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Quasi Conformal Curvature Tensor on a Lorentzian Para-Sasakian Manifold

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Abstract

In this paper, we consider quasi-conformally flat, quasi-conformally conservative and ϕ-quasi conformally flat Lorentzian para-Sasakian manifold. It has also been proved that an Einstein Lorentzian para-Sasakian manifold satisfying the relation $R(X, Y).\widetilde{C} = 0$, where $\widetilde{C}$ is quasi-conformal curvature tensor is locally isometric with a unit sphere.

Keywords : LPS manifold, Quasi conformal curvature tensor, ϕ-quasi conformally flat.

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1. Introduction

An n-dimensional differentiable manifold M^n is a Lorentzian para-Sasakian (LP-Sasakian) manifold, if it admits a (1, 1)-tensor field ϕ, vector field ξ, 1-form η and a Lorentzian metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi, \tag{1.1}$$

$$\eta(\xi) = -1, \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{1.3}$$

$$g(X, \xi) = \eta(X), \tag{1.4}$$

$$(D_X\phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1.5}$$

and $$D_X\xi = \phi X, \tag{1.6}$$

for arbitrary vector fields X and Y; where D denotes covariant differentiation with respect to g, (Matsumoto, (1989) and Matsumoto and Mihai, (1988)).

In an LP-Sasakian manifold M^n with structure (ϕ, ξ, η, g), it is easily seen that

$$\text{(a)} \quad \phi\xi = 0 \qquad \text{(b)} \quad \eta(\phi X) = 0 \qquad \text{(c)} \quad \text{rank } \phi = (n-1). \tag{1.7}$$

Let us put

$$F(X, Y) = g(\phi X, Y). \tag{1.8}$$

Then the tensor field F is symmetric (0, 2) tensor field

$$F(X, Y) = F(Y, X), \tag{1.9}$$

$$F(X, Y) = (D_X \eta)(Y), \tag{1.10}$$

and $$(D_X \eta)(Y) - (D_X \eta)(X) = 0. \tag{1.11}$$

An LP-Sasakian manifold M^n is said to be Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = kg(X, Y), \tag{1.12}$$

where $k = (n-1)$.

An LP-Sasakian manifold M^n is said to be η-Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y), \tag{1.13}$$

for any vector fields X and Y, where α, β are functions on M^n.

Let M^n be an n-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g). Then we have (Matsumoto and Mihai, (1988) and Mihai, Shaikh and De (1999)).

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{1.14}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{1.15(a)}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{1.15(b)}$$

$$R(X, Y)\,\xi = \eta(Y)\,X - \eta(X)\,Y, \tag{1.15)(c}$$

$$S(X, \xi) = (n-1)\,\eta(X), \tag{1.16}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\,\eta(X)\,\eta(Y), \tag{1.17}$$

for any vector fields X, Y, Z, where R(X, Y) Z is the Riemannian curvature tensor of type (1, 3), S is the Ricci-tensor of type (0, 2), Q is (1, 1) type Ricci tensor and r is the scalar curvature, g (QX, Y) = S(X, Y), for all X, Y.

Quasi-conformal curvature tensor $\tilde{C}$ on a Riemannian manifold (M^n, g), $(n > 3)$ of type (1, 3) is defined as follows (Yano and Sawaki, (1968)).

$$\tilde{C}(X, Y)Z = a\,R(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]$$
$$-\frac{r}{n}\left[\frac{a}{(n-1)} + 2b\right][g(Y, Z)X - g(X, Z)Y], \tag{1.18}$$

where a, b are constants such that $a, b \neq 0$.

If $a = 1$ and $b = -\frac{1}{(n-2)}$, then (1.18) takes the form

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{1}{(n-2)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX$$
$$- g(X, Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] = C(X, Y)Z,$$

where C is the conformal curvature tensor. Thus the conformal curvature tensor C is a particular case of the tensor $\tilde{C}$. For this reason $\tilde{C}$ is called the quasi-conformal curvature tensor.

Let
$$L(X, Y) = S(X, Y) - \frac{r}{2(n-1)}\,g(X, Y), \tag{1.19}$$

and
$$g(NX, Y) = L(X, Y), \tag{1.20}$$

where L and N are tensor field of type (0, 2) and (1, 1) respectively.

From (1.19) and (1.20), we get

$$N(X) = QX - \frac{r}{2(n-1)}\,X. \tag{1.21}$$

Using (1.19) and (1.20), we can write (1.18) as follows

$$\tilde{C}(X, Y) Z = a R(X, Y) Z + b[L(Y, Z) X - L(X, Z) Y + g(Y, Z) NX - g(X, Z) NY] - \lambda r [g(Y, Z) X - g(X, Z) Y], \quad (1.22)$$

where $\lambda = \frac{a + (n-2) b}{n (n-1)}$.

2. An Einstein LP-Sasakian manifold satisfying $\tilde{C}(X, Y) Z = 0$.

In this section we assume that $\tilde{C}(X, Y) Z = 0$.

Then from (1.18), we get

$$a R(X, Y) Z = - b[S(Y, Z) X - S(X, Z) Y + g(Y, Z) QX - g(X, Z) QY] + \frac{r}{n}\left[\frac{a}{(n-1)} + 2b\right] [g(Y, Z) X - g(X, Z) Y], \quad (2.1)$$

or $$a\, 'R(X, Y, Z, W) = - b[S(Y, Z) g(X, W) - S(X, Z) g(Y, W) + g(Y, Z) S(X, W) - g(X, Z) S(Y, W)] + \frac{r}{n}\left[\frac{a}{(n-1)} + 2b\right] [g(Y, Z) g(X, W) - g(X, Z) g(Y, W)], \quad (2.2)$$

where $'R(X, Y, Z, W) = g(R(X, Y) Z, W)$.

Putting $X = W = \xi$ in (2.2) and using (1.12), we get

$$[a + 2b (n-1)][r - n (n-1)] g(\phi Y, \phi Z) = 0. \quad (2.3)$$

Thus we see that $g(\phi Y, \phi Z) \neq 0$.

Hence from (2.3), we get $r = n (n-1)$, provided $a + 2b (n-1) \neq 0$.

Hence, we can state the following theorem :

Theorem 1. An Einstein LP-Sasakian manifold satisfying the condition $\tilde{C}(X, Y) Z = 0$, has constant curvature $r = n (n-1)$, provided $a + 2b (n-1) \neq 0$.

Contracting equation (1.18) with respect to X, we get

$$(S_1^1 \tilde{C})(Y, Z) = a\, S(Y, Z) + b[S(Y, Z)\, n - S(X, Z) + g(Y, Z)\, r - S(Y, Z)]$$
$$- \frac{r}{n}\left[\frac{a}{(n-1)} + 2b\right][g(Y, Z)\, n - g(Y, Z)\,]$$
$$= [a + (n-2)\, b]\,[S(Y, Z) - \frac{r}{n}\, g(Y, Z)], \tag{2.4}$$

where $(S_1^1 \tilde{C})(Y, Z)$ is the contraction of $\tilde{C}(X, Y) Z$ with respect to X.

If $(S_1^1 \tilde{C})(Y, Z) = 0$, we get

$$S(Y, Z) = \frac{r}{n}\, g(Y, Z), \text{ provided } a + (n-2)\, b \neq 0. \tag{2.5}$$

Hence from (2.2) it follows that

$$a\,['R(X, Y, Z, W) - \frac{r}{n\,(n-1)}\,\{g(Y, Z)\, g(X, W) - g(X, Z)\, g(Y, W)\}] = 0. \tag{2.6}$$

Therefore, from (2.6), we get

$$'R(X, Y, Z, W) = \frac{r}{n\,(n-1)}\,[g(Y, Z)\, g(X, W) - g(X, Z)\, g(Y, W)], \tag{2.7}$$

provided $a \neq 0$.

Hence, we can state the following theorem :

Theorem 2. A quasi-conformally flat (M^n, g) $(n > 3)$, LP-Sasakian manifold satisfying $(S_1^1 \tilde{C})(Y, Z) = 0$ is a manifold of constant curvature provided $a \neq 0$.

Using (2.5) and (2.7) in (2.2), we get

$$'\tilde{C}(X, Y, Z, W) = 0.$$

From this it follows that

$$C(X, Y)\, Z = 0.$$

Hence the manifold is quasi-conformally flat.

Hence, we can state the following theorem :

Theorem 3. An LP-Sasakian manifold (M^n, g) $(n > 3)$ satisfying $(S_1^1 \tilde{C})(Y, Z) = 0$, of constant curvature is quasi-conformally flat.

3. Einstein LP-Sasakian manifold satisfying $(\text{div}\, \tilde{C})(X, Y) Z = 0$.

Definition. A manifold (M^n, g) $(n > 3)$ is called quasi-conformally conservative if (Hicks, N. J. (1969)), $\text{div}\, \tilde{C} = 0$.

In this section we assume that

$$\text{div}\, \tilde{C} = 0, \tag{3.1}$$

where div denotes divergence.

From (1.21), we get

$$N = Q - \frac{rI}{2(n-1)}.$$

Hence

$$\text{div}\, N = \text{div}\, Q - \frac{dr}{2(n-1)}. \tag{3.2}$$

But $\text{div}\, Q = \frac{1}{2}\, dr$, therefore

$$\text{div}\, N = \frac{(n-2)}{2(n-1)}\, dr. \tag{3.3}$$

Now differentiating (1.22) covariantly, we get

$$\begin{aligned}(D_W \tilde{C})(X, Y) Z = {} & a (D_W R)(X, Y) Z + b[(D_W L)(Y, Z) X - (D_W L)(X, Z) Y \\ & + g(Y, Z)(D_W N)(X) - g(X, Z)(D_W N)(Y)] \\ & - \lambda (D_W r)[g(Y, Z) X - g(X, Z) Y],\end{aligned} \tag{3.4}$$

which gives on contraction

$$\begin{aligned}(\text{div}\, \tilde{C})(X, Y) Z = {} & a (\text{div}\, R)(X, Y) Z + b[(D_X L)(Y, Z) - (D_Y L)(X, Z)] \\ & + \left[\frac{(n-2)\, b}{2(n-1)} - \lambda\right][g(Y, Z)\, dr(X) - g(X, Z)\, dr(Y)].\end{aligned} \tag{3.5}$$

We have (Eisenhart, L. P. (1926)).

$$(\text{div } R)(X, Y) Z = (D_X S)(Y, Z) - (D_Y S)(X, Z)$$
$$= (D_X L)(Y, Z) - (D_Y L)(X, Z) + \frac{1}{2(n-1)} [g(Y, Z)\, dr(X) - g(X, Z)\, dr(Y)]. \quad (3.6)$$

Hence (3.5) takes the form

$$(\text{div } \tilde{C})(X, Y) Z = (a + b) [(D_X L)(Y, Z) - (D_Y L)(X, Z)]$$
$$+ \frac{(n-2)[a + b(n-2)]}{2n(n-1)} [g(Y, Z)\, dr(X) - g(X, Z)\, dr(Y)]. \quad (3.7)$$

If LP-Sasakian manifold is an Einstein manifold, then we have

$$(D_X L)(Y, Z) - (D_Y L)(X, Z) = 0,$$

which gives from (3.7) that

$$(\text{div } \tilde{C})(X, Y) Z = \frac{(n-2)[a + b(n-2)]}{2n(n-1)} [g(Y, Z)\, dr(X) - g(X, Z)\, dr(Y)]. \quad (3.8)$$

Hence if div $\tilde{C} = 0$, then $g(Y, Z)\, dr(X) - g(X, Z)\, dr(Y) = 0$, provided $a + (n-2) b \neq 0$. Consequently r is constant. Again if r is constant then from (3.8) it follows that $(\text{div } \tilde{C})(X, Y, Z) = 0$.

Hence, we can state the following theorem :

Theorem 4. An Einstein LP-Sasakian manifold (M^n, g) $(n > 3)$ is quasi conformally conservative if and only if the scalar curvature is constant, provided $a + (n-2) b \neq 0$.

4. LP-Sasakian manifold satisfying $\phi^2 \tilde{C}(\phi X, \phi Y) \phi Z = 0$.

Definition. A differentiable manifold (M^n, g) $(n > 3)$, satisfying the condition $\phi^2 \tilde{C}(\phi X, \phi Y) \phi Z = 0$ is called f-quasi conformally flat (Cabreizo, Fernandez, Fernandez and Zhen (1999)).

In this section we assume that LP-Sasakian manifold (M^n, g) $(n > 3)$, is ϕ-quasi conformally flat, then $\phi^2 \tilde{C}(\phi X, \phi Y) \phi Z = 0$ implies

$$g(\widetilde{C}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{4.1}$$

for any vector fields X, Y, Z, W.

So by the use of (1.18), ϕ-quasi conformally flat means

$$a\,'R(\phi X, \phi Y, \phi Z, \phi W) = -b\,[S(\phi Y, \phi Z)\,g(\phi X, \phi W) - S(\phi X, \phi Z)\,g(\phi Y, \phi W)$$
$$+ g(\phi Y, \phi Z)\,S(\phi X, \phi W) - g(\phi X, \phi Z)\,S(\phi Y, \phi W)] + \frac{r}{n}\left[\frac{a}{(n-1)} + 2b\right]$$
$$[g(\phi Y, \phi Z)\,g(\phi X, \phi W) - g(\phi X, \phi Z)\,g(\phi Y, \phi W)], \tag{4.2}$$

where $'R(\phi X, \phi Y, \phi Z, \phi W) = g(R(\phi X, \phi Y)\phi Z, \phi W)$.

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthogonal basis of vector fields in M^n by using the fact that $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (4.2) and sum up with respect to i, then we have

$$\sum_{i=1}^{(n-1)} a\,'R(\phi e_i, \phi Y, \phi Z, \phi e_i) = -b \sum_{i=1}^{(n-1)} [S(\phi Y, \phi Z)\,g(\phi e_i, \phi e_i) - S(\phi e_i, \phi Z)\,g(\phi Y, \phi e_i)$$
$$+ g(\phi Y, \phi Z)\,S(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)\,S(\phi Y, \phi e_i)] + \frac{r}{n}\left[\frac{a}{(n-1)} + 2b\right]$$
$$\sum_{i=1}^{(n-1)} [g(\phi Y, \phi Z)\,g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)\,g(\phi Y, \phi e_i)]. \tag{4.3}$$

On an LP-Sasakian manifold, we have (Özgür (2003))

$$\sum_{i=1}^{(n-1)} {'R}(\phi e_i, \phi Y, \phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{4.4}$$

$$\sum_{i=1}^{(n-1)} S(\phi e_i, \phi e_i) = r + n - 1, \tag{4.5}$$

$$\sum_{i=1}^{(n-1)} g(\phi e_i, \phi Z)\,S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \tag{4.6}$$

$$\sum_{i=1}^{(n-1)} g(\phi e_i, \phi e_i) = n + 1, \tag{4.7}$$

$$\sum_{i=1}^{(n-1)} g(\phi e_i, \phi Z)\, g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{4.8}$$

So by virtue of (4.4) - (4.8), the equation (4.3) takes the form

$$[a + b(n-1)]\, S(\phi Y, \phi Z) = [\frac{ar}{(n-1)} + br - bn + b - a]\, g(\phi Y, \phi Z). \tag{4.9}$$

Then by making use of (1.3) and (1.17), the equation (4.9) takes the form

$$[a + b(n-1)]\, [S(Y, Z) - \left(\frac{r}{n-1} - 1\right) g(Y, Z) - \left(\frac{r}{n-1} - n\right) \eta(Y)\, \eta(Z)] = 0,$$

which gives

$$S(Y, Z) = \left(\frac{r}{n-1} - 1\right) g(Y, Z) + \left(\frac{r}{n-1} - n\right) \eta(Y)\, \eta(Z),$$

provided $a + (n-1)\, b \neq 0$.

Which shows that M^n is an η-Einstein manifold, provided $a + (n-1)\, b \neq 0$, with constants α and β are same as in η-Einstein manifold of an LP-Sasakian manifold, given by $\alpha = \left(\frac{r}{n-1} - 1\right)$ and $\beta = \left(\frac{r}{n-1} - n\right)$.

Hence, we can state the following theorem :

Theorem 5. Let M^n be an n-dimensional ($n > 3$), ϕ-quasi conformally flat LP-Sasakian manifold. Then M^n is an η-Einstein manifold, provided $a + (n-1)\, b \neq 0$, with constants $\alpha = \left(\frac{r}{n-1} - 1\right)$ and $\beta = \left(\frac{r}{n-1} - n\right)$.

5. An Einstein LP-Sasakian manifold satisfying $R(X, Y).\tilde{C} = 0$.

In this section we assume that $R(X, Y).\tilde{C}(U, V)\, W = 0$. (5.1)

Let the Riemannian manifold M^n be an Einstein manifold, then (1.18) gives

$$'\tilde{C}(X, Y, Z, W) = a\,'R(X, Y, Z, W) + \left[2bk - \frac{r}{n}\left\{\frac{a}{(n-1)} + 2b\right\}\right][g(Y, Z)\, g(X, W) - g(X, Z)\, g(Y, W)], \qquad (5.2)$$

where $'\tilde{C}(X, Y, Z, W) = g(\tilde{C}(X, Y) Z, W)$.

Putting $W = \xi$ in (5.2) and using (1.14), we get

$$\eta(\tilde{C}(X, Y) Z) = \left[a + 2bk - \frac{r}{n}\left\{\frac{a}{(n-1)} + 2b\right\}\right][g(Y, Z)\, \eta(X) - g(X, Z)\, \eta(Y)]. \quad (5.3)$$

Taking $X = \xi$ in (5.3), we get

$$\eta(\tilde{C}(\xi, Y) Z) = \left[a + 2bk - \frac{r}{n}\left\{\frac{a}{(n-1)} + 2b\right\}\right][- g(Y, Z) - \eta(Y)\, \eta(Z)]. \qquad (5.4)$$

$$\eta(\tilde{C}(X, Y)\, \xi) = 0. \qquad (5.5)$$

Now,

$$(R(X, Y).\tilde{C})(U, V) W = R(X, Y)\, \tilde{C}(U, V) W - \tilde{C}(R(X, Y) U, V) W - \tilde{C}(U, R(X, Y) V) W - \tilde{C}(U, V) R(X, Y) W. \qquad (5.6)$$

In view of (5.1), we get

$$R(X,Y)\tilde{C}(U,V)W - \tilde{C}(R(X,Y)U, V)W - \tilde{C}(U,R(X,Y)V)W - \tilde{C}(U,V)R(X,Y)W = 0.$$

Putting $X = \xi$ and taking the inner product of the above equation with ξ, we get

$$g(R(\xi, Y)\tilde{C}(U, V)W, \xi) - g(\tilde{C}(R(\xi, Y)U, V)W, \xi) - g(\tilde{C}(U, R(\xi, Y)V)W, \xi) - g(\tilde{C}(U, V)\, R(\xi, Y)W, \xi) = 0.$$

From this it follows that

$$- '\tilde{C}(U, V, W, Y) - \eta(Y)\, \eta(\tilde{C}(U, V)W) + \eta(U)\, \eta(\tilde{C}(Y,V)W) + \eta(V)\, \eta(\tilde{C}(U, Y)W) + \eta(W)\, \eta(\tilde{C}(U, V)Y) - g(Y, U)\eta(\tilde{C}(\xi,V)W) - g(Y, V)\, \eta(\tilde{C}(U, \xi)W) - g(Y, W).\eta(\tilde{C}(U,V)\, \xi) = 0. \qquad (5.7)$$

Putting Y = U in (5.7), we get

$$-{}'\widetilde{C}(U, V, W, U) - \eta(U)\,\eta(\widetilde{C}(U, V)W) + \eta(U)\,\eta(\widetilde{C}(U,V)W) + \eta(V)\,\eta(\widetilde{C}(U, U)W)$$
$$+\,\eta(W)\,\eta(\widetilde{C}(U, V)U) - g(U, U)\,\eta(\widetilde{C}(\xi,V)W) - g(U, V)\,\eta(\widetilde{C}(U, \xi)W)$$
$$-\,g(U, V).\eta(\widetilde{C}\,(U, \xi)\,W) - g(U, W).\eta(\widetilde{C}\,(U, V)\,\xi) = 0. \qquad (5.8)$$

Let $\{e_i\}$, i = 1, 2, 3, ..., n be an orthogonal basis of the tangent space at any point. Then the sum for $1 \le i \le n$ of relation (5.8), for $U = e_i$, gives

$$\eta(\widetilde{C}\,(\xi, V)W) = \frac{1}{(n-1)}\left[-a\,S(V, W) - \left\{2bk - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right\}(n-1)\,g(V, W)\right.$$
$$\left. - \left\{a + 2bk - \frac{r}{n}\left(\frac{a}{(n-1)} + 2b\right)\right\}(n-1)\,\eta(V)\,\eta(W)\right]. \qquad (5.9)$$

Using (5.3) and (5.9), it follows from (5.7) that

$${}'\widetilde{C}\,(U, V, W, Y) = \left[2bk - \frac{r}{n}\left\{\frac{a}{n-1} + 2b\right\}\right][g(V, W)\,g(Y, U) - g(V, Y)\,g(U, W)]$$
$$+\frac{a}{(n-1)}\,[S(V, W)\,g(Y, U) - S(U, W)\,g(V, Y)]. \qquad (5.10)$$

Using (1.12) in (5.10), we get

$${}'\widetilde{C}\,(U, V, W, Y) = \left[a + 2bk - \frac{r}{n}\left\{\frac{a}{n-1} + 2b\right\}\right][g(V, W)\,g(Y, U) - g(V, Y)\,g(U, W)]\,. \qquad (5.11)$$

From equation (5.2) and (5.11), we get

$${}'R(U, V, W, Y) = g(V, W)\,g(Y, U) - g(U, W)\,g(V, Y), \text{ provided } a \neq 0.$$

Hence, we can state the following theorem :

Theorem 6. If in an Einstein LP-Sasakian manifold, the relation $R(X, Y).\widetilde{C} = 0$ holds, then it is locally isometric with a unit sphere, provided $a \neq 0$.

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