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A Study of Nuclear Operators, Riesz Operators and their Application in the Eigen Values of Linear Operators

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Abstract

In the present paper we study the concept of Nuclear operators, Reisz operators in context with eigen values of Linear Operators. We obtained the results with the help of these operators in various context. Some important theorems related to the topics, are also the part of the study.

Keywords : Riesz operator, Quasi-Banach operator, Canonical operator.

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1. Nuclear Operators

Definition. An operator $T \in L(E, F)$ is said to be P-nuclear if there exists a so called P-nuclear representation $T = \sum_{i=1}^{\infty} a_i \otimes Y_i$ such that $a_1, a_2, ... \in E$, and $Y_1, Y_2, ... \in F$ and $(\|a_i\| \, \|Y_i\|) \in l_p$. The set of these operators $\beta_p(E, F)$. Then, $\|T \mid \beta_p\|$ $:= \inf \left(\sum_{i=1}^{\infty} \|a_i\|^p \, \|Y_i\|^p \right)^{1/p}$.

Preposition. The Quasi-Banach operator ideal β_p is approximative. More precisely, $\beta_p = G \circ \beta_p \circ G$.

Proof. Let $T \in \beta_p(E, F)$ and a factorization $T = YD_t A$. The sequence $t \in l_p$ is $t = y\, s\, a$ such that $S \in l_p$ and $a, Y \in Co$. Then it follows that $T = (YD_y) D_s (D_a A)$, where $(D_a A) \in G(E, l_\infty)$, $D_s \in \beta_p(l_\infty, l_1)$ and $YD_y \in G(l_1, F)$. Hence, $T \in G \circ \beta_p \circ G(E, F)$.

2. Nuclear Representation

Definition. An operator $T \in L(E, F)$ is said to be (r, 2)-nuclear if there exists a so-called (r, 2)-nuclear representation. $T = \sum_{i=1}^{\infty} a_i \otimes Y_i$ such that $(a_i) \in [l_r, E]$ and $(Y_1) \in (w_2, F)$. The set of these operators is $\beta_{r, 2}(E, F)$, then

$$\| T \mid \beta_{r, 2} \| := \inf \{\| (a_i) \mid l_r \| \, \| (y_i) \mid w_2 \|\}.$$

Theorem 1. The Quasi-Banach operator ideal $\beta_{r, 2}$ is stable with respect to the tensor norm ε. More precisely, $\| S \tilde{\otimes}_\varepsilon T \mid \beta_{r, 2}\| \le \|S \mid \beta_{r, 2}\| \, \|T \mid \beta_{r, 2}\|$ for $S \in \beta_{r, 2}(E, E_0)$ and $T \in \beta_{r, 2}(F, F_0)$.

Proof. Given $\delta > 0$, then $S = \sum_{i=1}^{\infty} a_i \otimes X_i$ and $T = \sum_{i=1}^{\infty} b_j \otimes Y_j$ such that

$$\|(a_i) \mid [l_r, E)\| \le S \mid \beta_{r, 2}\| \text{ and } \|(b_j) \mid [l_r, F) \| \le \|T \mid \beta_{r, 2}\|.$$

$$\| (x_i) \mid [w_2, E_0] \| \le 1 + \delta \text{ and } \|(Y_j) \mid [w_2, F_0] \| \le 1 + \delta.$$

Obviously,
$$\|(a_i \otimes b_j) \mid [l_r, (N \times N), (E \tilde{\otimes}_\varepsilon F)] \|$$

$$\le \|(a_i) \mid [l_r, E)\| \, \| (b_j) \mid [l_r, F] \|$$

$$\le \|S \mid \beta_{r, 2}\| \;\; \|T \mid \beta_{r, 2}\|.$$

Furthermore, for all $(Y_{ij}) \in l_2 (N \times N)$, then

$$\| \sum_{i=1}^{\infty} = \sum_{j=1}^{\infty} Y_{ij} X_i \otimes Y_i \|$$

$$= \sup \{| \sum_{i=1}^{\infty} = \sum_{j=1}^{\infty} Y_{ij} <X_i, a> <Y_i, b > | : \|(a)\| \le 1, \|(b)\| \le 1\}$$

$$\le (1 + \delta)^2 (\sum_{i=1}^{\infty} = \sum_{j=1}^{\infty} |Y_{ij}|^2)^{1/2}.$$

Thus the double sequence $(x_i \tilde{\otimes} Y_j)$ is weakly 2-summable in $E_0 \tilde{\otimes}_\varepsilon F_0$ and $\|(x_i \otimes y_j) \mid [w_2 (N \times N), (E_0 \tilde{\otimes}_\varepsilon F_0] \| \le (1 + \delta)^2$. Therefore, $S \tilde{\otimes}_\varepsilon T = \sum_{i=1}^{\infty} = \sum_{j=1}^{\infty} (a_i \otimes b_j) \otimes (x_i \otimes Y_j)$ is an (r, 2)-nuclear operator. Moreover, letting $\delta \to 0$, yields

$$\| S \tilde{\otimes}_\varepsilon T \mid \beta_{r,2} \| \le \|S \mid \beta_{r,2}\| \, \|T \mid \beta_{r,2}\|.$$

Hence the proof is complete.

Theorem 2. The norms on both sides are equal.

Proof. Let $T \in \beta_2 (E, F)$. Given $\varepsilon > 0$, there exists $T = \sum_{i=1}^{\infty} a_i \otimes Y_i$ such that $\|(a_i) \mid l_2\| \le (1 + \varepsilon) \|T \mid \beta_2\|$ and $\|(Y_i) \mid w_2\| \le 1$. Then for $x \in E$ and $b \in F$, we have

$$|\langle Tx, b \rangle| = \left| \sum_{i=1}^{\infty} \langle x, a_i \rangle \langle Y_i, b \rangle \right|$$

$$\le \left(\sum_{i=1}^{\infty} |\langle x, a_i \rangle|^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} |\langle Y_i, b \rangle|^2 \right)^{1/2}.$$

This implies that $\|Tx\| \le \left(\sum_{i=1}^{\infty} |\langle x, a_i \rangle|^2 \right)^{1/2}$ for all $X \in E$. Hence, $T \in \beta_2$ (E, F) and $\|T \mid \beta_2\| \le \|(a_i) \mid l_2\| \le (1 + \varepsilon) \| T \mid \beta_2 \|$.

Letting $\varepsilon \to 0$ yields $\|T \mid \beta_2\| \le \| T \mid \beta_2 \|$ for all $T \in \beta_2$ (E, F).

Let $T \in L(E, F)$, there exist a factorization

$$\begin{array}{ccc} & T & \\ E & \rightarrow & F \\ A \downarrow & & \uparrow Y \\ C(X) & \rightarrow & L_2(X, \mu) \\ & I & \end{array}$$

such that $\|A\| = \| T \mid \beta_2 \|$ and $\| Y \| = 1$. Moreover Y has finite rank. Let $Y = \sum_{i=1}^{m} f_i \otimes Y_i$ be any representation, given $\varepsilon > 0$, there exist a simple function $S_1, S_2, \ldots, S_m$ such that

$$\sum_{i=1}^{m} \|f_i - s_i \mid L_2\| \, \|Y_i\| \le \varepsilon.$$

Since, $\quad Z := \sum_{i=1}^{m} S_i \otimes Y_i \,.\, \|Y - Z\| \le \|Y - Z \mid \beta_2\| \le \| Y - Z \mid \beta \| \le \varepsilon$

and $\quad \| Z \| \le \| Y \| + \|Z - Y\| \le 1 + \varepsilon.$

Therefore, $S_i = \sum_{j=1}^{n} \alpha_{ij} h_j$, where $h_1, h_2, \ldots\ldots, h_n$ are characteristic functions of pairwise disjoint measurable subsets $X_1, \ldots\ldots, X_n$ of X. If $U_j := \mu(X_j)^{-1/2} h_j$ and $Z_j := \mu(X_j)^{1/2} \sum_{i=1}^{m} \alpha_{ij} Y_i$, then $Z = \sum_{i=1}^{n} U_j \otimes Z_j$, and $Zu_j = Z_j$, since (U_j) is an orthonormal family in $L_2(X, \mu)$.

Therefore, $\|(Z_j) \mid w_2\| \le \| Z \| \, \| (u_j) \mid w_2 \| \le 1 + \varepsilon.$

Since I denote the cannonical operator from C(X) into $L_2(X, \mu)$. Thus $I\mu_j$ is the function defined by u_j on C(X), and then

$$\| (I\mu_j) \mid l_2\| = \sum_{j=1}^{n} \| u_j \mid L_1\|^2)^{1/2} = \sum_{j=1}^{n} \mu(X_j))^{1/2} \le \mu(X)^{1/2} = 1.$$

Therefore, it follows from

$$ZI = \sum_{j=1}^{n} I\, u_j \otimes z_i \quad \text{that} \quad \| ZI \mid \beta_2 \| \le 1 + \varepsilon.$$

Next, $\quad \|YI \mid \beta_2\| \le \| ZI \mid \beta_2 \| + \| (Y - Z) I \mid \beta_2 \| \le 1 + 2\varepsilon.$

Therefore,

$$\|T \mid \beta_2\| \leq \|T \mid \beta_2\| = \|YIA \mid \beta_2\| \leq (1 + 2\varepsilon)\|A\| \leq (1 + 2\varepsilon)\|T \mid \beta_2\|.$$

Letting, $\varepsilon \to 0$, yields

$\|T \mid \beta_2\| = \|T \mid \beta_2\|$ for all $T \in L(E, F)$. Finally let, any operator $T \in \beta_2$ (E, F). Then there exists a sequence of operator $T_k \in L$ (E, F) converging to T with respect to the norm of β_2. Therefore, $\|(T_h - T_k) \mid \beta_2\| = \|T_h - T_k\| \, \| \beta_2\|$ then (T_k) is a sequence in the Banach operator ideal β_2 (E, F). Since T is the only possible limit, it follows that $T \in \beta_2$ (E, F). Moreover,

$$\|T \mid \beta_2\| = \lim_k \|T_k \mid \beta_2\| = \lim_k \|T_k \mid \beta_2\| = \|T \mid \beta_2\|.$$

Hence, the proof is complete.

3. Riesz Operators

An operator $T \in L(E)$ is said to be iteratively compact if for every $\varepsilon > 0$ there exists an exponent n and element $u_1, \ldots\ldots, u_k \in E$ such that $T^n (U) \subseteq U_{h=1}^{k}\{u_h + \varepsilon^n U\}$, where U denotes the closed unit ball of the underlying Banach space E.

Lemma. Let $T \in L(E)$ is iteratively compact. Let (X_1) be any sequence in U. Then for every $\varepsilon > 0$ there exist an exponent n and an infinite subset 1 of N such that $\|T^n x_i - T^n X^j\| \leq \varepsilon$ for all $i, j \in I$.

Proof. Let n and $u_1, \ldots\ldots, u_k \in E$ such that $T^n (U) \subseteq U_{h=1}^{k}\{u_h + \varepsilon/2U\}$. Setting $I_n := \{i \in N : T^n X_i \in u_h + \varepsilon^n U\}$ for $h = 1, \ldots\ldots, k$. Therefore, $\|T^n x_i - T^n X_j\| \leq \varepsilon$ for all $i, j \in I_h$. Furthermore, it follows from $U_{h=1}^{k} I_h = N$ that at least one of the sets $I_1, \ldots, I_k$ is infinite.

Preposition 1. If $T \in L(E)$ is iteratively compact, then all null spaces Nk $(I - T)$ are finite dimensional.

Proof. Since the smallest k for which $N_k (I - T)$ is infinite dimensional. Therefore from Reisz lemma with $\varepsilon = 1/3$, there exist an elements $X_i \in N_k(I - T)$ such that $\|X_i\| = 1$ and $\|X_1 - X\| \geq 3/4$ for all $x \in$ span $(x_1, \ldots, x_{i-1}) + N_{k-1} (I - T)$. It follows from $X - T^n X = (1 + T + \ldots. T^{n-1})(I - T) X$ that $X - T^n X \in N_{k-1}(I - T)$

for all $X \in N_k (I - T)$ and $n = 1, 2, \ldots\ldots\ldots$ Hence, $T^n X_i - T^n X_j \in X_i - X_j + N_{k-1}$ $(I - T)$, which implies that $\|T^n X_i - T^n X_j\| \geq 3/4$ whenever $i > j$ and $n = 1, 2, \ldots\ldots$ Therefore by the Pigeon-hole principle, there exist an exponent n and different indices I and J such that $\|T^n X_i - T^n X_j\| \leq 1/2$, which is a contradiction, and the theorem is proved.

Preposition 2. If $T \in L(E)$ is iteratively compact, then $I - T$ has finite ascent.

Proof. From Riesz lemma with $\varepsilon = 1/3$. Let $X_k \in N_k (I - T)$ such that $\| X_k\| = 1$ and $\|X_k - X\| \geq 3/4$ for all $X \in N_k (I - T)$. It follows from $X = (1 + T + \ldots\ldots T^{n-1})$ $(I - T) X$ that $X_k - T^n X_k \in N_{k-1} (I - T)$ for all $n = 1, 2, \ldots\ldots\ldots$ Hence $T^n X_h - T^n X_k \in X_h - X_k + N_{h-1} (I - T) + N_{k-1} (I - T)$, which implies that $\|T^n X_h - T^n X_j k\| \geq 3/4$ whenever $h > k$ and $n = 1, 2, \ldots\ldots$ Therefore, by the Pigeon-hole principle, there exists an exponent n and different iidicates h and k such that $\|T^n X_h - T^n X_k\| \leq 1/2$, which is a contradiction and the theorem is proved.

Lemma. Let $T \in L(E)$ is iteratively compact, then every bounded sequence (X_i) for which $((I - T X_i)$ is convergent has a convergent subsequence.

Proof. Let (X_i) is contained in U. Given $\varepsilon > 0$. By the Pigeon-hole principle, there exists an exponent n and an infinite subset I such that $\|T^n X_i - T^n X_j\| \leq \varepsilon$ for all $i, j \in I$. It follows from $X \in T^n X |(I | T | \ldots. | T^{n-1}) (I - T) X$ that

$$\| X_i - X_j \| \leq \|T^n X_i - T^n X_j\| + (I + T + \ldots\ldots + T^{n-1}) \| \; \| (I - T) X_i - (I - T) X_j \|.$$

Thus, since $((I - T) X_i)$ is an infinite subset I_0 of I such that $\| X_i - X_j \| \leq 2\varepsilon$ for all $i, j \in I_0$. Let $(X_i^0) = (X_i)$ and $\varepsilon_m := 2^{-m-1}$ for $m = 1, 2, \ldots.$, Therefore a sequence (X_i^m) each of which is a subsequence of its predecessor (X_i^{m-1}) and such that $\|X_i^m - X_j^m \| \leq 2\varepsilon_m$ for all i and j. Then the diagonal (X_i^i) is the desired convergence subsequence, because $\|X_i^i - X_j^j \| \leq 2^{-m}$ whenever $i, j \geq m$.

Preposition 3. If $T \in L(E)$ is iteratively compact, then all ranges $M_h (I - T)$ are closed.

Proof. Let $Y = \lim_i Y_i$ where (Y_i) is contained in $M_h (I - T)$. Set $\rho_i = \inf \{\| X \| : (I - T)^h X = Y_i\}$, and let $X_i \in E$ such that $(I - T)^h X_i = Y_i$ and $\|X_i\| \leq 2\rho_i$. Let

$\rho_i \to \infty$, then $u_i := \rho_i^{-1} X_i$ and $v_i = \rho_i^{-1} Y_i$. Then $\| u_i \| \leq 2$ ard (V_i) tends to zero. A subsequence of (u_i) which converges to some $u \in E$. Therefore, $(I - T)^h U_i = V_i$ implies $(I - T)^h u = 0$. Hence $(I - T)^h (X_i - \rho i\, u) = Y_i$. Thus, by the definition of ρ_i, we have $\|X_i - \rho_i u\| \geq \rho_i$ or $\| u_i - u \| \geq 1$, which shows that (ρ_i) has a bounded subsequence.

Hence the theorem is proved.

Preposition 4. If $T \in L(E)$ is iteratively compact, then $I - T$ has finite descent.

Proof. From Riesz lemma with $\varepsilon = 1/3$. Let $Y_h \in M_h (I - T)$ such that $\|Y_h\| = 1$ and $\|Y_h - Y\| \geq 3/4$ for all $Y \in M_{h+1} (I - T)$. It follows from, $Y - T^n Y = (I + T + ... + T^{n-1}) (I - T) Y$ that $Y_h - T^n Y_h \in M_{h+1} (I - T)$, for $n = 1, 2, ...$

Hence $T^n Y_h - T^n Y_k \in Y_h - Y_k + M_{h+1} (I - T) + M_{k+1} (I - T)$, which implies that, $\| T^n Y_h - T^n Y_k \| \geq 3/4$ whenever $h < k$ and $n = 1, 2, ...$ Therefore, by the Pigeon-hole principle, there exists an exponent n and different indices h and k such that $\| T^n Y_h - T^n Y_k \| \geq 1/2$, which is a contradiction and the theorem is proved.

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