, U_n)$$

$$+ a^2(\nabla_n\nabla_{n-1} \ldots\ldots \nabla_3\nabla_2 \,{}'P)(X,\overline{Y},Z,(\nabla_1 F)(T,U_1),U_2, U_3, \ldots\ldots, U_n)$$

$$- (\nabla_{n-2} \ldots\ldots \nabla_1 \,{}'P)((\nabla_n F)(\overline{X},U_n),(\nabla_{n-1} F)(Y,U_{n-1}),Z,\overline{T},U_1, \ldots\ldots, U_{n-2})$$

$$- (\nabla_{n-2} \ldots\ldots \nabla_1 \,{}'P)((\nabla_{n-1} F)(\overline{X},U_{n-1}),(\nabla_n F)(Y,U_n),Z,\overline{T},U_1, \ldots\ldots, U_{n-2})$$

$$- (\nabla_{n-2} \ldots\ldots \nabla_1 \,{}'P)((\nabla_n F)(\overline{X},U_n)\overline{Y},Z,(\nabla_{n-1} F)(T,U_{n-1}),U_1, \ldots\ldots, U_{n-2})$$

$$- (\nabla_{n-2} \ldots\ldots \nabla_1 \,{}'P)((\nabla_{n-1} F)(\overline{X},U_{n-1}),\overline{Y},Z,(\nabla_n F)(T,U_n),U_1, \ldots\ldots, U_{n-2})$$

$$+ a^2 (\nabla_{n-2} \ldots\ldots \nabla_2\nabla_1 \,{}'P)(X,(\nabla_n F)(Y,U_n),Z,(\nabla_{n-1} F)(\overline{T},U_{n-1}),U_1, \ldots\ldots, U_{n-2})$$

$$+ a^2 (\nabla_{n-2} \ldots\ldots \nabla_2\nabla_1 \,{}'P)(X,(\nabla_{n-1} F)(Y,U_{n-1}),Z,(\nabla_n F)(T,U_n),U_1, \ldots\ldots, U_{n-2})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- (\nabla_n\nabla_{n-1} \ldots\ldots \nabla_3 \,{}'P)((\nabla_1 F)(\overline{X},U_1),(\nabla_2 F)(Y,U_2),Z,\overline{T},U_3, \ldots\ldots, U_n)$$

$$- (\nabla_n\nabla_{n-1} \ldots\ldots \nabla_3 \,{}'P)((\nabla_2 F)(\overline{X},U_2),(\nabla_1 F)(Y,U_1),Z,\overline{T},U_3, \ldots\ldots, U_n)$$

$$- (\nabla_n\nabla_{n-1} \ldots\ldots \nabla_3 \,{}'P)((\nabla_1 F)(\overline{X},U_1),\overline{Y},Z,(\nabla_2 F)(T,U_2),U_3, \ldots\ldots, U_n)$$

$$- (\nabla_n\nabla_{n-1} \ldots\ldots \nabla_3 \,{}'P)((\nabla_2 F)(\overline{X},U_2),\overline{Y},Z,(\nabla_1 F)(T,U_1),U_3, \ldots\ldots, U_n)$$

$$+ a^2 (\nabla_n \ldots\ldots \nabla_3 \,{}'P)(X,(\nabla_1 F)(Y,U_1),Z,(\nabla_2 F)(T,U_2),U_3, \ldots\ldots, U_n)$$

$$+ a^2 (\nabla_n \ldots\ldots \nabla_3 \,{}'P)(X,(\nabla_2 F)(Y,U_2),Z,(\nabla_1 F)(T,U_1),U_3, \ldots\ldots, U_n)$$

$$- (\nabla_{n-2} \ldots\ldots \nabla_2\nabla_1 \,{}'P)((\nabla_n\nabla_{n-1} F)(\overline{X},U_{n-1},U_n),\overline{Y}, Z,\overline{T},U_1, , \ldots\ldots, U_{n-2})$$

$$+ a^2(\nabla_{n-2} \ldots\ldots \nabla_2\nabla_1 \,{}'P)(X,(\nabla_n\nabla_{n-1} F)(Y,U_{n-1}, U_n),Z,\overline{T},U_1, \ldots\ldots, U_{n-2})$$

$$+ a^2(\nabla_{n-2} \ldots\ldots \nabla_2\nabla_1 \,{}'P)(X,\overline{Y},Z,(\nabla_n\nabla_{n-1} F)(T,U_{n-1}, U_n),U_1, \ldots\ldots, U_{n-2})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- (\nabla_n \ldots\ldots \nabla_3 \,{}'P)((\nabla_2\nabla_1 F)(\overline{X},U_1,U_2),\overline{Y}, Z, \overline{T}, U_3, \ldots\ldots, U_n)$$

$$+ a^2 (\nabla_n \nabla_{n-1} \dots\dots \nabla_3 {}'P)(X,(\nabla_2 \nabla_1 F)(Y,U_1,U_2),Z,\overline{T}, U_3, \dots\dots, U_n)$$

$$+ a^2 (\nabla_n \nabla_{n-1} \dots\dots \nabla_3 {}'P)(X,\overline{Y},Z,(\nabla_2 \nabla_1 F)(T,U_1,U_2),U_3, \dots\dots, U_n)$$

$$- (\nabla_{n-3} \dots\dots \nabla_1 {}'P)((\nabla_n F)(\overline{X},U_n),(\nabla_{n-1} \nabla_{n-2} F)(Y,U_{n-2},U_{n-1}),Z,\overline{T},U_1, \dots\dots,U_{n-3})$$

$$- (\nabla_{n-3} \dots\dots \nabla_1 {}'P)((\nabla_{n-1} \nabla_{n-2} F)(\overline{X},U_{n-2},U_{n-1}),(\nabla_n F),(Y,U_n),Z,\overline{T},U_1,\dots\dots,U_{n-3})$$

$$- (\nabla_{n-3} \dots\dots \nabla_1 {}'P)((\nabla_n F)(\overline{X},U_n),\overline{Y},Z,(\nabla_{n-1} \nabla_{n-2} F)(T,U_{n-2},U_{n-1}),U_1, \dots\dots,U_{n-3})$$

$$- (\nabla_{n-3} \dots\dots \nabla_1 {}'P)((\nabla_{n-1} \nabla_{n-2} F)(\overline{X},U_{n-2},U_{n-1}),\overline{Y},Z,(\nabla_n F),(T,U_n),U_1,\dots\dots,U_{n-3})$$

$$+ a^2 (\nabla_{n-3} \dots\dots \nabla_1 {}'P)(X,(\nabla_n F)(Y,U_n),Z,(\nabla_{n-1} \nabla_{n-2} F)(T,U_{n-2},U_{n-1}),U_1, \dots\dots,U_{n-3})$$

$$+ a^2 (\nabla_{n-3} \dots\dots \nabla_1 {}'P)(X,(\nabla_{n-1} \nabla_{n-2} F)(Y,U_{n-2},U_{n-1}),Z,(\nabla_n F)(T,U_n),U_1, \dots\dots,U_{n-3})$$

....

$$- (\nabla_n \dots\dots \nabla_4 {}'P)((\nabla_1 F)(\overline{X},U_1),(\nabla_3 \nabla_2 F)(Y,U_2,U_3),Z,\overline{T},U_4, \dots\dots,U_n)$$

$$- (\nabla_n \dots\dots \nabla_4 {}'P)((\nabla_3 \nabla_2 F)(\overline{X},U_2,U_3),(\nabla_1 F)(Y,U_1),Z,\overline{T},U_4, \dots\dots,U_n)$$

$$- (\nabla_n \dots\dots \nabla_4 {}'P)((\nabla_1 F)(\overline{X},U_1),\overline{Y},Z,(\nabla_3 \nabla_2 F)(T,U_2,U_3),U_4, \dots\dots,U_n)$$

$$- (\nabla_n \dots\dots \nabla_4 {}'P)((\nabla_3 \nabla_2 F)(\overline{X},U_2,U_3),\overline{Y},Z,(\nabla_1 F)(T,U_1),U_4, \dots\dots,U_n)$$

$$+ a^2 (\nabla_n \nabla_{n-1} \dots\dots \nabla_4 {}'P)(X,(\nabla_1 F)(Y,U_1),Z,(\nabla_3 \nabla_2 F)(T,U_2,U_3),U_4, \dots\dots,U_n)$$

$$+ a^2 (\nabla_n \nabla_{n-1} \dots\dots \nabla_4 {}'P)(X,(\nabla_3 \nabla_2 F)(Y,U_2,U_3),Z,(\nabla_1 F)(T,U_1),U_4, \dots\dots,U_n)$$

$$- (\nabla_{n-3} \dots\dots \nabla_1 {}'P)((\nabla_n F)(\overline{X},U_n),(\nabla_{n-1} F)(Y,U_{n-1}),Z,(\nabla_{n-2} F)(T,U_{n-2}),U_1,\dots\dots,U_{n-3})$$

....

$$- (\nabla_n \dots\dots \nabla_4 {}'P)((\nabla_1 F)(\overline{X},U_1),(\nabla_2 F)(Y,U_2),Z,(\nabla_3 F)(T,U_3),U_4, \dots\dots,U_n)$$

$$- (\nabla_{n-3} \dots\dots \nabla_1 {}'P)((\nabla_n \nabla_{n-1} \nabla_{n-2} F)(\overline{X},U_{n-2},U_{n-1},U_n),\overline{Y},Z,\overline{T},U_4,\dots\dots,U_n)$$

$$+ a^2 (\nabla_{n-3} \dots\dots \nabla_1 {}'P)(X,(\nabla_n \nabla_{n-1} \nabla_{n-2} F)(Y,U_{n-2},U_{n-1},U_n),Z,\overline{T},U_4,\dots\dots,U_n)$$

$$+ a^2 (\nabla_{n-3} \dots\dots \nabla_1 {}'P)(X,\overline{Y},Z,(\nabla_n \nabla_{n-1} \nabla_{n-2} F)(T,U_{n-2},U_{n-1},U_n),U_4,\dots\dots,U_n)$$

....

$$- (\nabla_n \dots\dots \nabla_4 {}'P)((\nabla_3 \nabla_2 \nabla_1 F)(\overline{X},U_1,U_2,U_3),\overline{Y}, Z,\overline{T}, U_4, \dots\dots,U_n)$$

$$+ a^2 (\nabla_n \dots\dots \nabla_4 {}'P)(X,(\nabla_3 \nabla_2 \nabla_1 F)(Y,U_1,U_2,U_3),Z,\overline{T}, U_4,\dots\dots,U_n)$$

$$+ a^2 (\nabla_n \ldots\ldots \nabla_4 {}'P)(X,\overline{Y},Z,(\nabla_3\nabla_2\nabla_1 F)(T,U_1,U_2,U_3),U_4,\ldots\ldots,U_n)$$

$$-(\nabla_{n-4}\ldots\nabla_1 {}'P)((\nabla_n\nabla_{n-1}F)(\overline{X},U_{n-1},U_n),(\nabla_{n-2}\nabla_{n-3}F),(Y,U_{n-3},U_{n-2}),Z,\overline{T},U_1,\ldots,U_{n-4})$$

$$-(\nabla_{n-4}\ldots\nabla_1 {}'P)((\nabla_{n-2}\nabla_{n-3}F)(\overline{X},U_{n-3},U_{n-2}),(\nabla_n\nabla_{n-1}F),(Y,U_{n-1},U_n),Z,\overline{T},U_1,\ldots,U_{n-4})$$

$$-(\nabla_{n-4}\ldots\nabla_1 {}'P)((\nabla_n\nabla_{n-1}F)(\overline{X},U_{n-1},U_n),\overline{Y},Z,(\nabla_{n-2}\nabla_{n-3}F),(T,U_{n-3},U_{n-2}),U_1,\ldots,U_{n-4})$$

$$-(\nabla_{n-4}\ldots\nabla_1 {}'P)((\nabla_{n-2}\nabla_{n-3}F)(\overline{X},U_{n-3},U_{n-2}),\overline{Y},Z,(\nabla_n\nabla_{n-1}F),(T,U_{n-1},U_n),U_1,\ldots,U_{n-4})$$

$$+a^2(\nabla_{n-4}\ldots\nabla_1 {}'P)(X,(\nabla_n\nabla_{n-1}F)(Y,U_{n-1},U_n),Z,(\nabla_{n-2}\nabla_{n-3}F),(T,U_{n-3},U_{n-2}),U_1,\ldots,U_{n-4})$$

$$+a^2(\nabla_{n-4}\ldots\nabla_1 {}'P)(X,(\nabla_{n-2}\nabla_{n-3}F)(Y,U_{n-3},U_{n-2}),Z,(\nabla_n\nabla_{n-1}F),(T,U_{n-1},U_n),U_1,\ldots,U_{n-4})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(\nabla_n\ldots\nabla_5 {}'P)((\nabla_2\nabla_1F)(\overline{X},U_1,U_2),(\nabla_4\nabla_3F)(Y,U_3,U_4),Z,\overline{T},U_5,\ldots,U_{n-1},U_n)$$

$$-(\nabla_n\nabla_{n-1}\ldots\nabla_5 {}'P)((\nabla_4\nabla_3F)(\overline{X},U_3,U_4),(\nabla_2\nabla_1F)(Y,U_1,U_2),Z,\overline{T},U_5,\ldots,U_n)$$

$$-(\nabla_n\ldots\nabla_5 {}'P)((\nabla_2\nabla_1F)(\overline{X},U_1,U_2),\overline{Y},Z,(\nabla_4\nabla_3F)(T,U_3,U_4),U_5,\ldots,U_{n-1},U_n)$$

$$-(\nabla_n\nabla_{n-1}\ldots\nabla_5 {}'P)((\nabla_4\nabla_3F)(\overline{X},U_3,U_4),\overline{Y},Z,(\nabla_2\nabla_1F)(T,U_1,U_2),U_5,\ldots,U_n)$$

$$+a^2(\nabla_n\ldots\nabla_5 {}'P)(X,(\nabla_2\nabla_1F)(Y,U_1,U_2),Z,(\nabla_4\nabla_3F)(T,U_3,U_4),U_5,\ldots,U_n)$$

$$+a^2(\nabla_n\ldots\nabla_5 {}'P)(X,(\nabla_4\nabla_3F)(Y,U_3,U_4),Z,(\nabla_2\nabla_1F)(T,U_1,U_2),U_5,\ldots,U_n)$$

$$-(\nabla_n\nabla_{n-1}\ldots\nabla_5 {}'P)((\nabla_4\nabla_3F)(\overline{X},U_3,U_4),\overline{Y},Z,(\nabla_2\nabla_1F)(T,U_1,U_2),U_5,\ldots,U_n)$$

$$-(\nabla_{n-4}\ldots\nabla_1 {}'P)((\nabla_nF)(\overline{X},U_n),(\nabla_{n-1}F)(Y,U_{n-1}),Z,(\nabla_{n-2}\nabla_{n-3}F),(T,U_{n-3},U_{n-2}),U_1,\ldots,U_{n-4})$$

$$-(\nabla_{n-4}\ldots\nabla_1 {}'P)((\nabla_nF)(\overline{X},U_n),(\nabla_{n-2}\nabla_{n-3}F)(Y,U_{n-3}U_{n-2}),Z,(\nabla_{n-1}F),(T,U_{n-1}),U_1,\ldots,U_{n-4})$$

$$-(\nabla_{n-4}\ldots\nabla_1 {}'P)((\nabla_{n-2}\nabla_{n-3}F)(\overline{X},U_{n-3}U_{n-2}),(\nabla_nF)(Y,U_n),Z,(\nabla_{n-1}F),(T,U_{n-1}),U_1,\ldots,U_{n-4})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(\nabla_n\ldots\nabla_5 {}'P)((\nabla_1F)(\overline{X},U_1),(\nabla_2F)(Y,U_2),Z,(\nabla_4\nabla_3F),(T,U_3,U_4),U_5,\ldots,U_n)$$

$$-(\nabla_n\ldots\nabla_5 {}'P)((\nabla_1F)(\overline{X},U_1),(\nabla_4\nabla_3F),(Y,U_3,U_4),Z,(\nabla_2F)(T,U_2),U_5,\ldots,U_n)$$

$$-(\nabla_n\ldots\nabla_5 {}'P)((\nabla_4\nabla_3F)(\overline{X},U_3,U_4),(\nabla_1F)(Y,U_1),Z,(\nabla_2F)(T,U_2),U_5,\ldots,U_n)$$

$$-(\nabla_{n-4}\ldots\nabla_1 {}'P)((\nabla_nF)(\overline{X},U_n),(\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F),(Y,U_{n-3},U_{n-2},U_{n-1}),Z,\overline{T},U_1,\ldots,U_{n-4})$$

$$-(\nabla_{n-4}\ldots\ldots\nabla_1{}'P)((\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F)(\overline{X},U_{n-3},U_{n-2},U_{n-1}),(\nabla_nF),(Y,U_n),Z,\overline{T},U_1,\ldots\ldots,U_{n-4})$$

$$-(\nabla_{n-4}\ldots\ldots\nabla_1{}'P)((\nabla_nF)(\overline{X},U_n),\overline{Y},Z,(\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F),(T,U_{n-3},U_{n-2},U_{n-1}),U_1,\ldots\ldots,U_{n-4})$$

$$-(\nabla_{n-4}\ldots\ldots\nabla_1{}'P)((\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F)(\overline{X},U_{n-3},U_{n-2},U_{n-1}),\overline{Y},Z,(\nabla_nF),(T,U_n),U_1,\ldots\ldots,U_{n-4})$$

$$+a^2(\nabla_{n-4}\ldots\ldots\nabla_1{}'P)(X,(\nabla_nF)(Y,U_n),Z,(\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F),(T,U_{n-3},U_{n-2},U_{n-1}),U_1,\ldots\ldots,U_{n-4})$$

$$+a^2(\nabla_{n-4}\ldots\ldots\nabla_1{}'P)(X,(\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F)(Y,U_{n-3},U_{n-2},U_{n-1}),Z,(\nabla_nF),(T,U_n),U_1,\ldots\ldots,U_{n-4})$$

....

$$-(\nabla_n\ldots\ldots\nabla_5{}'P)((\nabla_1F)(\overline{X},U_1),(\nabla_4\nabla_3\nabla_2F)(Y,U_2,U_3,U_4),Z,\overline{T},U_5,\ldots\ldots,U_n)$$

$$-(\nabla_n\ldots\ldots\nabla_5{}'P)((\nabla_4\nabla_3\nabla_2F)(\overline{X},U_2,U_3,U_4),(\nabla_1F)(Y,U_1),Z,\overline{T},U_5,\ldots\ldots,U_n)$$

$$-(\nabla_n\ldots\ldots\nabla_5{}'P)((\nabla_1F)(\overline{X},U_1),\overline{Y},Z,(\nabla_4\nabla_3\nabla_2F)(T,U_2,U_3,U_4),U_5,\ldots\ldots,U_n)$$

$$-(\nabla_n\ldots\ldots\nabla_5{}'P)((\nabla_4\nabla_3\nabla_2F)(\overline{X},U_2,U_3,U_4),\overline{Y},Z,(\nabla_1F)(T,U_1),U_5,\ldots\ldots,U_n)$$

$$+a^2(\nabla_n\ldots\ldots\nabla_5{}'P)(X,(\nabla_1F)(Y,U_1),Z,(\nabla_4\nabla_3\nabla_2F)(T,U_2,U_3,U_4),U_5,\ldots\ldots,U_n)$$

$$+a^2(\nabla_n\ldots\ldots\nabla_5{}'P)(X,(\nabla_4\nabla_3\nabla_2F)(Y,U_2,U_3,U_4),Z,(\nabla_1F)(T,U_1)\,U_5,\ldots\ldots,U_n)$$

$$-(\nabla_{n-4}\ldots\ldots\nabla_1{}'P)((\nabla_n\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F)(\overline{X},U_{n-3},U_{n-2},U_{n-1},U_n),\overline{Y},Z,\overline{T},U_1,\ldots\ldots,U_{n-4})$$

$$+a^2(\nabla_{n-4}\ldots\ldots\nabla_1{}'P)(X,(\nabla_n\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F),(Y,U_{n-3},U_{n-2},U_{n-1},U_n),Z,\overline{T},U_1,\ldots\ldots,U_{n-4})$$

$$+a^2(\nabla_{n-4}\ldots\ldots\nabla_1{}'P)(X,\overline{Y},Z,(\nabla_n\nabla_{n-1}\nabla_{n-2}\nabla_{n-3}F)(T,U_{n-3},U_{n-2},U_{n-1},U_n),U_1,\ldots\ldots,U_{n-4})$$

....

$$-(\nabla_n\ldots\ldots\nabla_5{}'P)((\nabla_4\nabla_3\nabla_2\nabla_1F)(\overline{X},U_1,U_2,U_3,U_4),\overline{Y},Z,\overline{T},U_5,\ldots\ldots,U_n)$$

$$+a^2(\nabla_n\ldots\ldots\nabla_5{}'P)(X,(\nabla_4\nabla_3\nabla_2\nabla_1F)(Y,U_1,U_2,U_3,U_4),Z,\overline{T},U_5,\ldots\ldots,U_n)$$

$$+a^2(\nabla_n\ldots\ldots\nabla_5{}'P)(X,\overline{Y},Z,(\nabla_4\nabla_3\nabla_2\nabla_1F)(T,U_1,U_2,U_3,U_4),U_5,\ldots\ldots,U_n)$$

....

$$-(\nabla_n{}'P)((\nabla_{n-1}F)(\overline{X},U_{n-1}),(\nabla_{n-2}\ldots\ldots\nabla_1F)(Y,U_1,\ldots\ldots,U_{n-2}),Z,\overline{T},U_n)$$

$$-(\nabla_n{}'P)((\nabla_{n-2}\ldots\ldots\nabla_1F)(\overline{X},U_1,\ldots\ldots,U_{n-2}),(\nabla_{n-1}F)(Y,U_{n-1}),Z,\overline{T},U_n)$$

$$-(\nabla_n{}'P)((\nabla_{n-1}F)(\overline{X},U_{n-1}),\overline{Y},Z,(\nabla_{n-2}\ldots\ldots\nabla_1F)(T,U_1,\ldots\ldots,U_{n-2}),U_n)$$

$$-(\nabla_n{}'P)((\nabla_{n-2}\ldots\ldots\nabla_1F)(\overline{X},U_1,\ldots\ldots,U_{n-2}),\overline{Y},Z,(\nabla_{n-1}F)(T,U_{n-1}),U_n)$$

$+ a^2(\nabla_n{}'P)(X,(\nabla_{n-1}F)(Y,U_{n-1}),Z,(\nabla_{n-2}\ldots\ldots\nabla_1F)(T,U_1,\ldots\ldots,U_{n-2}),U_n)$

$+ a^2(\nabla_n{}'P)(X,(\nabla_{n-2}\ldots\ldots\nabla_1F)(Y,U_1,\ldots\ldots,U_{n-2}),Z,(\nabla_{n-1}F)(T,U_{n-1}),U_n)$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$-(\nabla_1{}'P)((\nabla_2F)(\overline{X},U_2),(\nabla_n\ldots\ldots\nabla_3F)(Y,U_3,\ldots\ldots,U_n),Z,\overline{T},U_1)$

$-(\nabla_1{}'P)((\nabla_n\ldots\ldots\nabla_3F)(\overline{X},U_3,\ldots\ldots,U_n),(\nabla_2F)(Y,U_2),Z,\overline{T},U_1)$

$-(\nabla_1{}'P)((\nabla_2F)(\overline{X},U_2),\overline{Y},Z,(\nabla_n\ldots\ldots\nabla_3F)(T,U_3,\ldots\ldots,U_n),U_1)$

$-(\nabla_1{}'P)((\nabla_n\ldots\ldots\nabla_3F)(\overline{X},U_3,\ldots\ldots,U_n),\overline{Y},Z,(\nabla_2F)(T,U_2),U_1)$

$+a^2(\nabla_1{}'P)(X,(\nabla_2F)(Y,U_2),Z,(\nabla_n\ldots\ldots\nabla_3F)(T,U_3,\ldots\ldots,U_n),U_1)$

$+a^2(\nabla_1{}'P)(X,(\nabla_n\ldots\ldots\nabla_3F)(Y,U_3,\ldots\ldots,U_n),Z,(\nabla_2F)(T,U_2),U_1)$

$-{}'P((\nabla_n\nabla_{n-1}\ldots\ldots\nabla_1F)(\overline{X},U_1,\ldots\ldots,U_{n-1},U_n),\overline{Y},Z,\overline{T})$

$+a^2\,{}'P(X,(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_1F)(Y,U_1,\ldots\ldots,U_{n-1},U_n),Z,\overline{T})$

$+a^2\,{}'P(X,\overline{Y},Z,(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_1F)(T,U_1,\ldots\ldots,U_{n-1},U_n))$

$$= a^2 B_n(U_1,\ldots\ldots,U_n)\,{}'P(X,\overline{Y},Z,\overline{T}). \tag{2.3}$$

Barring X and Y or X and T in the equation (2.3), we obtain the other forms of the definition.

Similarly we can define (123), (134), (234) and (1234) n-recurrent HGF-metric manifold.

Definition (2.4). The n-recurrent in 'P-(4), (14), (124) and (1234) n-recurrent HGF-structure manifold is said to be 'P-symmetric, if $B_n(U_1,\ldots\ldots,U_n) = 0$, in the equations (2.1), (2.2) and (2.3).

Similarly, we can state and prove theorems for (1), (2), (3),(12),(13),(23), (24), (34), (123), (234), (134) and (1234) n-recurrent HGF-structure be 'P-symmetric.

Theorem (2.1). In the (4) n-recurrent HGF-structure manifold, if any two of the following conditions hold for the same recurrence parameter, then the third also holds :

(a) It is associated Pseudo concircular (4)n-recurrent,

(b) It is associated Pseudo conharmonic (4)n-recurrent,

(c) It is associated Pseudo conformal (4)n-recurrent.

Proof. From the equations (1.27), (1.28) and (1.29), we have

$$'L^*(X,Y,Z,T) = 'C^*(X,Y,Z,T) + \frac{n}{(n-2)}\{'K(X,Y,Z,T) - 'V^*(X,Y,Z,T)\}. \quad (2.4)$$

Barring T in equation (2.6), we have

$$'L^*(X,Y,Z,T) = 'C^*(X,Y,Z,\overline{T}) + \frac{n}{(n-2)}\{'K(X,Y,Z,\overline{T}) - 'V^*(X,Y,Z,\overline{T})\}. \quad (2.5)$$

Now from equations (1.1)(a) and (2.5), we have

$$a^2 B_n (U_1, U_2, \ldots\ldots\ldots, U_n)\, 'L^*(X,Y,Z,T) = a^2 B_n (U_1,U_2,\ldots\ldots\ldots, U_n)\, 'C^*(X,Y,Z,T)$$
$$+\ a^2 B_n (U_1,U_2,\ldots\ldots\ldots, U_n)\frac{n}{(n-2)}\{'K(X,Y,Z,T) - 'V^*(X,Y,Z,T)\}. \quad (2.6)$$

Differentiating (2.5) w.r.t. $U_1, U_2, U_3, \ldots, U_{n-1}$ and U_n.

Using the equation (2.5) and then barring T in the resulting equation, we get

$$a^2(\nabla_n\nabla_{n-1}\ldots\ldots\ldots\ldots\nabla_2\nabla_1{}'L^*)(X,Y,Z,T,U_1,U_2,\ldots\ldots\ldots,U_{n-1},U_n)$$
$$-(\nabla_{n-1}\ldots\ldots\ldots\ldots\nabla_2\nabla_1{}'L^*)(X,Y,Z,(\nabla_n F)(\overline{T},U_n),U_1,U_2,\ldots\ldots\ldots,U_{n-1})$$
$$-(\nabla_n\nabla_{n-2}\ldots\ldots\ldots\ldots\nabla_2\nabla_1{}'L^*)(X,Y,Z,(\nabla_{n-1}F)(\overline{T},U_{n-1}),U_1,U_2,\ldots\ldots\ldots,U_n)$$
$$-(\nabla_n\nabla_{n-1}\nabla_{n-3}\ldots\ldots\ldots\nabla_2\nabla_1{}'L^*)(X,Y,Z,(\nabla_{n-2}F)(\overline{T},U_{n-2}),U_1,U_2,\ldots\ldots,U_{n-3},U_{n-1},U_n)$$

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....

$$-(\nabla_n\nabla_{n-1}\ldots\ldots\ldots\ldots\nabla_3\nabla_2{}'L^*)(X,Y,Z,(\nabla_1 F)(\overline{T},U_1),U_2,\ldots\ldots\ldots,U_{n-1},U_n)$$
$$-(\nabla_{n-2}\ldots\ldots\ldots\ldots\nabla_1{}'L^*)(X,Y,Z,(\nabla_n\nabla_{n-1}F)(\overline{T},U_{n-1},U_n),U_1,\ldots\ldots\ldots,U_{n-2})$$

....

....

$$-(\nabla_n\ldots\ldots\ldots\ldots\nabla_3{}'L^*)(X,Y,Z,(\nabla_2\nabla_1 F)(\overline{T},U_1,U_2),U_3,\ldots\ldots\ldots,U_n)$$
$$-(\nabla_{n-3}\ldots\ldots\ldots\nabla_1{}'L^*)(X,Y,Z,(\nabla_n\nabla_{n-1}\nabla_{n-2}F)(\overline{T},U_{n-2},U_{n-1},U_n),U_1,\ldots\ldots\ldots U_{n-3})$$

....

$$-(\nabla_n\ldots\ldots\ldots\ldots\nabla_4{}'L^*)(X,Y,Z,(\nabla_3\nabla_3\nabla_1 F)(\overline{T},U_1,U_2,U_3),U_4,\ldots\ldots\ldots,U_n)$$

....

....

$$-(\nabla_n{}'L^*)(X,Y,Z,(\nabla_{n-1}\nabla_{n-2}\ldots\ldots\ldots\nabla_2\nabla_1 F)(\overline{T},U_1,U_2,\ldots\ldots\ldots,U_{n-1}),U_n)$$

$$
\begin{aligned}
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-(\nabla_1{}'L^*)(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2F)(\bar{T},U_2,\cdots\cdots U_{n-1},U_n),U_1)\\
&-{}'L^*(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1F)(\bar{T},U_1,U_2,\cdots\cdots,U_{n-1},U_n)\\
&=a^2(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1{}'C^*)(X,Y,Z,T,U_1,U_2,\cdots\cdots,U_{n-1},U_n)\\
&-(\nabla_{n-1}\nabla_{n-2}\cdots\cdots\nabla_2\nabla_1{}'C^*)(X,Y,Z,(\nabla_nF)(\bar{T},U_n),U_1,U_2,\cdots\cdots,U_{n-1})\\
&-(\nabla_n\nabla_{n-2}\cdots\cdots\nabla_2\nabla_1{}'C^*)(X,Y,Z,(\nabla_{n-1}F)(\bar{T},U_{n-1}),U_1,U_2,\cdots\cdots,U_n)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_3\nabla_2{}'C^*)(X,Y,Z,(\nabla_1F)(\bar{T},U_1),U_2,U_3,\cdots\cdots,U_n)\\
&-(\nabla_{n-2}\cdots\cdots\nabla_1{}'C^*)(X,Y,Z,(\nabla_n\nabla_{n-1}F)(\bar{T},U_{n-1},U_n),U_1,\cdots\cdots U_{n-2})\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-(\nabla_n\cdots\cdots\nabla_3{}'C^*)(X,Y,Z,(\nabla_2\nabla_1F)(\bar{T},U_1,U_2),U_3\cdots\cdots,U_n)\\
&-(\nabla_{n-3}\cdots\cdots\nabla_1{}'C^*)(X,Y,Z,(\nabla_n\nabla_{n-1},\nabla_{n-2}F)(\bar{T},U_{n-2},U_{n-1},U_n),U_1\cdots\cdots,U_{n-3})\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_4{}'C^*)(X,Y,Z,(\nabla_3\nabla_2\nabla_1F)(\bar{T},U_1,U_2,U_3),U_4,\cdots\cdots,U_n)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-(\nabla_n{}'C^*)(X,Y,Z,(\nabla_{n-1}\nabla_{n-2}\cdots\cdots\nabla_2\nabla_1F)(\bar{T},U_1,U_2,\cdots\cdots,U_{n-1}),U_n)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-(\nabla_1{}'C^*)(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_3\nabla_2F)(\bar{T},U_2,U_3\cdots\cdots,U_n),U_1)\\
&-{}'C^*(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1F)(\bar{T},U_1,U_2,\cdots\cdots,U_{n-1}),U_n)\\
&+\frac{n}{(n-2)}a^2(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1{}'K)(X,Y,Z,T,U_1,U_2,\cdots\cdots U_{n-1},U_n)\\
&-(\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1{}'K)(X,Y,Z,(\nabla_nF)(\bar{T},U_n),U_1,U_2,\cdots\cdots,U_{n-1})\\
&-(\nabla_n\nabla_{n-2}\cdots\cdots\nabla_2\nabla_1{}'K)(X,Y,Z,(\nabla_{n-1}F)(\bar{T},U_{n-1}),U_1,U_2,\cdots\cdots U_{n-2},U_n)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_3\nabla_2{}'K)(X,Y,Z,(\nabla_1F)(\bar{T},U_1),U_2,\cdots\cdots,U_n)\\
&-(\nabla_{n-2}\cdots\cdots\nabla_2\nabla_1{}'K)(X,Y,Z,(\nabla_n\nabla_{n-1}F)(\bar{T},U_{n-1},U_n),U_1,U_2,\cdots\cdots U_{n-2})
\end{aligned}
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_3{}'K)(X,Y,Z,(\nabla_2\nabla_1F)(\overline{T},U_1,U_2),U_3,\cdots\cdots,U_n)$$

$$-(\nabla_{n-3}\cdots\cdots\nabla_1{}'K)(X,Y,Z,(\nabla_n\nabla_{n-1}\nabla_{n-2}F)(\overline{T},U_{n-2},U_{n-1},U_n),U_1,U_2,\cdots\cdots U_{n-3})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-(\nabla_n{}'K)(X,Y,Z,(\nabla_{n-1}\nabla_{n-2}\cdots\cdots\nabla_2\nabla_1F)(\overline{T},U_1,U_2,\cdots\cdots,U_{n-1}),U_n)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-(\nabla_1{}'K)(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_3\nabla_2F)(\overline{T},U_2,U_3\cdots\cdots,U_n),U_1)$$

$$-{}'K(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1F)(\overline{T},U_1,U_2,\cdots\cdots,U_{n-1},U_n))$$

$$-a^2(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1{}'V^*)(X,Y,Z,T,U_1,U_2,\cdots\cdots U_{n-1},U_n)$$

$$+{}'V^*(\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1{}'V^*)(X,Y,Z,(\nabla_nF)(\overline{T},U_n),U_1,U_2,\cdots\cdots U_{n-1})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(\nabla_n\cdots\cdots\nabla_2{}'V^*)(X,Y,Z,(\nabla_1F)(\overline{T},U_1),U_2,\cdots\cdots,U_n)$$

$$+(\nabla_{n-2}\cdots\cdots\nabla_1{}'V^*)(X,Y,Z,(\nabla_n\nabla_{n-1}F)(\overline{T},U_{n-1},U_n),U_1,\cdots\cdots,U_{n-2})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_3{}'V^*)(X,Y,Z,(\nabla_2\nabla_1F)(\overline{T},U_1,U_2),U_3\cdots\cdots,U_n)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(\nabla_n{}'V^*)(X,Y,Z,(\nabla_{n-1}\cdots\cdots\nabla_1F)(\overline{T},U_1,U_2,\cdots\cdots,U_{n-1}),U_n)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(\nabla_1{}'V^*)(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_3\nabla_2F)(\overline{T},U_2,U_3\cdots\cdots,U_{n-1},U_n),U_1)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+{}'V^*(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1F)(\overline{T},U_1,U_2,\cdots\cdots,U_n)) \tag{2.7}$$

Subtracting (2.6) from (2.7), we have

$$a^2(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1{}'L^*)(X,Y,Z,T,U_1,U_2,\cdots\cdots U_{n-1},U_n)$$

$$-(\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1{}'L^*)(X,Y,Z,(\nabla_nF)(\overline{T},U_n),U_1,U_2,\cdots\cdots U_{n-1})$$

$-(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1{}'L^*)(X,Y,Z,(\nabla_{n-1}F)(\overline{T},U_{n-1}),U_1,U_2,\ldots\ldots U_n)$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$-(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2{}'L^*)(X,Y,Z,(\nabla_1F)(\overline{T},U_1),U_2,\ldots\ldots,U_n)$

$-(\nabla_{n-2}\ldots\ldots\nabla_1{}'L^*)(X,Y,Z,(\nabla_n\nabla_{n-1}F)(\overline{T},U_{n-1},U_n),U_1,\ldots\ldots U_{n-2})$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$-(\nabla_n{}'L^*)(X,Y,Z,(\nabla_{n-1}\nabla_{n-2}\ldots\ldots\nabla_2\nabla_1F)(\overline{T},U_1,U_2,\ldots\ldots,U_{n-1}),U_n)$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$-(\nabla_1{}'L^*)(X,Y,Z,(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2F)(\overline{T},U_2,\ldots\ldots,U_n),U_1)$

$-{}'L^*(X,Y,Z,(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1F)(\overline{T},U_1,U_2,\ldots\ldots,U_{n-1},U_n)$

$-a^2B_n((\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1)'L^*(X,Y,Z,T,U_1,U_2,\ldots\ldots,U_{n-1},U_n)$

$=a^2(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1{}'C^*)(X,Y,Z,T,U_1,U_2,\ldots\ldots U_{n-1},U_n)$

$-(\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1{}'C^*)(X,Y,Z,(\nabla_nF)(\overline{T},U_n),U_1,U_2,\ldots\ldots U_{n-1})$

$-(\nabla_n\nabla_{n-2}\ldots\ldots\nabla_1{}'C^*)(X,Y,Z,(\nabla_{n-1}F)(\overline{T},U_{n-1}),U_1,U_2,\ldots\ldots U_{n-2},U_n)$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$-(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2{}'C^*)(X,Y,Z,(\nabla_1F)(\overline{T},U_1),U_2,\ldots\ldots U_{n-1},U_n)$

$-(\nabla_{n-2}\ldots\ldots\nabla_2\nabla_1{}'C^*)(X,Y,Z,(\nabla_n\nabla_{n-1}F)(\overline{T},U_{n-1},U_n),U_1,U_2\ldots\ldots U_{n-2})$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$-(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_3{}'C^*)(X,Y,Z,(\nabla_2\nabla_1F)(\overline{T},U_1,U_2),U_3\ldots\ldots,U_n)$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$-(\nabla_n{}'C^*)(X,Y,Z,(\nabla_{n-1}\nabla_{n-2}\ldots\ldots\nabla_2\nabla_1F)(\overline{T},U_1,U_2,\ldots\ldots,U_{n-1}),U_n)$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$-(\nabla_1{}'C^*)(X,Y,Z,(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_3\nabla_2F)(\overline{T},U_2,U_3\ldots\ldots,U_n),U_1)$

$-{}'C^*(X,Y,Z,(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1F)(\overline{T},U_1,U_2,\ldots\ldots,U_{n-1},U_n))$

$-a^2B_n(U_1,U_2,\ldots\ldots,U_{n-1},U_n)'C^*(X,Y,Z,T)$

$=\frac{n}{(n-2)}a^2(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1{}'K)(X,Y,Z,T,U_1,U_2,\ldots\ldots U_{n-1},U_n)$

$$-(\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1{}'K)(X,Y,Z,(\nabla_n F)(\bar{T},U_{n-1}),U_1,U_2,\cdots\cdots U_{n-2},U_1)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2{}'K)(X,Y,Z,(\nabla_1 F)(\bar{T},U_1),U_2,\cdots\cdots U_{n-1},U_n)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-(\nabla_n{}'K)(X,Y,Z,(\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1 F)(\bar{T},U_1,U_2,\cdots\cdots,U_{n-1}),U_n)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-(\nabla_1{}'K)(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1 F)(\bar{T},U_1,U_2\cdots\cdots,U_{n-1},U_n)$$

$$-{}'K(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1 F)(\bar{T},U_1,U_2,\cdots\cdots,U_{n-1},U_n)$$

$$-a^2(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1{}'V^*)(X,Y,Z,T,U_1,U_2,\cdots\cdots U_{n-1},U_n)$$

$$+(\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1{}'V^*)(X,Y,Z,(\nabla_n F)(\bar{T},U_n),U_1,U_2,\cdots\cdots U_{n-1})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(\nabla_n\cdots\cdots\nabla_2{}'V^*)(X,Y,Z,(\nabla_1 F)(\bar{T},U_1),U_2,\cdots\cdots,U_n)$$

$$+(\nabla_{n-2}\cdots\cdots\nabla_1{}'V^*)(X,Y,Z,(\nabla_n\nabla_{n-1}F)(\bar{T},U_{n-1},U_n),U_1,\cdots\cdots,U_{n-2})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_3{}'V^*)(X,Y,Z,(\nabla_2\nabla_1 F)(\bar{T},U_1,U_2),U_3\cdots\cdots,U_n)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(\nabla_n{}'V^*)(X,Y,Z,(\nabla_{n-1}\cdots\cdots\nabla_1 F)(\bar{T},U_1,U_2,\cdots\cdots,U_{n-1}),U_n)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(\nabla_1{}'V^*)(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_3\nabla_2 F)(\bar{T},U_2,U_3\cdots\cdots,U_{n-1},U_n),U_1)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+{}'V^*(X,Y,Z,(\nabla_n\nabla_{n-1}\cdots\cdots\nabla_2\nabla_1 F)(\bar{T},U_1,U_2,\cdots\cdots,U_n))$$

$$+a^2 B_n(U_1,U_2,\cdots\cdots\cdots\cdots,U_{n-1},U_n){}'V^*(X,Y,Z,T). \tag{2.8}$$

Now, if the (4) n-recurrent HGF-metric manifold is associated Pseudo concircular (4) n-recurrent and associated Pseudo conformal (4) n-recurrent for the same recurrence parameter, then from equation (2.10), we get

$$a^2(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1{}'L^*)(X,Y,Z,T,U_1,U_2,\ldots\ldots,U_{n-1},U_n)$$
$$-(\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1{}'L^*)(X,Y,Z,(\nabla_nF)(\overline{T},U_n),U_1,U_2,\ldots\ldots,U_{n-1})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$-(\nabla_n\ldots\ldots\nabla_2{}'L^*)(X,Y,Z,(\nabla_1F)(\overline{T},U_1),U_2,\ldots\ldots,U_n)$$
$$-(\nabla_{n-2}\ldots\ldots\nabla_1{}'L^*)(X,Y,Z,(\nabla_n\nabla_{n-1}F)(\overline{T},U_{n-1},U_n),U_1,\ldots\ldots U_{n-2})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_3{}'L^*)(X,Y,Z,(\nabla_2\nabla_1F)(\overline{T},U_1,U_2),U_3,\ldots\ldots,U_n)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$-(\nabla_n{}'L^*)(X,Y,Z,(\nabla_{n-1}\ldots\ldots\nabla_1F)(\overline{T},U_1,U_2,\ldots\ldots,U_{n-1}),U_n)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$-(\nabla_1{}'L^*)(X,Y,Z,(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_3\nabla_2F)(\overline{T},U_2,U_3,\ldots\ldots,U_{n-1},U_n),U_1)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$-{}'L^*(X,Y,Z,(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1F)(\overline{T},U_1,U_2,\ldots\ldots,U_n))$$
$$=a^2B_n(U_1,U_2,\ldots\ldots,U_n)\,{}'L^*(X,Y,Z,T). \qquad (2.9)$$

which shows that the manifold is associated Pseudo conharmonic (4) n-recurrent.

Similarly, it can be shown that if the (4) n-recurrent HGF-metric manifold is either associated Pseudo conharmonic (4) n-recurrent and associated concircular (4) n-recurrent or associated Pseudo conharmonic (4) n-recurrent and associated Pseudo conformal (4) n-recurrent then it is either associated Pseudo conformal (4) n-recurrent or associated Pseudo concircular (4) n-recurrent.

Theorem (2.2). In the (4) n-recurrent-symmetric HGF-structure metric manifold, if any two of the following conditions hold for the some recurrence parameter, then the third also holds :

(a) It is associated Pseudo concircular (4) n-recurrent symmetric,

(b) It is associated Pseudo conharmonic (4) n-recurrent symmetric,

(c) It is associated Pseudo conformal (4) n-recurrent symmetric.

Proof. Let the (4) n-recurrent symmetric HGF-structure metric manifold be associated Pseudo concircular (4) n-recurrent symmetric and associated Pseudo conformal (4) n-recurrent symmetric for the same n-recurrence parameter then from equation (2.9), we have

$$a^2(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1{'L^*})(X,Y,Z,T,U_1,U_2,\ldots\ldots,U_{n-1},U_n)$$

$$-(\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1{'L^*})(X,Y,Z,(\nabla_nF)(\overline{T},U_n),U_1,U_2,\ldots\ldots,U_{n-1})$$

$$-(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1{'L^*})(X,Y,Z,(\nabla_{n-1}F)(\overline{T},U_{n-1}),U_1,U_2,\ldots\ldots,U_{n-1})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(\nabla_n\ldots\ldots\nabla_2{'L^*})(X,Y,Z,(\nabla_1F)(\overline{T},U_1),U_2,\ldots\ldots,U_n)$$

$$-(\nabla_{n-2}\ldots\ldots\nabla_1{'L^*})(X,Y,Z,(\nabla_n\nabla_{n-1}F)(\overline{T},U_{n-1},U_n),U_1,\ldots\ldots,U_{n-2})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_3{'L^*})(X,Y,Z,(\nabla_2\nabla_1F)(\overline{T},U_1,U_2),U_3,\ldots\ldots,U_n)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(\nabla_n{'L^*})(X,Y,Z,(\nabla_{n-1}\ldots\ldots\nabla_1F)(\overline{T},U_1,U_2,\ldots\ldots,U_{n-1}),U_n)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(\nabla_1{'L^*})(X,Y,Z,(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_3\nabla_2F)(\overline{T},U_2,U_3,\ldots\ldots,U_{n-1},U_n),U_1)$$

$$-{'L^*}(X,Y,Z,(\nabla_n\nabla_{n-1}\ldots\ldots\nabla_2\nabla_1F)(\overline{T},U_1,U_2,\ldots\ldots,U_n))$$

$$=0,$$

which shows that the manifold is associated Pseudo conharmonic (4) n-recurrent symmetric.

Similarly, it can be shown that the (4) n-recurrent symmetric HGF-structure metric manifold is either associated Pseudo conharmonic (4) n-recurrent symmetric and associated Pseudo concircular (4) n-recurrent symmetric or associated Pseudo conharmonic (4) n-recurrent symmetric and associated Pseudo conformal (4) n-recurrent symmetric then it is either associated Pseudo conformal (4) n-recurrent symmetric or associated Pseudo concircular (4) n-recurrent symmetric.

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